Mergeable Heaps

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cs302
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Admin

- Homework 7?

Binary heap

A binary tree where the value of a parent is greater than or equal to the value of it's children

Additional restriction: all levels of the tree are **complete** except the last

Max heap vs. min heap

Binary heap - operations

Maximum(S) - return the largest element in the set

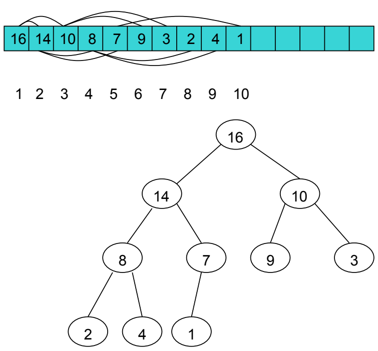
ExtractMax(S) – Return and remove the largest element in the set

Insert(S, val) – insert val into the set

IncreaseElement(S, x, val) – increase the value of element x to val

BuildHeap(A) – build a heap from an array of elements

Binary heap representations



Heapify

Assume left and right children are heaps, turn current set into a valid heap

```
HEAPIFY(A, i)
  l ← LEFT(i)
  r ← RIGHT(i)
  largest ← i
  if l ≤ heap-size[A] and A[l] > A[i]
        largest ← l
  if r ≤ heap-size[A] and A[r] > A[largest]
        largest ← r
  if largest ≠ i
        swap A[i] and A[largest]
        HEAPIFY(A, largest)
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find out which is largest: current, left of right

Heapify

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$l \leftarrow$ LEFT(i)
$r \leftarrow$ RIGHT(i)
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if $largest \neq i$
  swap $A[i]$ and $A[largest]$
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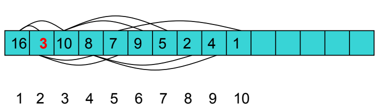
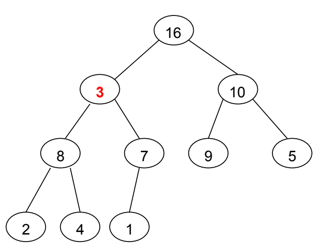
if a child is larger, swap and recurse

Heapify





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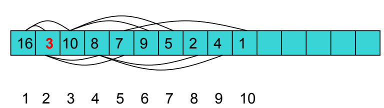

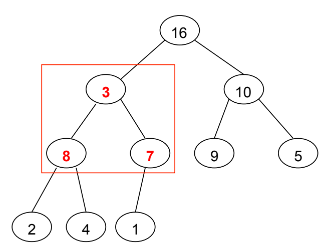

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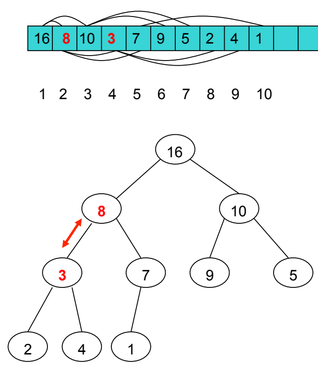


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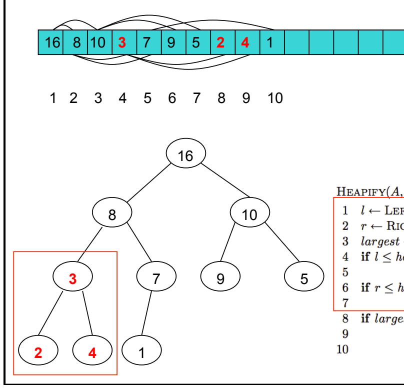


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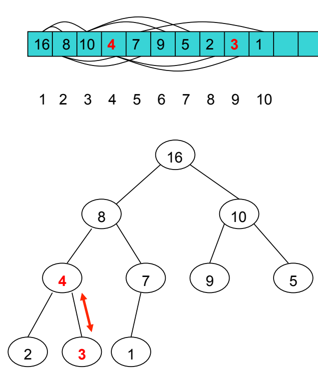


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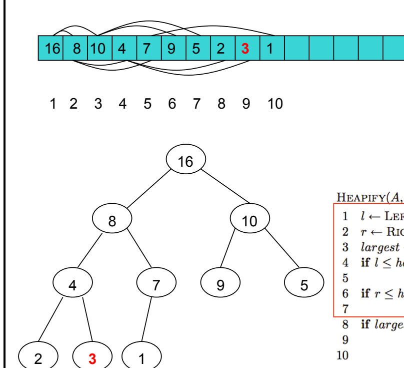


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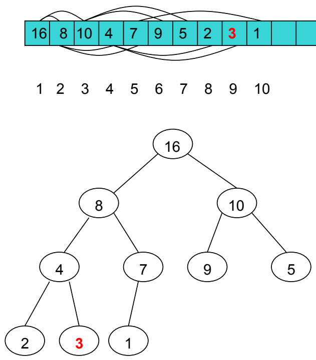


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Correctness of Heapify

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Correctness of Heapify

Base case:

- Heap with a single element
- Trivially a heap

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if largest ≠ i
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```

Correctness of Heapify

Both children are valid heaps

Three cases:

Case 1: A[i] (current node) is the largest

```
if largest ≠ i
        swap A[i] and A[largest]
        HEAPIFY(A, largest)
```

- parent is greater than both children
- both children are heaps
- current node is a valid heap

Correctness of Heapify

Case 2: left child is the largest

```
if largest ≠ i
        swap A[i] and A[largest]
        HEAPIFY(A, largest)
```

- When Heapify returns:
  - Left child is a valid heap
  - Right child is unchanged and therefore a valid heap
  - Current node is larger than both children since we selected the largest node of current, left and right
  - current node is a valid heap

Case 3: right child is largest

- similar to above

Running time of Heapify

What is the cost of each individual call to Heapify (not counting recursive calls)?

- Θ(1)

How many calls are made to Heapify?

- O(height of the tree)

What is the height of the tree?

- Complete binary tree, except for the last level

$$2^h \le n$$

$$h \le \log_2 n$$

O(log n)

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Binary heap - operations

Maximum(S) - return the largest element in the set

ExtractMax(S) – Return and remove the largest element in the set

Insert(S, val) – insert val into the set

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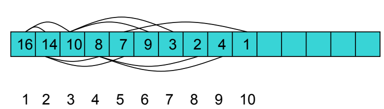
Maximum

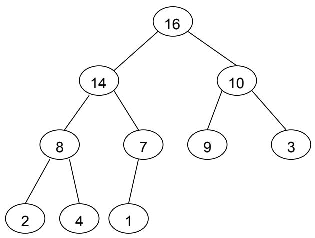
Return the largest element from the set

Return A[1]



ExtractMax

Return and remove the largest element in the set



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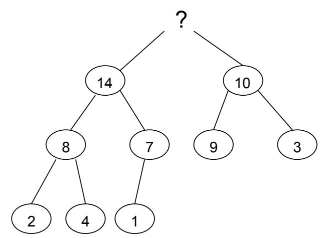

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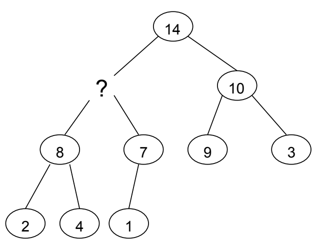

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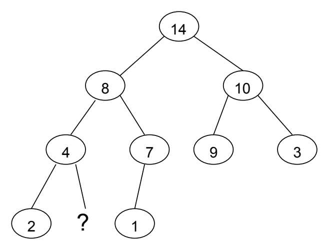

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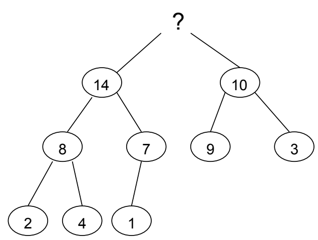

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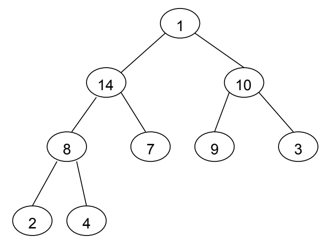

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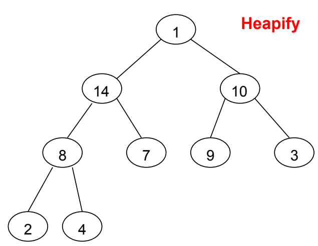

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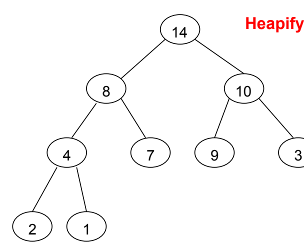

ExtractMax

Return and remove the largest element in the set

```
EXTRACT-MAX(A)
  if heap-size[A] < 1
        error
  max ← A[1]
  A[1] ← A[HEAP-SIZE[A]] − 1
  heap-size[A] ← heap-size[A] − 1
  HEAPIFY(A, 1)
  return max
```

ExtractMax running time

Constant amount of work plus one call to Heapify – O(log n)

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        error
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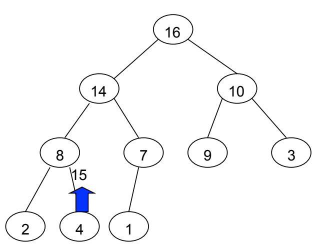
IncreaseElement

Increase the value of element *x* to *val*



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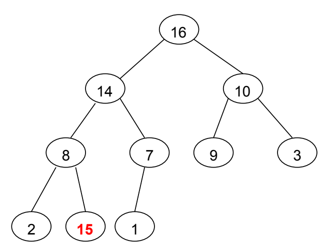

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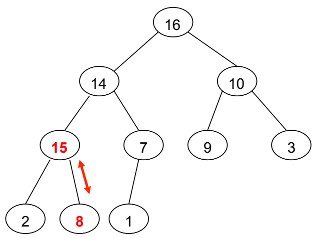

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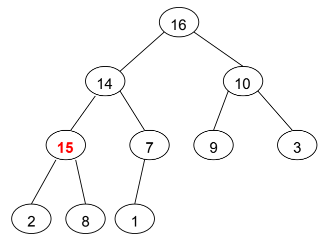

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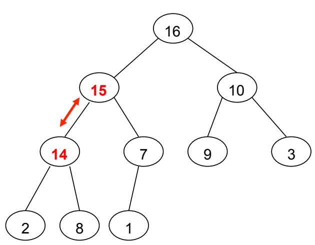

IncreaseElement

Increase the value of element *x* to *val*

```
INCREASE-ELEMENT(A, i, val)
  if val < A[i]
        error
  A[i] ← val
  while i > 1 and A[PARENT(i)] < A[i]
        swap A[i] and A[PARENT(i)]
        i ← PARENT(i)
```

Correctness of IncreaseElement

Why is it ok to swap values with parent?

```
INCREASE-ELEMENT(A, i, val)
if val < A[i]
        error
A[i] ← val
while i > 1 and A[PARENT(i)] < A[i]
        swap A[i] and A[PARENT(i)]
        i ← PARENT(i)
```

Correctness of IncreaseElement

Stop when heap property is satisfied

```
INCREASE-ELEMENT(A, i, val)
if val < A[i]
        error
A[i] ← val
while i > 1 and A[PARENT(i)] < A[i]
        swap A[i] and A[PARENT(i)]
        i ← PARENT(i)
```

Running time of IncreaseElement

Follows a path from a node to the root

Worst case O(height of the tree)

O(log n)

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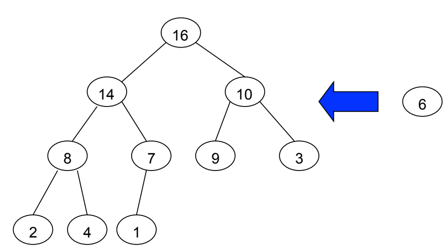
Insert

Insert *val* into the set



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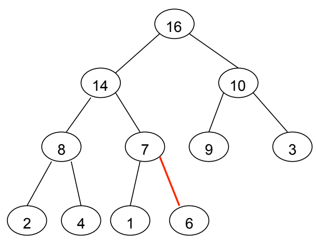

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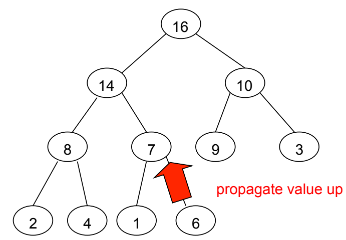

Insert

$\text{INSERT}(A, val)$

$heap\text{-}size[A] \leftarrow heap\text{-}size[A] + 1$
$A[heap\text{-}size[A]] \leftarrow -\infty$
$\text{INCREASE-ELEMENT}(A, heap\text{-}size[A], val)$

Running time of Insert

Constant amount of work plus one call to IncreaseElement – O(log n)

$\text{INSERT}(A, val)$

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$\text{INCREASE-ELEMENT}(A, heap\text{-}size[A], val)$

Building a heap

Can we build a heap using the functions we have so far?

- Maximum(S)
- ExtractMax(S)
- Insert(S, val)|
- IncreaseElement(S, x, val)

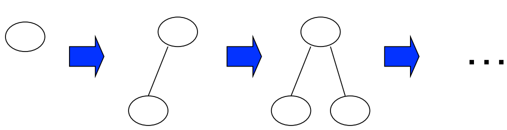
Building a heap

```
BUILD-HEAP1(A)
  copy A to B
  heap-size[A] ← 0
  for i ← 1 to length[B]
          INSERT(A, B[i])
```

Running time of BuildHeap1

n calls to Insert – O(n log n)

Can we do better?



Building a heap: take 2

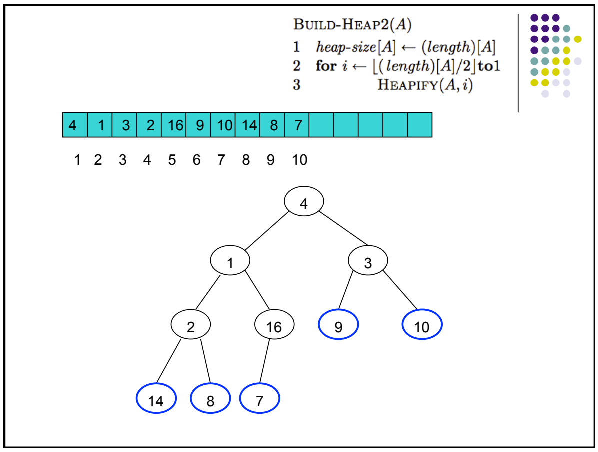
```
BUILD-HEAP2(A)
  heap-size[A] ← (length)[A]
  for i ← ⌊(length)[A]/2⌋ to 1
          HEAPIFY(A, i)
```

Start with n/2 “simple” heaps

call Heapify on element n/2-1, n/2-2, n/2-3 …

all children have smaller indices

building from the bottom up, makes sure that all the children are heaps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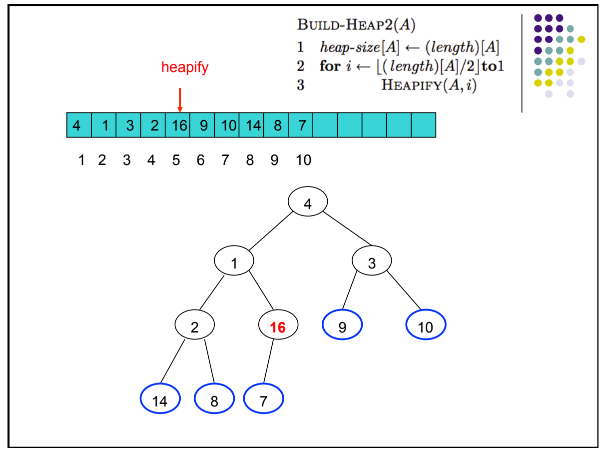
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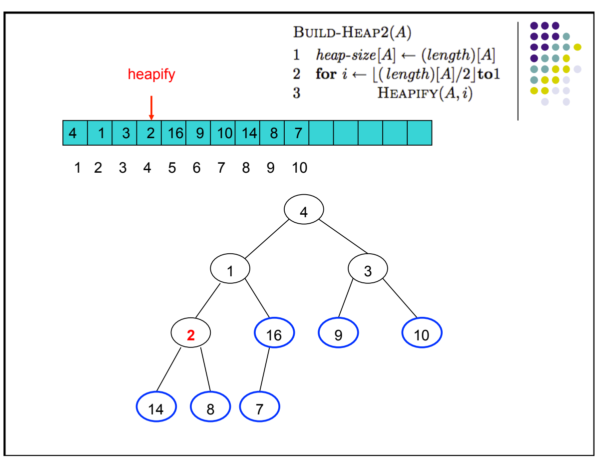
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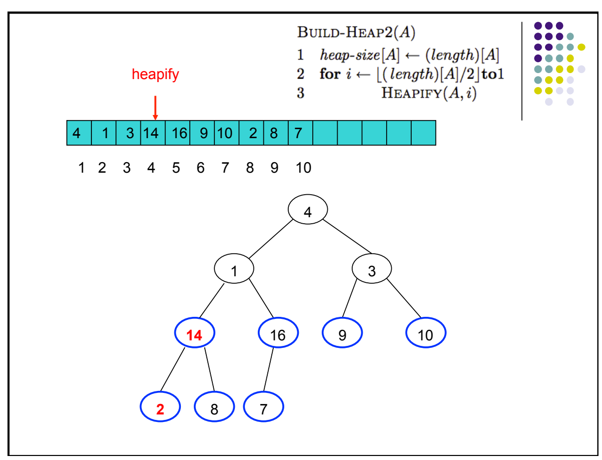
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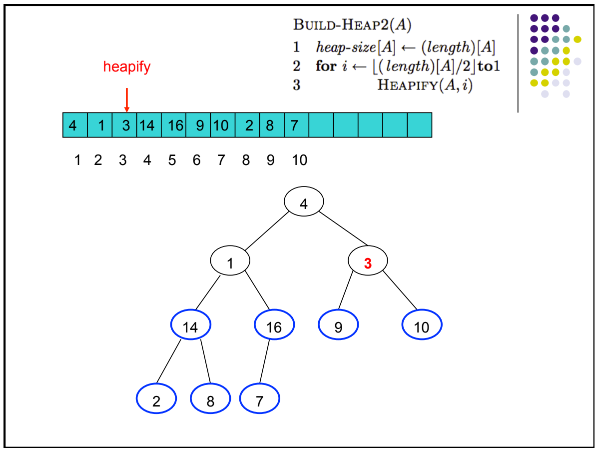
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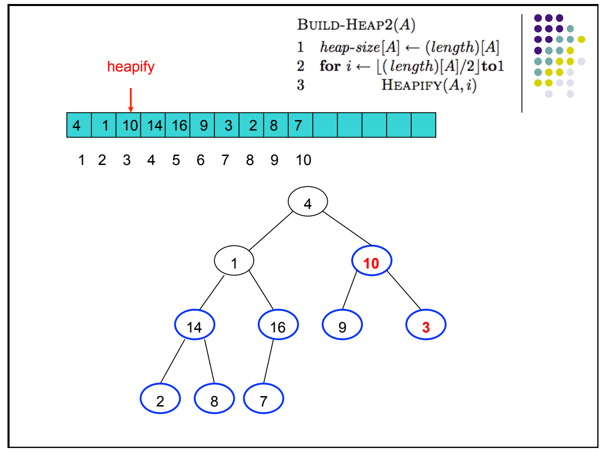
4 1 3 14 16 9 10 2 8 7
1 2 3 4 5 6 7 8 9 10
4
1 3
14 16 9 10
2 8 7


BUILD-HEAP2(A)
1 heap-size[A] ← (length)[A]
2 for i ← ⌊(length)[A]/2⌋ to 1
3 HEAPIFY(A, i)
heapify
4 1 10 14 16 9 3 2 8 7
1 2 3 4 5 6 7 8 9 10
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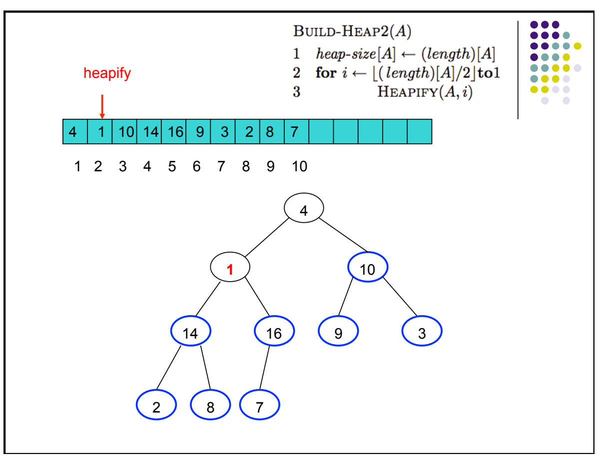
BUILD-HEAP2(A)
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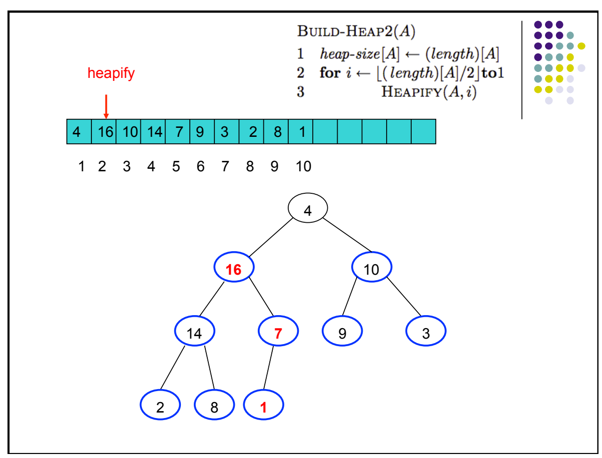
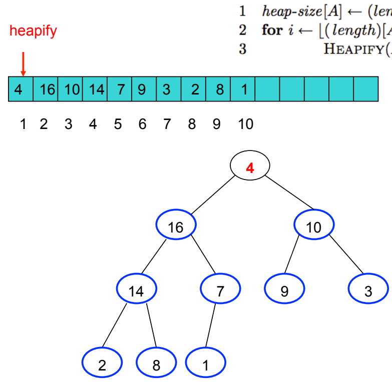
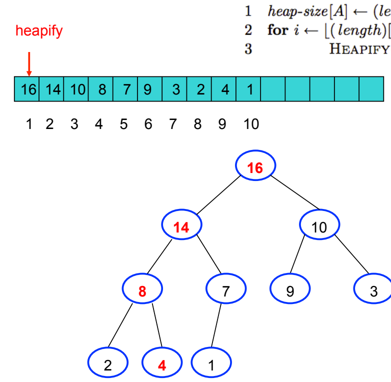

BUILD-HEAP2(A)
1 heap-size[A] ← (length)[A]
2 for i ← ⌊(length)[A]/2⌋ to 1
3 HEAPIFY(A, i)
heapify
4 16 10 14 7 9 3 2 8 1
1 2 3 4 5 6 7 8 9 10
4
16 10
14 7 9 3
2 8 1

BUILD-HEAP2(A)
$heap\text{-}size[A] \leftarrow length[A]$
for $i \leftarrow \lfloor length[A]/2 \rfloor$ **to** 1
  HEAPIFY(A, i)



BUILD-HEAP2(A)
$heap\text{-}size[A] \leftarrow length[A]$
for $i \leftarrow \lfloor length[A]/2 \rfloor$ **to** 1
  HEAPIFY(A, i)



Correctness of BuildHeap2

BUILD-HEAP2(A)
$heap\text{-}size[A] \leftarrow length[A]$
for $i \leftarrow \lfloor length[A]/2 \rfloor$ **to** 1
  HEAPIFY(A, i)

Invariant:

Correctness of BuildHeap2

BUILD-HEAP2(A)
$heap\text{-}size[A] \leftarrow length[A]$
for $i \leftarrow \lfloor length[A]/2 \rfloor$ **to** 1
  HEAPIFY(A, i)

Invariant: elements A[i+1…n] are all heaps

Base case: i = floor(n/2). All elements i+1, i+2, …, n are "simple" heaps

Inductive case: We know i+1, i+2, .., n are all heaps, therefore the call to Heapify(A,i) generates a heap at node i

Termination?

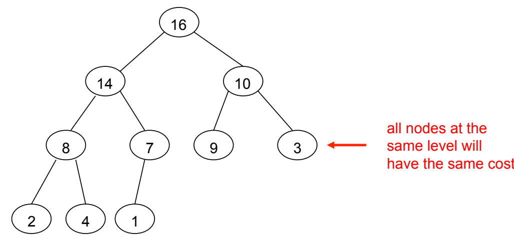
Running time of BuildHeap2

n/2 calls to Heapify – O(n log n)

Can we get a tighter bound?

```
BUILD-HEAP2(A)
heap-size[A] ← (length)[A]
for i ← ⌊(length)[A]/2⌋ to 1
        HEAPIFY(A, i)
```

Running time of BuildHeap2



How many nodes are at level *d*? 2^d

Running time of BuildHeap2

$$T(n) = \sum_{d=0}^{\log n} 2^d O(d)$$

?

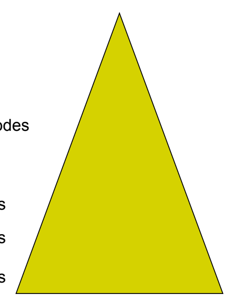
Nodes at height *h*

h < ceil($n/2^{h+1}$) nodes

h=2 < ceil(n/8) nodes

h=1 < ceil(n/4) nodes

h=0 < ceil(n/2) nodes



Running time of BuildHeap2

$$T(n) = \sum_{h=0}^{\log n} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h)$$

$$= O\left(n \sum_{h=0}^{\log n} \left\lceil \frac{1}{2^{h+1}} \right\rceil h \right)$$

$$= O\left(n \sum_{h=0}^{\log n} \frac{h}{2^h} \right)$$

$$= O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right)$$

$$= O(n)$$

$$\boxed{\sum_{h=0}^{\infty} \frac{h}{2^h} = \frac{1/2}{(1-1/2)^2} = 2}$$

BuildHeap1 vs. BuildHeap2

```
BUILD-HEAP1(A)
copy A to B
heap-size[A] ← 0
for i ← 1 to length[B]
        INSERT(A, B[i])
```

```
BUILD-HEAP2(A)
heap-size[A] ← (length)[A]
for i ← ⌊(length)[A]/2⌋ to 1
        HEAPIFY(A, i)
```

Runtime

- O(n) vs. O(n log n)

Memory

- Both O(n)
- BuildHeap1 requires an additional array, i.e. 2n memory

Complexity/Ease of implementation

Heap uses

Could we use a heap to sort?

Heap uses

Heapsort

- Build a heap
- Call ExtractMax for all the elements
- O(n log n) running time

Priority queues

- scheduling tasks: jobs, processes, network traffic
- A* search algorithm

Binary heaps

Procedure	Binary heap (worst-case)
BUILD-HEAP	$\Theta(n)$
INSERT	$\Theta(\log n)$
MAXIMUM	$\Theta(1)$
EXTRAC-MAX	$\Theta(\log n)$
UNION	
INCREASE-ELEMENT	$\Theta(\log n)$
DELETE	$\Theta(\log n)$

(adapted from Figure 19.1, pg. 456 [1])

Mergeable heaps

Procedure	Binary heap (worst-case)
BUILD-HEAP	$\Theta(n)$
INSERT	$\Theta(\log n)$
MAXIMUM	$\Theta(1)$
EXTRAC-MAX	$\Theta(\log n)$
UNION	
INCREASE-ELEMENT	$\Theta(\log n)$
DELETE	$\Theta(\log n)$

(adapted from Figure 19.1, pg. 456 [1])

- Mergeable heaps support the union operation
- Allows us to combine two heaps to get a single heap
- Union runtime for binary heaps?

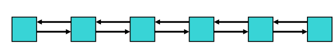
Union for binary heaps

Procedure	Binary heap (worst-case)
BUILD-HEAP	$\Theta(n)$
INSERT	$\Theta(\log n)$
MAXIMUM	$\Theta(1)$
EXTRAC-MAX	$\Theta(\log n)$
UNION	$\boxed{\Theta(n)}$
INCREASE-ELEMENT	$\Theta(\log n)$
DELETE	$\Theta(\log n)$

(adapted from Figure 19.1, pg. 456 [1])

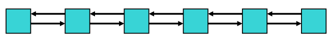
concatenate the arrays and then call Build-Heap

Linked-list heap



- Store the elements in a doubly linked list
- Insert:
- Max:
- Extract-Max:
- Increase:
- Union:

Linked-list heap



- Store the elements in a doubly linked list
- Insert: add to the end/beginning
- Max: search through the linked list
- Extract-Max: search and delete
- Increase: increase value
- Union: concatenate linked lists

Linked-list heap

Procedure	Binary heap (worst-case)	Linked-list
BUILD-HEAP	$\Theta(n)$	Θ(n)
INSERT	$\Theta(\log n)$	Θ(1)
MAXIMUM	$\Theta(1)$	Θ(n)
EXTRAC-MAX	$\Theta(\log n)$	Θ(n)
UNION	$\Theta(n)$	Θ(1)
INCREASE-ELEMENT	$\Theta(\log n)$	Θ(1)
DELETE	$\Theta(\log n)$	Θ(1)

(adapted from Figure 19.1, pg. 456 [1])

Faster Union, Increase, Insert and Delete… but slower Max operations

Binomial Tree