CS054: Countability

The goal of this worksheet is to give you practice with cardinality, countability, and uncountability. It’s not for a grade—no need to turn it in! I’ll post solutions, but you’ll get the most out of it if you don’t peek.

1. Write an injective function from bool to RPS.

2. How many different injective functions from bool to RPS are there?

3. Write a surjective function from RPS to bool.

4. How many surjective functions from RPS to bool are there?
5. (a) Give an example of finite sets $A$ and $B$ where $|A \cup B| = |A| + |B|$.

(b) Give an example of finite sets $A$ and $B$ where $|A \cup B| \neq |A| + |B|$.

(c) Give a condition on finite sets $A$ and $B$ that characterizes when $|A \cup B| = |A| + |B|$. No need to prove it (but good to think about how you might!).
6. Prove that if $A$ is countable, then so is $\text{option}(A) = \{\text{some}(a) \mid a \in A\} \cup \{	ext{none}\}$. 

Pro tip: for practice, try to prove this using both injection and surjection!
7. The English alphabet has 26 letters, A through Z. Prove that the set of possible words—i.e., one or more letters—is countable.

(This is a hard question. Hint: can you use primes in a creative way? Feel free to assume useful facts about prime factoring.)
8. Prove that $\mathbb{bt}(N)$ is countable.
   (Hint: first do question (7); then try to use primes.)

9. In light of questions (6), (7), and (8), give a general argument (but not a proof) that every inductive data type over countable components is countable.
10.  (a) Give an example of a set $A$ where $A \rightarrow \mathbb{N}$ is countable. No need to prove it (but it’s good practice, of course!).

(b) Give an example of a set $A$ where $A \rightarrow \mathbb{N}$ is countable. No need to prove it (but it’s good practice, of course!).

(c) Characterize when $A \rightarrow \mathbb{N}$ is countable. No need to prove it (but how would you?).
11. Prove that the complex numbers \( \mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\} \) are uncountable.

12. Prove that \( N \to \text{base} \) is uncountable.