CS054: Countability

The goal of this worksheet is to give you practice with cardinality, countability, and uncountability. It's not for a grade—no need to turn it in! I'll post solutions, but you'll get the most out of it if you don't peek.

1. Write an injective function from `bool` to RPS.

2. How many different injective functions from `bool` to RPS are there?

3. Write a surjective function from RPS to `bool`.

4. How many surjective functions from RPS to `bool` are there?

5. (a) Give an example of finite sets A and B where $|A \cup B| = |A| + |B|$.

(b) Give an example of finite sets A and B where $|A \cup B| \neq |A| + |B|$.

(c) Give a condition on finite sets A and B that characterizes when $|A \cup B| = |A| + |B|$. No need to prove it (but good to think about how you might!).

6. Prove that if A is countable, then so is $\texttt{option}(A) = \{\texttt{some}(a) \mid a \in A\} \uplus \{\texttt{none}\}$.

Pro tip: for practice, try to prove this using both injection and surjection!

7. The English alphabet has 26 letters, A through Z. Prove that the set of possible words—i.e., one or more letters—is countable.

   (This is a hard question. Hint: can you use primes in a creative way? Feel free to assume useful facts about prime factoring.)

8. Prove that $\texttt{bt}(\mathbb{N})$ is countable.
   (Hint: first do question (7); then try to use primes.)

9. In light of questions (6), (7), and (8), give a general argument (but not a proof) that every inductive data type over countable components is countable.

10. (a) Give an example of a set A where $A \to \mathbb{N}$ is countable. No need to prove it (but it's good practice, of course!).

(b) Give an example of a set A where $A \to \mathbb{N}$ is countable. No need to prove it (but it's good practice, of course!).

(c) Characterize when $A \to \mathbb{N}$ is countable. No need to prove it (but how would you?).

11. Prove that the complex numbers $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ are uncountable.

12. Prove that $\mathbb{N} \to \texttt{base}$ is uncountable.