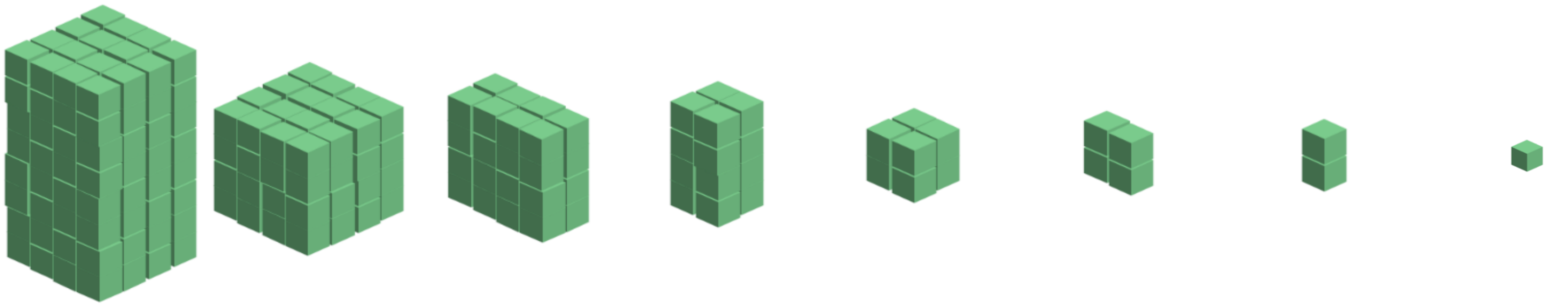
Lecture 4: Floats

CS 105 Spring 2021

Representing Integers

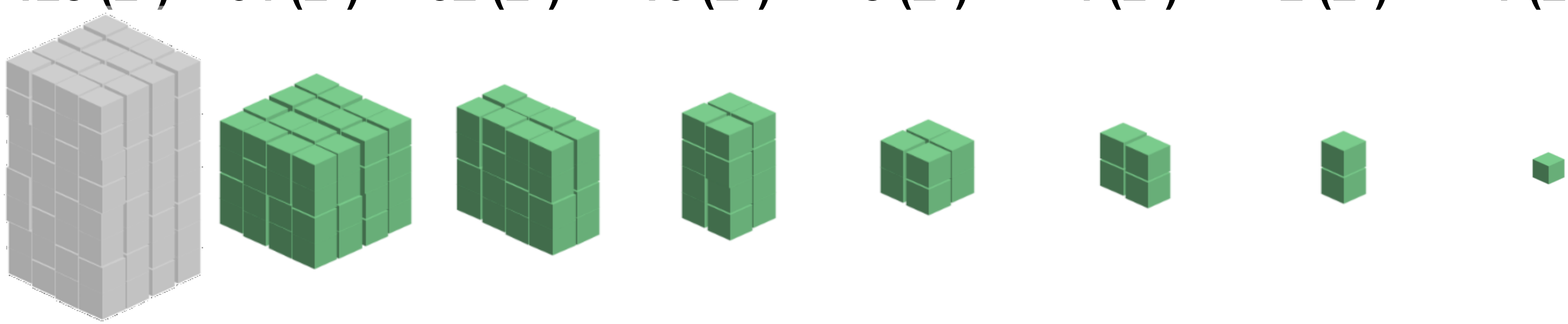
- unsigned:

128 (2^7) 64 (2^6) 32 (2^5) 16 (2^4) 8 (2^3) 4 (2^2) 2 (2^1) 1 (2^0)



- signed (two's complement):

-128 (2^7) 64 (2^6) 32 (2^5) 16 (2^4) 8 (2^3) 4 (2^2) 2 (2^1) 1 (2^0)



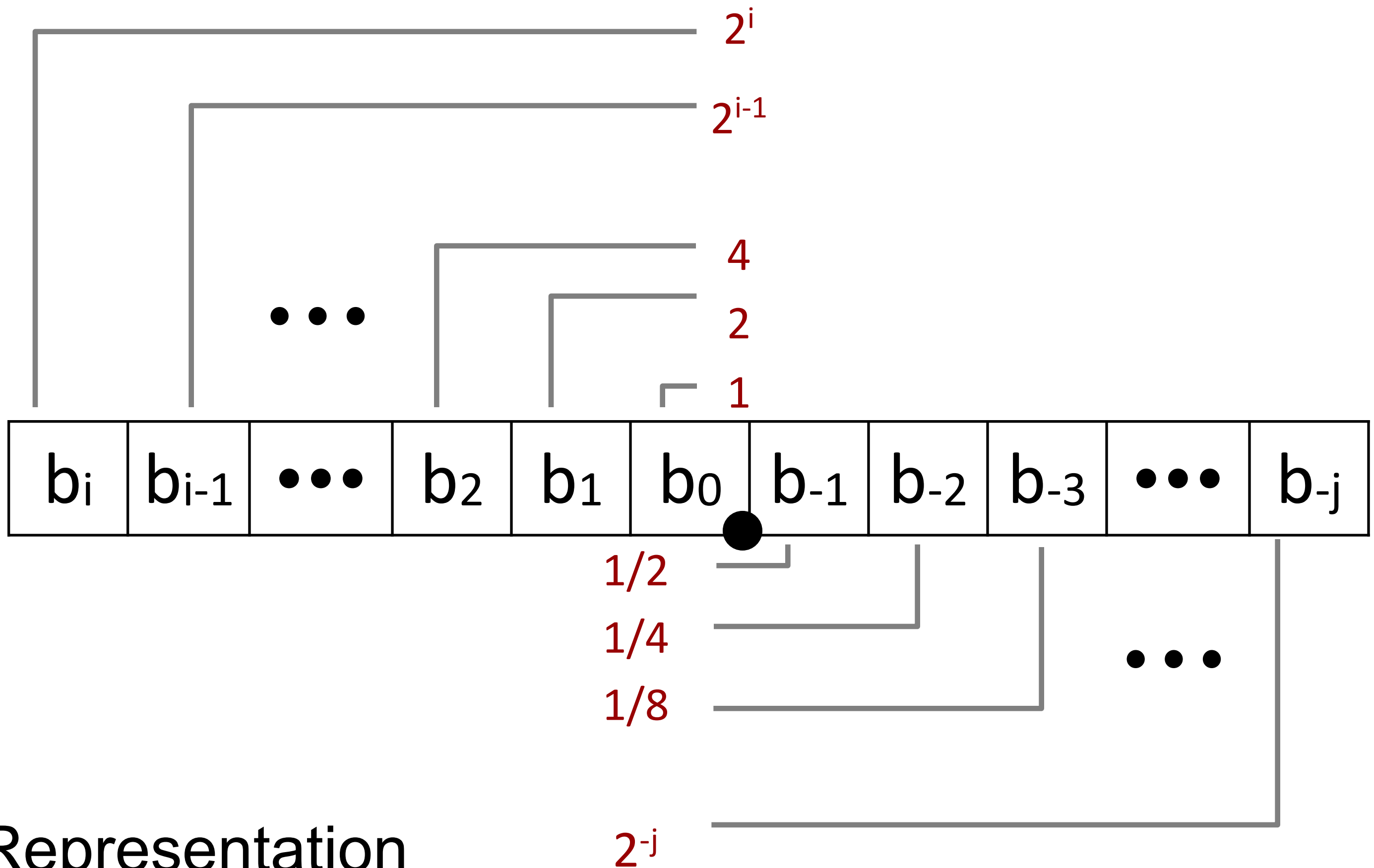
Note: to compute –x for a signed int x, flip all the bits, then add 1

$$x + \sim x = 11\ldots1 = -1, \text{ so } x + (\sim x + 1) = 0$$

Fractional binary numbers

- What is 1001.101_2?

Fractional Binary Numbers



- Representation
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number: $\sum_{k=-j}^{i}(b_k \cdot 2^k)$

Example: Fractional Binary Numbers

- What is 1001.101_2?

$$= 8 + 1 + \frac{1}{2} + \frac{1}{8} = 9\ \frac{5}{8} = 9.625$$

- What is the binary representation of 13 9/16?

`1101.1001`

Exercise 1: Fractional Binary Numbers

- Translate the following fractional numbers to their binary representation
  - 5 3/4
  - 2 7/8
  - 1 7/16

- Translate the following fractional binary numbers to their decimal representation
  - .011
  - .11
  - 1.1

Exercise 1: Fractional Binary Numbers

- Translate the following fractional numbers to their binary representation
  - 5 3/4 **101.11**
  - 2 7/8 **10.111**
  - 1 7/16 **1.0111**

- Translate the following fractional binary numbers to their decimal representation
  - .011 $= \frac{1}{4} + \frac{1}{8} = \frac{3}{8} = .375$
  - .11 $= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = .75$
  - 1.1 $= 1 + \frac{1}{2} = \frac{3}{2} = 1.5$

Representable Numbers

- Limitation #1
  - Can only exactly represent numbers of the form $x/2^k$
  - Other rational numbers have repeating bit representations
  - Value Representation
    - 1/3 `0.0101010101[01]`...$_2$
    - 1/5 `0.001100110011[0011]`...$_2$
    - 1/10 `0.0001100110011[0011]`...$_2$

- Limitation #2
  - Just one setting of binary point within the *w* bits
  - Limited range of numbers (very small values? very large?)

Floating Point Representation

- Numerical Form: $(-1)^s \cdot M \cdot 2^E$
  - Sign bit s determines whether number is negative or positive
  - Significand M normally a fractional value in range [1.0,2.0)
  - Exponent E weights value by power of two

Exercise 2: Floating Point Numbers

- For each of the following numbers, specify a binary fractional number M in [1.0,2.0) and a binary number E such that the number is equal to $M \cdot 2^E$
  - 5 3/4
  - 2 7/8
  - 1 1/2
  - 3/4

Exercise 2: Floating Point Numbers

- For each of the following numbers, specify a binary fractional number M in [1.0,2.0) and a binary number E such that the number is equal to $M \cdot 2^E$
  - 5 3/4 **`M = 1.0111`** **`E = 2`**
  - 2 7/8 **`M = 1.0111`** **`E = 1`**
  - 1 1/2 **`M = 1.1`** **`E = 0`**
  - 3/4 **`M = 1.1`** **`E = -1`**

Floating Point Representation

- Numerical Form: $(-1)^s \cdot M \cdot 2^E$
  - Sign bit s determines whether number is negative or positive
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- Encoding:

s	$\text{exp} = e_{k-1} \ldots e_1 e_0$	$\text{frac} = f_{n-1} \ldots f_1 f_0$

  - s is sign bit s
  - exp field encodes E (but is not equal to E)
    - normally $E = e_{k-1} \ldots e_1 e_0 - (2^{k-1} - 1)$ — **bias**
  - frac field encodes M (but is not equal to M)
    - normally $M = 1.f_{n-1} \ldots f_1 f_0$

Float (32 bits):
- k = 8, n = 23
- bias = 127

Double (64 bits)
- k=11, n = 52
- bias = 1023

Exercise 3: Floating Point Representations

- What are the values of s, exp, and frac that correspond to the float representation of 5 3/4, assuming 1-bit s, 3-bit exp, and 4-bit frac?

s	$\text{exp} = e_{k-1} \dots e_1 e_0$	$\text{frac} = f_{n-1} \dots f_1 f_0$

  - $(-1)^s \cdot M \cdot 2^E$, M = 1.0111, E = 2
  - s is sign bit s
  - exp field encodes E (but is not equal to E)
    - normally $E = e_{k-1} \dots e_1 e_0 - (2^{k-1} - 1)$
  - frac field encodes M (but is not equal to M)
    - normally $M = 1.f_{n-1} \dots f_1 f_0$
- Under those assumptions, what is the full representation of 5 3/4 as a one-byte floating point value? Assume big-endian order.

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  - $(-1)^s \cdot M \cdot 2^E$, M = 1.0111, E = 2
  - s is sign bit s
  - exp field encodes E (but is not equal to E)
    - normally $E = e_{k-1} \ldots e_1 e_0 - (2^{k-1} - 1)$
  - frac field encodes M (but is not equal to M)
    - normally $M = 1.f_{n-1} \ldots f_1 f_0$

s = 0

exp = 101

frac = 0111

- Under those assumptions, what is the full representation of 5 3/4 as a one-byte floating point value? Assume big-endian order. **01010111 = 0x57**

Example: Floats

- What fractional number is represented by the bytes 0x3ec00000? Assume big-endian order.

s	$\text{exp} = e_{k-1} \ldots e_1 e_0$	$\text{frac} = f_{n-1} \ldots f_1 f_0$

- s is sign bit s
- exp field encodes E (but is not equal to E)
  - normally $E = e_{k-1} \ldots e_1 e_0 - (2^{k-1} - 1)$
- frac field encodes M (but is not equal to M)
  - normally $M = 1.f_{n-1} \ldots f_1 f_0$

Float (32 bits):
- k = 8, n = 23
- bias = 127

$$(-1)^s \cdot M \cdot 2^E$$

0 | 011 1110 1 | 100 0000 0000 0000 0000 0000

s=0 exp=125 frac = $10000000000000000000000_2$

s=0 E = -2 M = $1.10000000000000000000000_2 = 1.5_{10}$

$$(-1)^0 \cdot 1.5_{10} \cdot 2^{-2} = 1 \cdot \frac{3}{2} \cdot \frac{1}{4} = \frac{3}{8} = \mathbf{.375_{10}}$$

$$(-1)^0 \cdot 1.1_2 \cdot 2^{-2} = .011_2 = \frac{1}{4} + \frac{1}{8} = \mathbf{.375_{10}}$$

Exercise 4: Floats

- What fractional number is represented by the bytes 0x423c0000? Assume big-endian order.

s	$\text{exp} = e_{k-1} \dots e_1 e_0$	$\text{frac} = f_{n-1} \dots f_1 f_0$

- s is sign bit s
- exp field encodes E (but is not equal to E)
  - normally $E = e_{k-1} \dots e_1 e_0 - (2^{k-1} - 1)$
- frac field encodes M (but is not equal to M)
  - normally $M = 1.f_{n-1} \dots f_1 f_0$

Float (32 bits):
- k = 8, n = 23
- bias = 127

$$(-1)^s \cdot M \cdot 2^E$$

0	100 0010 0	011 1100 0000 0000 0000 0000

s=0 exp=132 frac = $011110000000000000000000_2$

s=0 E = 5 M = $1.011110000000000000000000_2$

$$(-1)^0 \cdot 1.011110_2 \cdot 2^5 = 101111.0_2 == \mathbf{47_{10}}$$

s	exp	frac
1	8-bits	23-bits

Limitation so far…

- What is the smallest non-negative number that can be represented?

0 | 000 0000 0 | 000 0000 0000 0000 0000 0000 0000

s=0 exp=0

s=0 E = -127

frac = $00000000000000000000000_2$

M = $1.00000000000000000000000_2$

$$(-1)^0 \cdot 1.0_2 \cdot 2^{-127} = 2^{-127}$$

Normalized and Denormalized

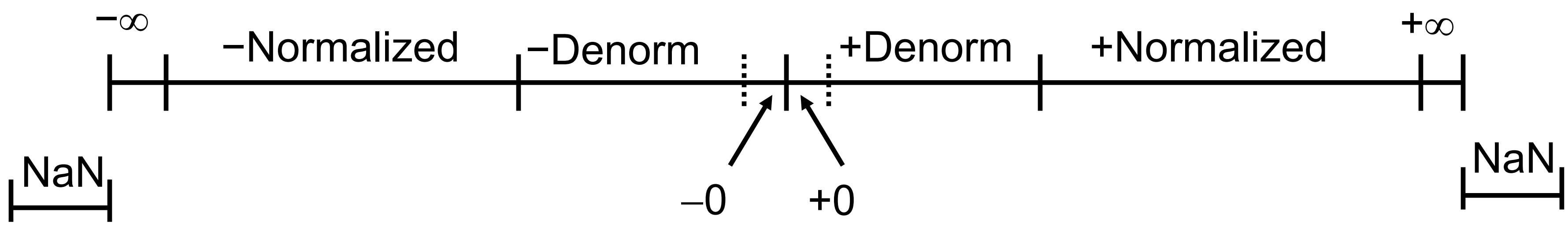
$$(-1)^s \cdot M \cdot 2^E$$

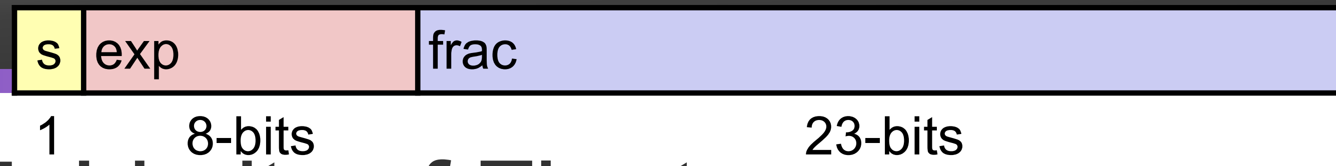
Normalized Values

- exp is neither all zeros nor all ones (normal case)
- exponent is defined as $\mathrm{E} = e_{k-1} \ldots e_1 e_0 - \mathrm{bias}$, where $\mathrm{bias} = 2^{k-1} - 1$ (e.g., 127 for float or 1023 for double)
- significand is defined as $M = 1.f_{n-1}f_{n-2} \ldots f_0$

- Denormalized Values
  - exp is either all zeros or all ones
  - if all zeros: $\mathrm{E} = 1 - \mathrm{bias}$ and $M = 0.f_{n-1}f_{n-2} \ldots f_0$
  - if all ones: infinity (if frac is all zeros) or NaN (if frac is non-zero)

Visualization: Floating Point Encodings





Exercise 5: Limits of Floats

- What is the difference between the largest (non-infinite) positive number that can be represented as a (normalized) float and the second-largest?

s	exp	frac
1	8-bits	23-bits

Exercise 5: Limits of Floats

- What is the difference between the largest (non-infinite) positive number that can be represented as a (normalized) float and the second-largest?

0	**111 1111 0**	**111 1111 1111 1111 1111 1111**
s=0	E = 127	$M = 1.11111111111111111111111_2$

$$\text{largest} = 1.11111111111111111111111_2 \cdot 2^{127}$$

$$\text{second\_largest} = 1.11111111111111111111110_2 \cdot 2^{127}$$

$$\text{diff} = 0.00000000000000000000001_2 \cdot 2^{127} = 1_2 \cdot 2^{127-23} = \mathbf{2^{104}}$$

Correctness

- **Example 1: Is (x + y) + z = x + (y + z)?**
  - Ints: Yes!
  - Floats:
    - (2^30 + -2^30) + 3.14 → 3.14
    - 2^30 + (-2^30 + 3.14) → 0.0

Floating Point in C

- C Guarantees Two Levels
  - **`float`** single precision (32 bits)
  - **`double`** double precision (64 bits)

- Conversions/Casting
  - Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`double`**/**`float`** → **`int`**
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - **`int`** → **`double`**
    - Exact conversion,
  - **`int`** → **`float`**
    - Will round

Exercise 6: Casting with Floats

- Assume you have three variables: an int x, a float f, and a double d. Assume that all three variables store numeric values (not $+\infty, -\infty$, or `NaN`). Which of the following expressions are guaranteed to evaluate to True?
  1. `x == (int)(double)(x)`
  2. `x == (int)(float)(x)`
  3. `d == (double)(float) d`
  4. `f == (float)(double) f`

Exercise 6: Casting with Floats

- Assume you have three variables: an int x, a float f, and a double d. Assume that all three variables store numeric values (not $+\infty, -\infty$, or `NaN`). Which of the following expressions are guaranteed to evaluate to True?
  1. `x == (int)(double)(x)` **True**
  2. `x == (int)(float)(x)` **False**
  3. `d == (double)(float) d` **False**
  4. `f == (float)(double) f` **True**

Floating Point Operations

- All of the bitwise and logical operations still work
- Float arithmetic operations done by separate hardware unit (FPU)

Exercise 7: Feedback

1. Rate how well you think this recorded lecture worked
   1. Better than an in-person class
   2. About as well as an in-person class
   3. Less well than an in-person class, but you still learned something
   4. Total waste of time, you didn't learn anything
2. How much time did you spend on this video lecture (including time spent on exercises)?
3. Are there particular questions you'd like me to discuss in the problem session?
4. Do you have any other comments or feedback?