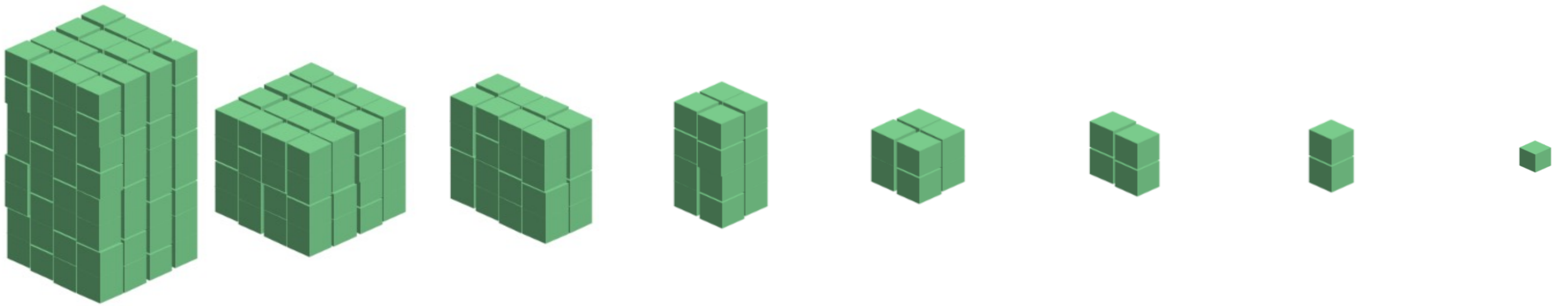
Lecture 4: Floats

CS 105 Spring 2024

Review: Representing Integers

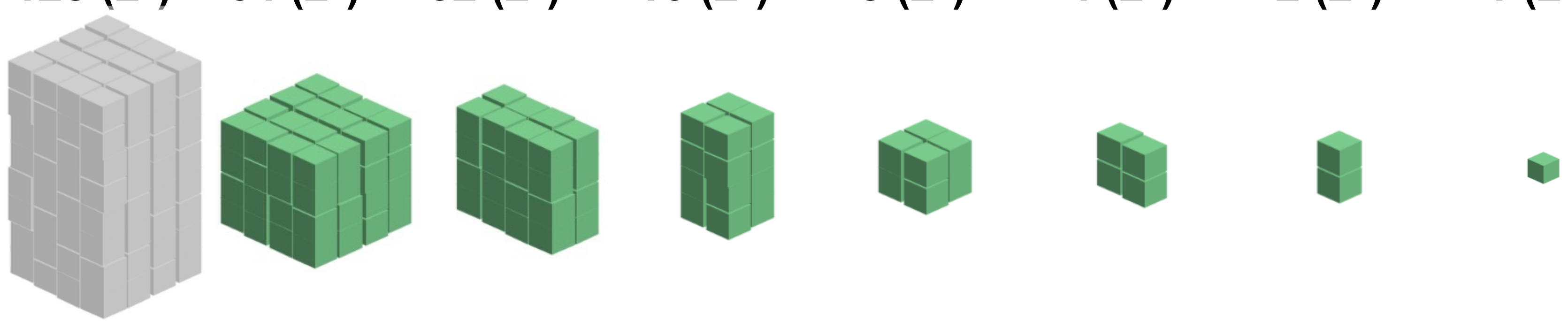
- unsigned:

128 (2^7) 64 (2^6) 32 (2^5) 16 (2^4) 8 (2^3) 4 (2^2) 2 (2^1) 1 (2^0)

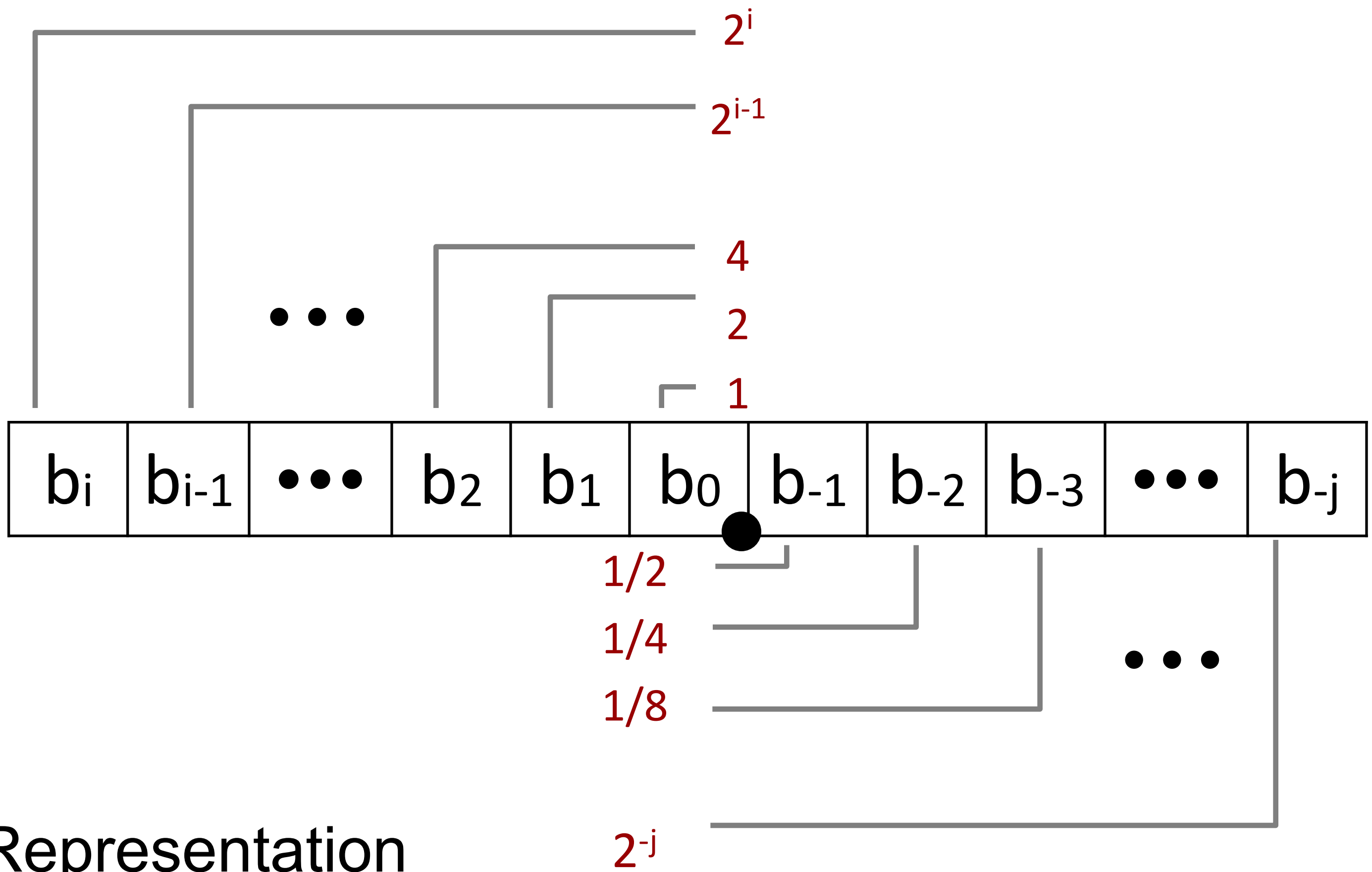


- signed (two's complement):

-128 (2^7) 64 (2^6) 32 (2^5) 16 (2^4) 8 (2^3) 4 (2^2) 2 (2^1) 1 (2^0)



Fractional binary numbers



- Representation
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number: $\sum_{k=-j}^{i}(b_k \cdot 2^k)$

Example: Fractional Binary Numbers

- What is 1001.101_2?

$$= 8 + 1 + \frac{1}{2} + \frac{1}{8} = 9\ \frac{5}{8} = 9.625$$

- What is the binary representation of 13 9/16?

`1101.1001`

Exercise 1: Fractional Binary Numbers

- Translate the following fractional numbers to their binary representation
  - 5 3/4
  - 2 7/8
  - 1 7/16

- Translate the following fractional binary numbers to their decimal representation
  - .011
  - .11
  - 1.1

Representable Numbers

- Limitation #1
  - Can only exactly represent numbers of the form $x/2^k$
  - Other rational numbers have repeating bit representations
  - Value Representation
    - 1/3 `0.0101010101[01]`…$_2$
    - 1/5 `0.001100110011[0011]`…$_2$
    - 1/10 `0.0001100110011[0011]`…$_2$

- Limitation #2
  - Just one setting of binary point within the *w* bits
  - Limited range of numbers (very small values? very large?)

Floating Point Representation

- Numerical Form: $(-1)^s \cdot M \cdot 2^E$
  - Sign bit s determines whether number is negative or positive
  - Significand M normally a (binary) fractional value in range [1.0,2.0)
  - Exponent E weights value by power of two

- Examples:
  - 1.0
  - 1.25
  - 64
  - -.625

Exercise 2: Floating Point Numbers

- For each of the following numbers, specify a binary fractional number M in [1.0,2.0) and a binary number E such that the number is equal to $M \cdot 2^E$
  - 5 3/4
  - 2 7/8
  - 1 1/2
  - 3/4

Floating Point Representation

- Numerical Form: $(-1)^s \cdot M \cdot 2^E$
  - Sign bit s determines whether number is negative or positive
  - Significand M normally a fractional value in range [1.0,2.0)
  - Exponent E weights value by power of two

- Encoding:

s	$\text{exp} = e_{k-1} \ldots e_1 e_0$	$\text{frac} = f_{n-1} \ldots f_1 f_0$

- s is sign bit s
- exp field encodes E (but is not equal to E)
  - normally $E = e_{k-1} \ldots e_1 e_0 - \boxed{(2^{k-1} - 1)}$ **bias**
- frac field encodes M (but is not equal to M)
  - normally $M = 1.f_{n-1} \ldots f_1 f_0$

Float (32 bits):
- k = 8, n = 23
- bias = 127

Double (64 bits)
- k=11, n = 52
- bias = 1023

Example: Floats

- What fractional number is represented by the bytes 0x3ec00000? Assume big-endian order.

s	$\text{exp} = e_{k-1} \ldots e_1 e_0$	$\text{frac} = f_{n-1} \ldots f_1 f_0$

- s is sign bit s
- exp field encodes E (but is not equal to E)
  - normally $E = e_{k-1} \ldots e_1 e_0 - (2^{k-1} - 1)$
- frac field encodes M (but is not equal to M)
  - normally $M = 1.f_{n-1} \ldots f_1 f_0$

Float (32 bits):
- k = 8, n = 23
- bias = 127

$$(-1)^s \cdot M \cdot 2^E$$

0	011 1110 1	100 0000 0000 0000 0000 0000

s=0 exp=125 frac = $10000000000000000000000_2$

s=0 E = -2 M = $1.10000000000000000000000_2 = 1.5_{10}$

$$(-1)^0 \cdot 1.5_{10} \cdot 2^{-2} = 1 \cdot \frac{3}{2} \cdot \frac{1}{4} = \frac{3}{8} = .375_{10}$$

$$(-1)^0 \cdot 1.1_2 \cdot 2^{-2} = .011_2 = \frac{1}{4} + \frac{1}{8} = .375_{10}$$

Exercise 3: Floats

- What fractional number is represented by the bytes 0x423c0000? Assume big-endian order.

s	$\text{exp} = e_{k-1} \ldots e_1 e_0$	$\text{frac} = f_{n-1} \ldots f_1 f_0$

- s is sign bit s
- exp field encodes E (but is not equal to E)
  - normally $E = e_{k-1} \ldots e_1 e_0 - (2^{k-1} - 1)$
- frac field encodes M (but is not equal to M)
  - normally $M = 1.f_{n-1} \ldots f_1 f_0$

Float (32 bits):
- k = 8, n = 23
- bias = 127

$$(-1)^s \cdot M \cdot 2^E$$

s	exp	frac
1	8-bits	23-bits

Limitation so far…

- What is the smallest non-negative number that can be represented?

0000 0000 0000 0000 0000 0000 0000 0000

s=0 exp=0

s=0 E = -127

frac = $00000000000000000000000_2$

M = $1.00000000000000000000000_2$

$$(-1)^0 \cdot 1.0_2 \cdot 2^{-127} = 2^{-127}$$

Normalized and Denormalized

s	exp	frac

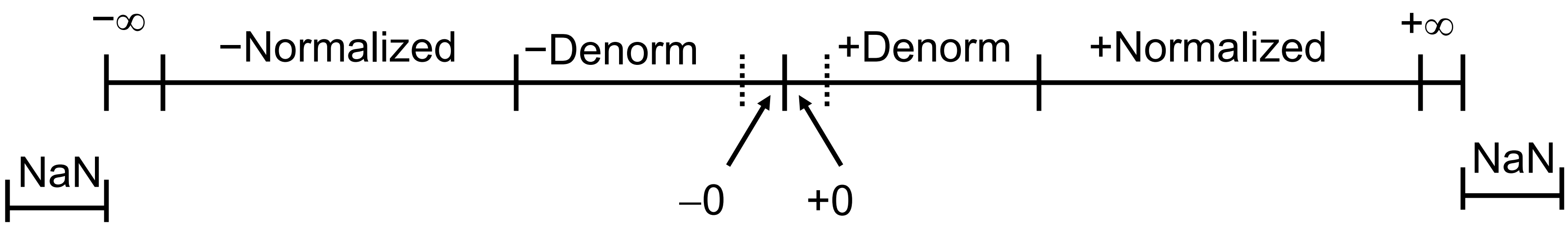
$$(-1)^s \cdot M \cdot 2^E$$

Normalized Values

- exp is neither all zeros nor all ones (normal case)
- exponent is defined as $\mathrm{E} = e_{k-1} \dots e_1 e_0 - \mathrm{bias}$, where $\mathrm{bias} = 2^{k-1} - 1$ (e.g., 127 for float or 1023 for double)
- significand is defined as $M = 1.f_{n-1}f_{n-2} \dots f_0$

- ## Denormalized Values
  - exp is either all zeros or all ones
  - if all zeros: $\mathrm{E} = 1 - \mathrm{bias}$ and $M = 0.f_{n-1}f_{n-2} \dots f_0$
  - if all ones: infinity (if frac is all zeros) or NaN (if frac is non-zero)

Visualization: Floating Point Encodings



s	exp	frac
1	8-bits	23-bits

Example: Limits of Floats

- What is the difference between the largest (non-infinite) positive number that can be represented as a (normalized) float and the second-largest?

s	exp	frac
1	8-bits	23-bits

Example: Limits of Floats

- What is the difference between the largest (non-infinite) positive number that can be represented as a (normalized) float and the second-largest?

0	111 1111 0	111 1111 1111 1111 1111 1111
s=0	E = 127	M = $1.11111111111111111111111_2$

$$\text{largest} = 1.11111111111111111111111_2 \cdot 2^{127}$$
$$\text{second\_largest} = 1.11111111111111111111110_2 \cdot 2^{127}$$

$$\text{diff} = 0.00000000000000000000001_2 \cdot 2^{127} = 1_2 \cdot 2^{127-23} = \mathbf{2^{104}}$$

Correctness

- **Example 1: Is (x + y) + z = x + (y + z)?**
  - Ints: Yes!
  - Floats:
    - (2^30 + -2^30) + 3.14 → 3.14
    - 2^30 + (-2^30 + 3.14) → 0.0

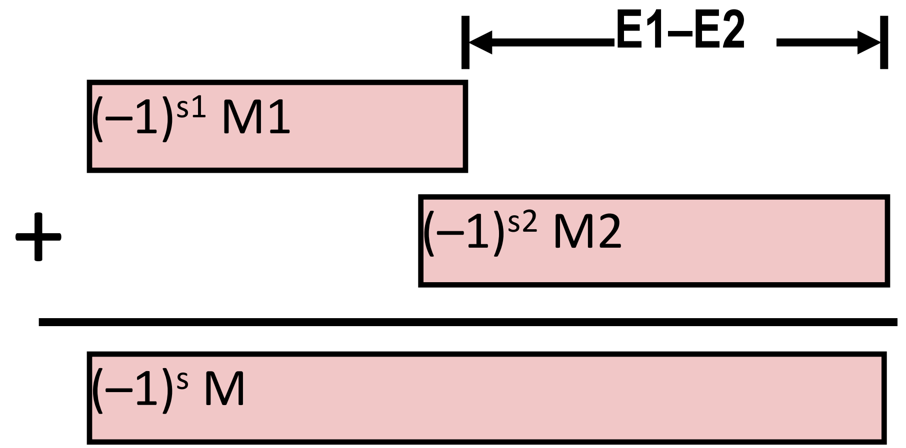
Floating Point Operations

- All of the bitwise and logical operations still work
- Float arithmetic operations done by separate hardware unit (FPU)

Floating Point Addition

- Float operations done by separate hardware unit (FPU)
- $F_1 + F_2 = (-1)^{s_1} \cdot M_1 \cdot 2^{E_1} + (-1)^{s_1} \cdot M_1 \cdot 2^{E_1}$
  - Assume E1 >= E2

Get binary points lined up



- Exact Result: $(-1)^{s} \cdot M \cdot 2^{E}$
  - Sign s, significand M:
    - Result of signed align & add
  - Exponent E: E1

- Fixing
  - If M ≥ 2, shift M right, increment E
  - if M < 1, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit **`frac`** precision

Floating Point Multiplication

- $F_1 \cdot F_2 = (-1)^{s_1} \cdot M_1 \cdot 2^{E_1} \cdot (-1)^{s_1} \cdot M_1 \cdot 2^{E_1}$
- Exact Result: $(-1)^s \cdot M \cdot 2^E$
  - Sign s: s1 ^ s2
  - Significand M: M1 x M2
  - Exponent E: E1 + E2

- Fixing
  - If M ≥ 2, shift M right, increment E
  - If E out of range, overflow
  - Round M to fit **`frac`** precision

- Implementation
  - Biggest chore is multiplying significands

Floating Point in C

- C Guarantees Two Levels
  - **`float`** single precision (32 bits)
  - **`double`** double precision (64 bits)

- Conversions/Casting
  - Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`double`**/**`float`** → **`int`**
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - **`int`** → **`double`**
    - Exact conversion,
  - **`int`** → **`float`**
    - Will round