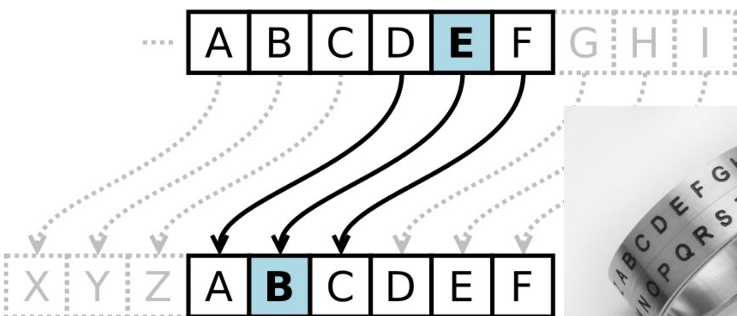
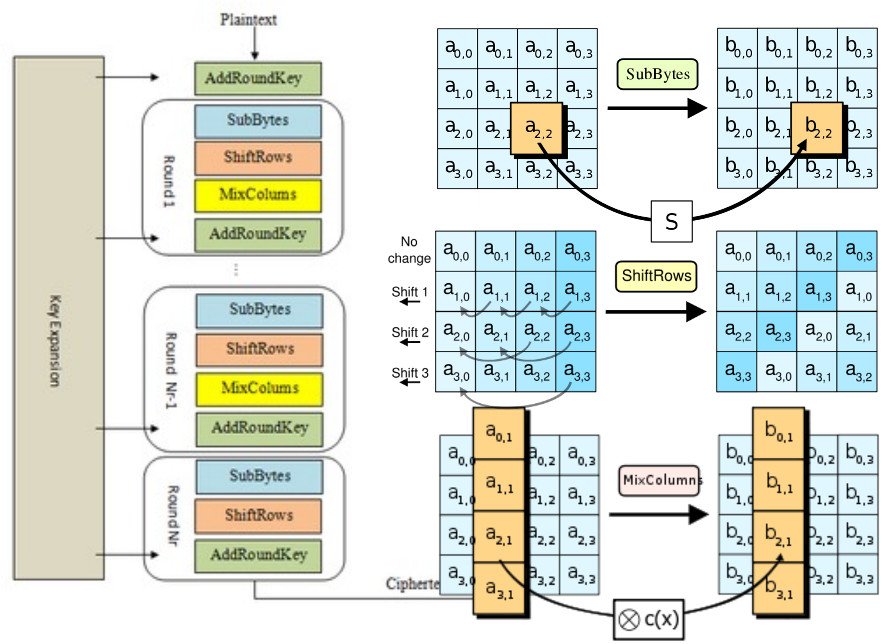
Lecture 7: Public-Key Cryptography

CS 181S

Spring 2024

Crypto Thus Far…



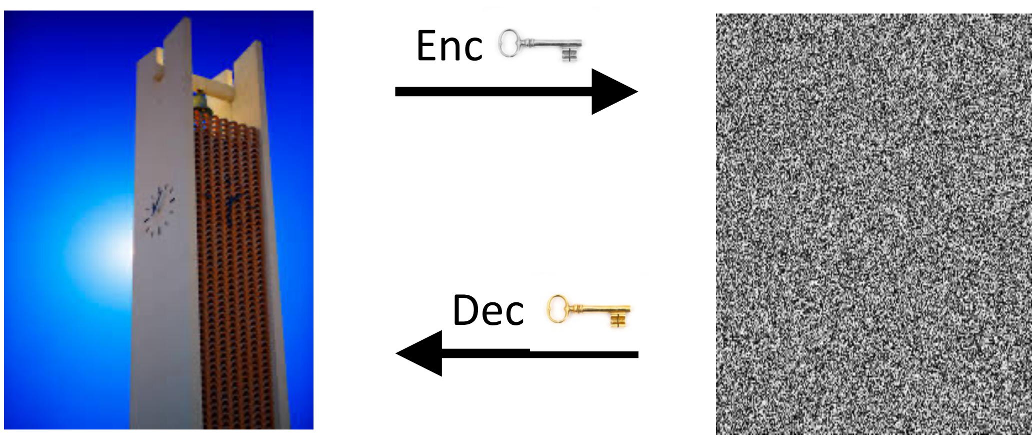


Key pairs

- Instead of sharing a key between pairs of principals...
- ...every principal has a pair of keys
  - **public key:** published for the world to see
  - **private key:** kept secret and never shared

(Public-Key) Encryption algorithms

- $\mathrm{Gen}(1^n)$: generate a **keypair** (pk, sk) of length n
- $\mathrm{Enc}(m; pk)$: encrypt **message** under public key pk
- $\mathrm{Dec}(c; sk)$: decrypt **ciphertext** c with secret key sk



$(\mathrm{Gen}, \mathrm{Enc}, \mathrm{Dec})$ is a public-key **encryption scheme** aka **cryptosystem**

RSA

[Rivest, Shamir, Adleman 1977]

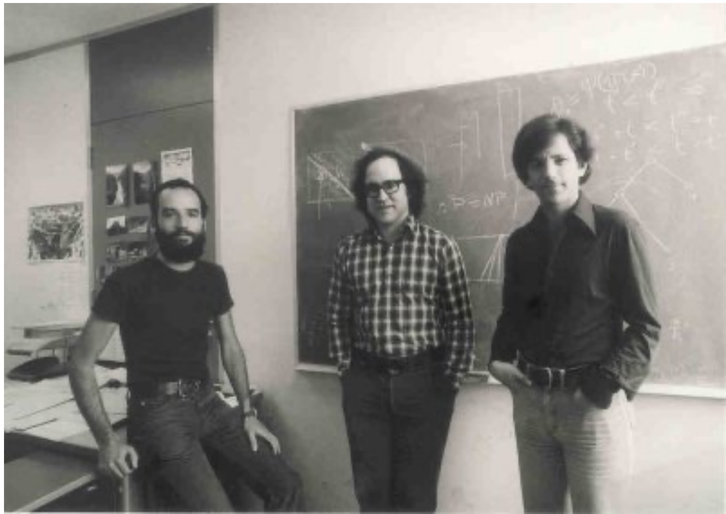
Shared Turing Award in 2002: *ingenious contribution to making public-key crypto*

- Gen(len):
  - Pick primes p, q, define $n = p \cdot q$
  - Choose e, d such that $ed = 1 \bmod (p-1)(q-1)$
  - $pk = (n, e), \;\; sk = (p, q, d)$
- Enc(m, pk)

$$c = m^e \bmod n$$

- Dec(c, sk):

$$m = c^d \bmod n$$



Exercise 1: RSA

- Let $pk = (n, e) = (21, 5)$ and $sk = (p, q, d) = (3, 7, 5)$
- Observe that $ed = 5 \cdot 5 = 25 = 1 \bmod 12$

1. Compute $c = \mathrm{Enc}(17; pk)$

2. Compute $\mathrm{Dec}(c; sk)$

RSA

- Theorem: RSA is a correct public-key encryption scheme.
- Theorem: $(m^e \bmod pq)^d \bmod pq == m$

$$\begin{aligned}\mathrm{Dec}(\mathrm{Enc}(m; pk); sk) &= (m^e \bmod pq)^d \bmod pq \\ &= (m^e)^d \bmod pq \\ &= m^{ed} \bmod pq \\ &= m \bmod pq\end{aligned}$$

$$\begin{aligned}m^{ed} \bmod p &= m^{1+k(p-1)(q-1)} \bmod p \\ &= m^1 \cdot (m^{p-1})^{k(q-1)} \bmod p \\ &= m \cdot (m^{p-1} \bmod p)^{k(q-1)} \bmod p \\ &= m \cdot (1)^{k(q-1)} \bmod p \\ &= m \bmod p\end{aligned}$$

$$\begin{aligned}m^{ed} \bmod q &= m^{1+k(p-1)(q-1)} \bmod q \\ &= m^1 \cdot (m^{q-1})^{k(p-1)} \bmod q \\ &= m \cdot (m^{q-1} \bmod p)^{k(p-1)} \bmod q \\ &= m \cdot (1)^{k(p-1)} \bmod q \\ &= m \bmod q\end{aligned}$$

RSA

- ~~**Theorem:** RSA is a secure public-key encryption scheme.~~
- ~~**Theorem:** If factoring is hard, then RSA is a secure public-key encryption scheme.~~

Problems with Textbook RSA

- *Insecure keys:* There are known efficient attacks for some choices of e, d, p, q
  - *pq is too small:* $n = pq$ can be factored

    make sure n is 2048 bits
  - *small values of e:* if you send the same message with multiple keys, plaintext message can be recovered
  - *small values of e:* if $m^e < n$ can compute log efficiently
  - *small values of d:* Boneh-Durfee attack for $d < n^{0.292}$

    e = 65537
  - *p or q is shared with other key:* can compute gcd to factor

    use high-quality randomness
  - *p similar to q:* $n = pq$ can be efficiently factored
  - *p or q are bad primes:* $n = pq$ can be efficiently factored
  - *gcd(e, lcm(p-1,q-1)) > 1*:

    use a good library, don't implement RSA yourself!

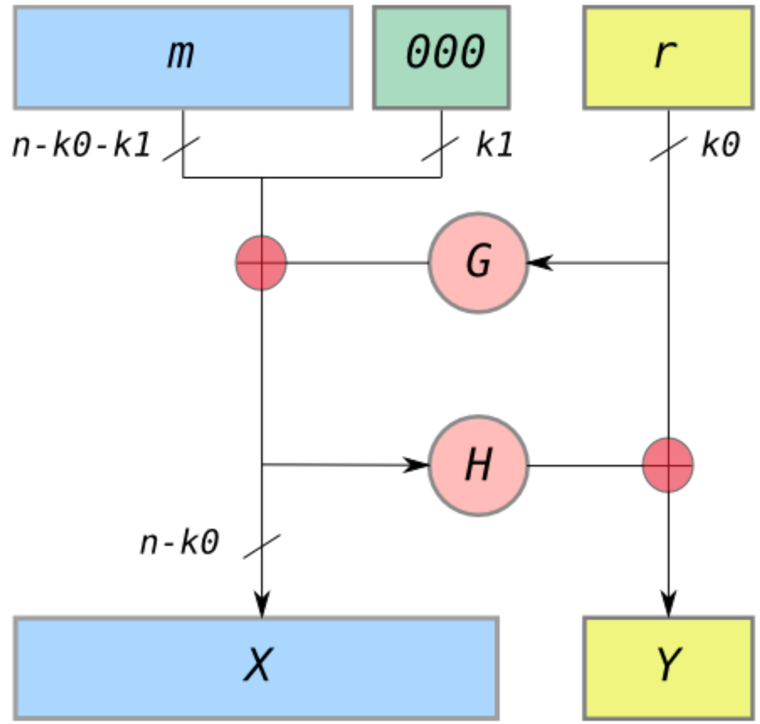
Problems with Textbook RSA

- *Insecure keys:* There are known efficient attacks for some choices of e, d, p, q

- *Deterministic*: given same plaintext and key, always produces the same ciphertext

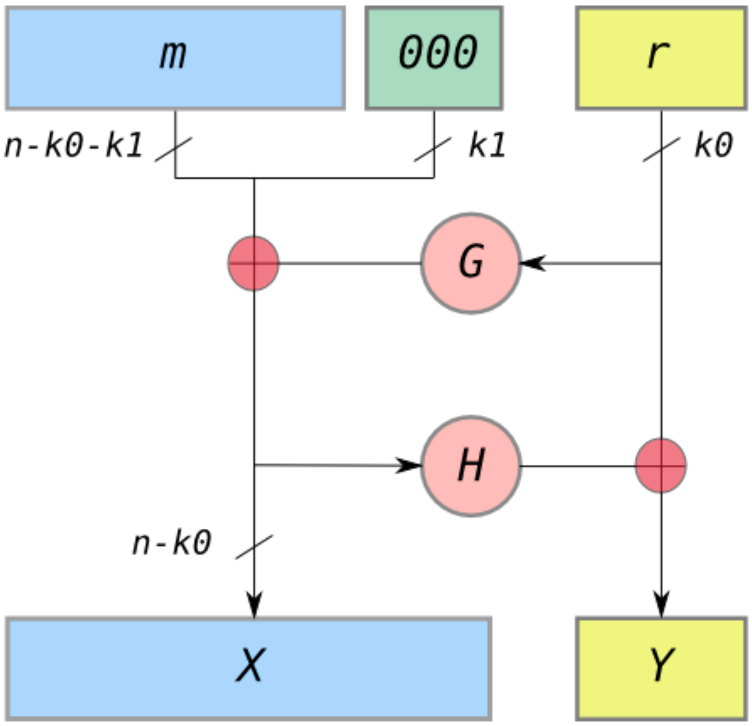
Solution 1: Padding

- PKCS#1 v1.5: 0x00 0x02 [non-zero bytes] 0x00 [message]
  - Vulnerable to a padding oracle attack!
- OAEP (Optimal Asymmetric Encryption Padding)
  - Security proof (with assumptions)



Exercise 2: OAEP

- Define an algorithm to compute m given values X and Y, constants k0 and k1, and hash functions G and H



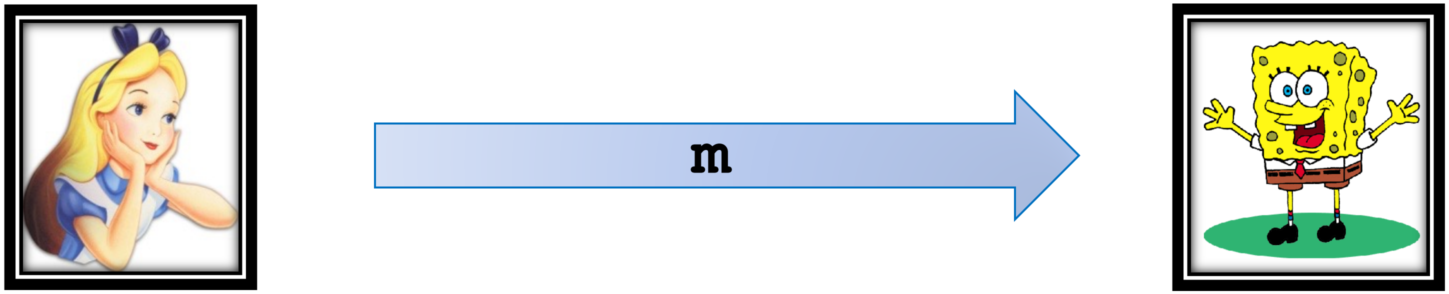
Problems with Textbook RSA

- *Insecure keys:* There are known efficient attacks for some choices of e, d, p, q
- *Deterministic*: given same plaintext and key, always produces the same ciphertext
- *Big numbers*: if m > n, can't compute do math mod n

Solution 2: Hybrid encryption

- Assume:
  - Symmetric encryption scheme (Gen_SE, Enc_SE, Dec_SE)
  - Public-key encryption scheme (Gen_PKE, Enc_PKE, Dec_PKE)
- Use public-key encryption to establish a shared session key
  - Avoids quadratic problem, assuming existence of phonebook
  - Avoids problem of key distribution
- Use symmetric encryption to exchange long plaintext encrypted under session key
  - Gain efficiency of block cipher and mode

Protocol to exchange encrypted message




```
0.   B: (pk_B, sk_B) = Gen_PKE(len_PKE)
        publish (B, pk_B)
1.   A: k_s = Gen_SE(len_SE)
        c1 = Enc_PKE(k_s; pk_B)
        c2 = Enc_SE(m; k_s)
2.   A -> B: c1, c2
3.   B: k_s = Dec_PKE(c1; sk_B)
        m = Dec_SE(c2; k_s)
```

Session keys

- If key compromised, only those messages encrypted under it are disclosed
- Used for a brief period then discarded
  - cryptoperiod*:* length of time for which key is valid
  - in this case, for a single (long) message
  - not intended for reuse in future messages
- only intended for unidirectional usage:
    - A->B, not B->A

Problems with Textbook RSA

- *Insecure keys:* There are known efficient attacks for some choices of e, d, p, q
- *Deterministic*: given same plaintext and key, always produces the same ciphertext
- *Big numbers*: if m > n, can't compute do math mod n
- *Side channel attacks*: interfaces can leak information about secret key

Square-and-Multiply

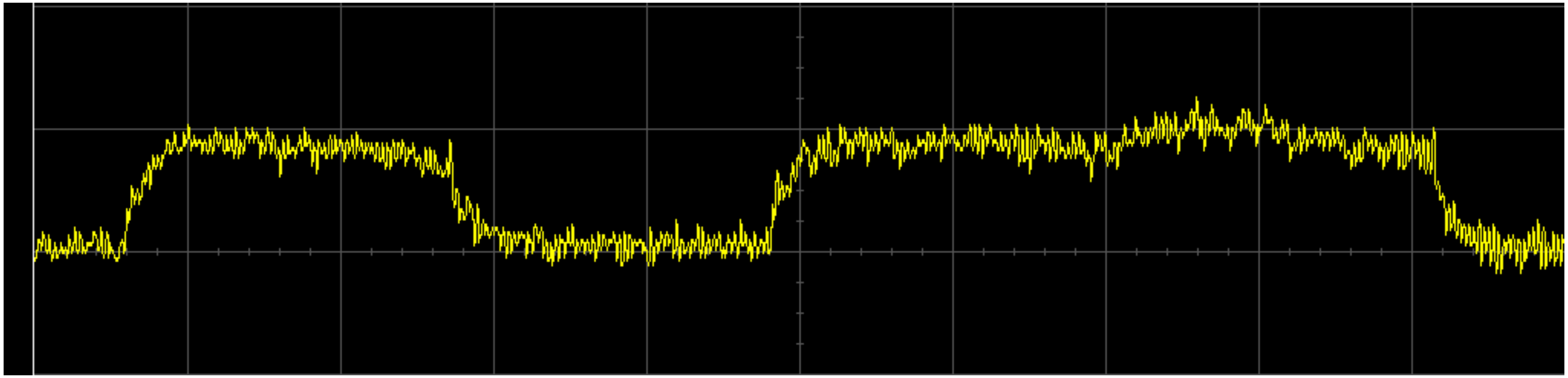
```
int modular_exp(x, n, p){
    res = 1;
    while (n > 0) {
        if (n % 2 == 1){
            res = res * x % p;
        }
        x = x^2 % p;
        n >> 1;
    }
    return res;
}
```

Exercise 3: Square-and Multiply

- Compute $3^5 \bmod 21$ using square and multiply

```
int modular_exp(x, n, p){
    res = 1;
    while (n > 0) {
        if (n % 2 == 1){
            res = res * x % p;
        }
        x = x^2 % p;
        n >> 1;
    }
    return res;
}
```

Side Channels



- Power
- Timing
- EM Radiation
- Acoustics

Solution 3: Blinded RSA

[Rivest, Shamir, Adleman 1977]

Shared Turing Award in 2002: *ingenious contribution to making public-key crypto*

- Gen(len):
  - Pick primes p, q
  - Choose e, d such that $ed = 1 \bmod \mathrm{lcm}(p-1, q-1)$
  - $pk = (n, e), \;\; sk = (p, q, d)$
- Enc(m, pk)

$$c = m^e \bmod n$$

- Dec(c, sk):

$$m = ((cr)^d \bmod n) \cdot r^{-d}$$

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- *Deterministic*: given same plaintext and key, always produces the same ciphertext
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- *Side channel attacks*: interfaces can leak information about secret key
- *Key Management:* no secure place to store the secret key

Solution 4: Key Management

- Store keys offline
- Store keys in protected files
- Memorize the keys (sort of)

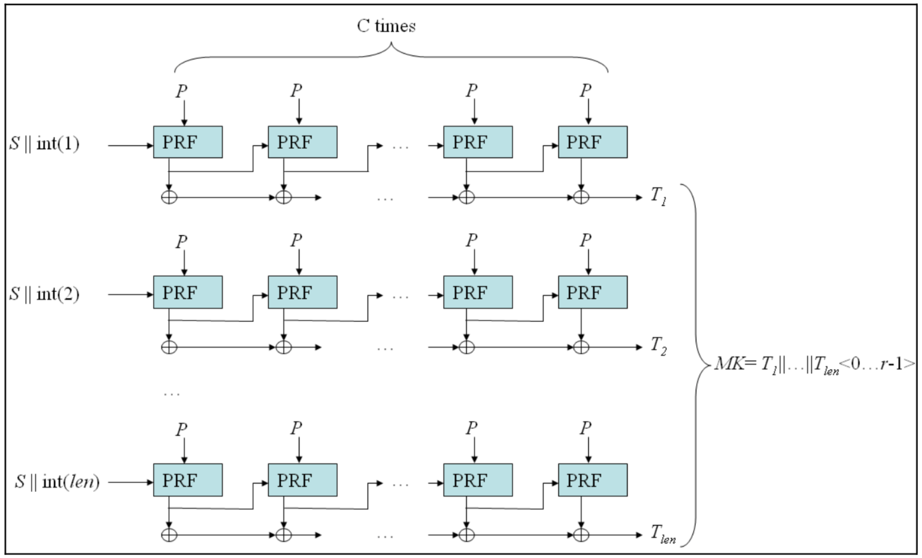
Password-Based Encryption

- PBKDF2: Password-based key derivation function [RFC 8018]
- **Output:** derived key k
- **Input:**
  - Password p
  - Salt s
  - Iteration count c
  - Key length len
  - Pseudorandom function (PRF): "looks random" to an adversary that doesn't know an input called the *seed* (commony instantiated with an HMAC)

PBKDF2

Algorithm:

- $F(p, s, i, c) = U(1)$ XOR ... XOR $U(c)$
  - $U(1) = PRF(s, i; p)$
  - $U(j) = PRF(U(j-1); p)$
  - F is in essence a salted iterated hash...
- $k = F(p, s, 1, c) \,||\, F(p, s, 2, c) \,||\, ... \,||\, F(p, s, n, c)$
  - enough copies to reach keylen
  - || denotes bit concatenation



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- *Side channel attacks*: interfaces can leak information about secret key
- *Key Management:* no secure place to store the secret key
- *Quantum Computers*: provably breakable with different hardware

Solution 5: Post-Quantum Cryptography