Greedy algorithms

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cs302
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Administrative

?

Greedy algorithms

What is a greedy algorithm?

Algorithm that makes a local decision with the goal of creating a globally optimal solution

Method for solving problems where optimal solutions can be defined in terms of optimal solutions to sub-problems

What does this mean? Where have we seen this before?

Greedy vs. divide and conquer

Divide and conquer

To solve the general problem:

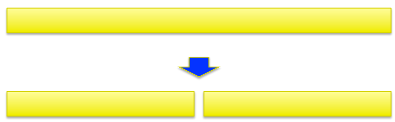
Break into sum number of sub problems, solve:

then possibly do a little work

Greedy vs. divide and conquer

Divide and conquer

To solve the general problem:

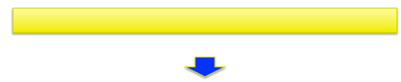


The solution to the general problem is solved with respect to solutions to sub-problems!

Greedy vs. divide and conquer

Greedy

To solve the general problem:



Pick a locally optimal solution and repeat

Greedy vs. divide and conquer

Greedy

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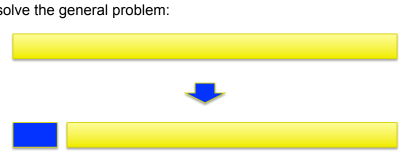

The solution to the general problem is solved with respect to solutions to sub-problems!

Slightly different than divide and conquer

Horn formula

A horn formula is a set of implications and negative clauses:

$$\Rightarrow x \qquad\qquad x \wedge u \Rightarrow z$$

$$\Rightarrow y \qquad\qquad \bar{x} \vee \bar{y} \vee \bar{z}$$

Goal

Given a horn formula, determine if the formula is satisfiable, i.e. an assignment of true/false to the variables that is consistent with all of the implications/causes

$$\Rightarrow x \qquad x \wedge u \Rightarrow z$$

$$\Rightarrow y \qquad \bar{x} \vee \bar{y} \vee \bar{z}$$

u	x	y	z
0	1	1	0

A greedy solution?

$$\Rightarrow x \qquad x \wedge z \Rightarrow w \qquad w \wedge y \wedge z \Rightarrow x$$

$$x \Rightarrow y \qquad x \wedge y \Rightarrow w \qquad \bar{w} \vee \bar{x} \vee \bar{y}$$

w	0
x	0
y	0
z	0

A greedy solution?

$$\boxed{\Rightarrow x} \qquad x \wedge z \Rightarrow w \qquad w \wedge y \wedge z \Rightarrow x$$

$$x \Rightarrow y \qquad x \wedge y \Rightarrow w \qquad \bar{w} \vee \bar{x} \vee \bar{y}$$

w	0
x	1
y	0
z	0

A greedy solution?

$$\Rightarrow x \qquad x \wedge z \Rightarrow w \qquad w \wedge y \wedge z \Rightarrow x$$

$$\boxed{x \Rightarrow y} \qquad x \wedge y \Rightarrow w \qquad \bar{w} \vee \bar{x} \vee \bar{y}$$

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A greedy solution?

$$\Rightarrow x \qquad x \wedge z \Rightarrow w \qquad w \wedge y \wedge z \Rightarrow x$$

$$x \Rightarrow y \qquad \boxed{x \wedge y \Rightarrow w} \qquad \overline{w} \vee \overline{x} \vee \overline{y}$$

w	1
x	1
y	1
z	0

A greedy solution?

$$\Rightarrow x \qquad x \wedge z \Rightarrow w \qquad w \wedge y \wedge z \Rightarrow x$$

$$x \Rightarrow y \qquad x \wedge y \Rightarrow w \qquad \boxed{\overline{w} \vee \overline{x} \vee \overline{y}}$$

w	1
x	1
y	1
z	0

not satisfiable

A greedy solution

```
HORN(H)
set all variables to false
for all implications i
    if EMPTY(LHS(i))
        RHS(i) ← true
changed ← true
while changed
    changed ← false
    for all implications i
        if LHS(i) = true and !RHS(i) = true
            RHS(i) ← true
            changed = true
for all negative clauses c
    if c = false
        return false
return true
```

A greedy solution

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```

set all variables of the implications of the form "⇒x" to true

A greedy solution

```
Horn(H)
set all variables to false
for all implications i
        if Empty(LHS(i))
                RHS(i) ← true
changed ← true
while changed
        changed ← false
        for all implications i
                if LHS(i) = true and !RHS(i) = true
                        RHS(i) ← true
                        changed = true
for all negative clauses c
        if c = false
                return false
return true
```

if the all variables of the lhs of an implication are true, then set the rhs variable to true

A greedy solution

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        if c = false
                return false
return true
```

see if all of the negative clauses are satisfied

Correctness of greedy solution

Two parts:

- If our algorithm returns an assignment, is it a valid assignment?
- If our algorithm does not return an assignment, does an assignment exist?

Correctness of greedy solution

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explicitly check all negative clauses

Correctness of greedy solution

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```

don't stop until all implications with all lhs elements true have rhs true

Correctness of greedy solution

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```

Our algorithm is "stingy". It only sets those variables that **have** to be true. All others remain false.

Running time?

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for all implications i
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?

n = number of variables

m = number of formulas

Running time?

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```

O(nm)

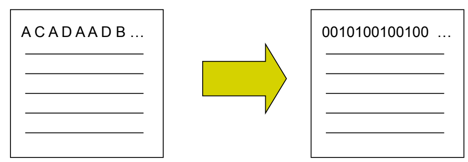
n = number of variables

m = number of formulas

Data compression

Given a file containing some data of a fixed alphabet Σ (e.g. A, B, C, D), we would like to pick a binary character code that minimizes the number of bits required to represent the data.

minimize the size of the encoded file



Compression algorithms

General purpose [edit]

- Run-length encoding (RLE) – a simple scheme that provides good compression of data containing lots of runs of the same value.
- Lempel-Ziv 1978 (LZ78), Lempel-Ziv-Welch (LZW) – used by GIF images and compress among many other applications
- DEFLATE – used by gzip, ZIP (since version 2.0), and as part of the compression process of Portable Network Graphics (PNG), Point-to-Point Protocol (PPP), HTTP, SSH
- bzip2 – using the Burrows–Wheeler transform, this provides slower but higher compression than DEFLATE
- Lempel–Ziv–Markov chain algorithm (LZMA) – used by 7zip, xz, and other programs; higher compression than bzip2 as well as much faster decompression.
- Lempel–Ziv–Oberhumer (LZO) – designed for compression/decompression speed at the expense of compression ratios
- Statistical Lempel Ziv – a combination of statistical method and dictionary-based method; better compression ratio than using single method.

Audio [edit]

- Free Lossless Audio Codec – FLAC
- Apple Lossless – ALAC (Apple Lossless Audio Codec)
- apt-X – Lossless
- Adaptive Transform Acoustic Coding – ATRAC
- Audio Lossless Coding – also known as MPEG-4 ALS
- MPEG-4 SLS – also known as HD-AAC
- Direct Stream Transfer – DST
- Dolby TrueHD
- DTS-HD Master Audio
- Meridian Lossless Packing – MLP
- Monkey's Audio – Monkey's Audio APE
- OptimFROG
- Original Sound Quality – OSQ
- RealPlayer – RealAudio Lossless
- Shorten – SHN
- TTA – True Audio Lossless
- WavPack – WavPack lossless
- WMA Lossless – Windows Media Lossless

Graphics [edit]

- ILBM – (lossless RLE compression of Amiga IFF images)
- JBIG2 – (lossless or lossy compression of B&W images)
- JPEG-LS – (lossless/near-lossless compression standard)
- JPEG 2000 – (includes lossless compression method, as proven by Sunil Kumar, Prof San Diego State University)
- JPEG XR – formerly *WMPhoto* and *HD Photo*, includes a lossless compression method
- PGF – Progressive Graphics File (lossless or lossy compression)
- PNG – Portable Network Graphics
- TIFF – Tagged Image File Format
- Gifsicle (GPL) – Optimize gif files
- Jpegoptim (GPL) – Optimize jpeg files

http://en.wikipedia.org/wiki/Lossless_data_compression

Simplifying assumption: frequency only

Assume that we only have character frequency information for a file

=

Symbol	Frequency
A	70
B	3
C	20
D	37

Fixed length code

Use ceil($\log_2|\Sigma|$) bits for each character

A =
B =
C =
D =

Fixed length code

Use ceil($\log_2|\Sigma|$) bits for each character

Symbol	Frequency
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A = 00 2 x 70 +
B = 01 2 x 3 +
C = 10 2 x 20 +
D = 11 2 x 37 =

260 bits

How many bits to encode the file?

Fixed length code

Use ceil($\log_2|\Sigma|$) bits for each character

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A	70
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A = 00 2 x 70 +
B = 01 2 x 3 +
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D = 11 2 x 37 =

260 bits

Can we do better?

Variable length code

What about:

Symbol	Frequency
A	70
B	3
C	20
D	37

A = 0 1 x 70 +
B = 01 2 x 3 +
C = 10 2 x 20 +
D = 1 1 x 37 =

153 bits

How many bits to encode the file?

Decoding a file

A = 0
B = 01
C = 10
D = 1

010100011010

What characters does this sequence represent?

Decoding a file

A = 0
B = 01
C = 10
D = 1

010100011010

A D or **B?**

What characters does this sequence represent?

Variable length code

What about:

A = 0
B = 100
C = 101
D = 11

Is it decodeable?

Symbol	Frequency
A	70
B	3
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Variable length code

What about:

A = 0 1 x 70 +
B = 100 3 x 3 +
C = 101 3 x 20 +
D = 11 2 x 37 =

213 bits
(18% reduction)

Symbol	Frequency
A	70
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How many bits to encode the file?

Prefix codes

A prefix code is a set of codes where no codeword is a **prefix** of any other codeword

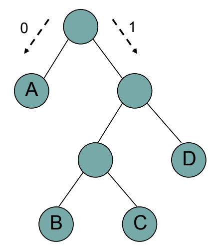
A = 0
B = 01
C = 10
D = 1

A = 0
B = 100
C = 101
D = 11

Prefix tree

We can encode a prefix code using a **full** binary tree where each leaf represents an encoding of a symbol

A = 0
B = 100
C = 101
D = 11



Decoding using a prefix tree

To decode, we traverse the graph until a leaf node is reached and output the symbol

A = 0
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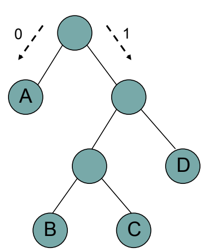
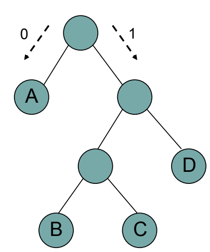


Decoding using a prefix tree

Traverse the graph until a leaf node is reached and output the symbol

1000111010100



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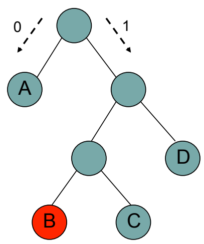
B



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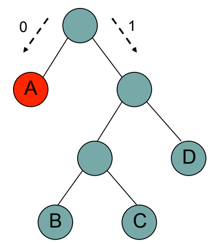
1000111010100

B A



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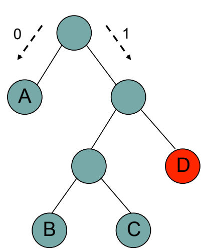
B A D



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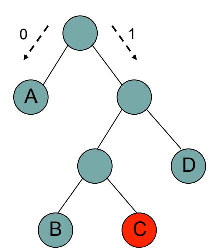

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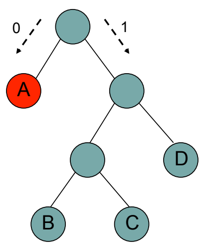
B A D C A



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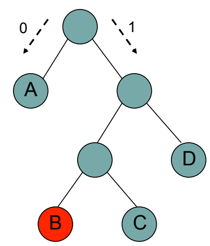
B A D C A B



Determining the cost of a file

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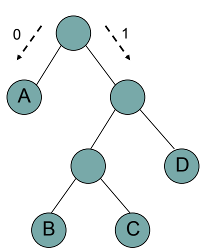

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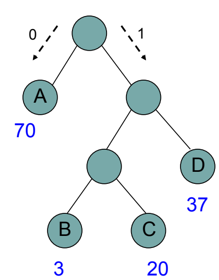
$$\text{cost}(T) = \sum_{i=1}^{n} f_i \,\text{depth}(i)$$



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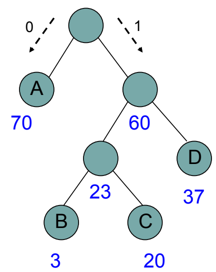

What if we label the internal nodes with the sum of the children?

Determining the cost of a file

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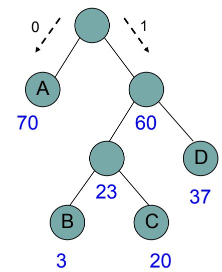


Cost is equal to the sum of the internal nodes and the leaf nodes

Determining the cost of a file

As we move down the tree, one bit gets read for every nonroot node

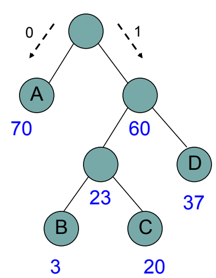
70 times we see a 0 by itself

60 times we see a prefix that starts with a 1

of those, 37 times we see an additional 1

the remaining 23 times we see an additional 0

of these, 20 times we see a last 1 and 3 times a last 0



A greedy algorithm?

Given file frequencies, can we come up with a prefix-free encoding (i.e. build a prefix tree) that minimizes the number of bits?

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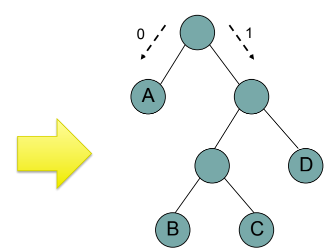

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```
HUFFMAN(F)
Q ← MAKEHEAP(F)
for i ← 1 to |Q| − 1
        allocate a new node z
        left[z] ← x ← EXTRACTMIN(Q)
        right[z] ← y ← EXTRACTMIN(Q)
        f[z] ← f[x] + f[y]
        INSERT(Q, z)
return EXTRACTMIN(Q)
```

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Heap

B	3
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A	70

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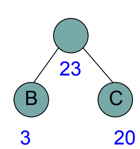
Heap

merging with this node will incur an additional cost of 23 →

BC	23
D	37
A	70



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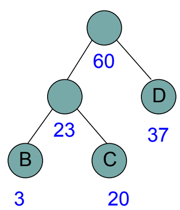

Heap

BCD 60
A 70

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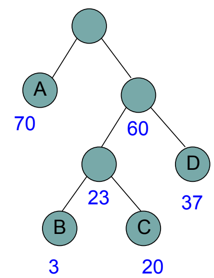

Heap

ABCD 130

Is it correct?

The algorithm selects the symbols with the two smallest frequencies first (call them f_1 and f_2)

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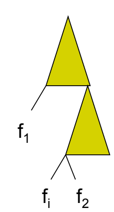
Consider a tree that did not do this (proof by contradiction):



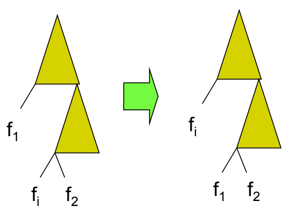
Is it optimal?

Is it correct?

The algorithm selects the symbols with the two smallest frequencies first (call them f_1 and f_2)

Consider a tree that did not do this:

$$\text{cost}(T) = \sum_{i=1}^{n} f_i \, \text{depth}(i)$$



- frequencies don't change
- cost will **decrease** since $f_1 < f_i$

contradiction

Runtime?

```
HUFFMAN(F)
Q ← MAKEHEAP(F)
for i ← 1 to |Q| − 1
        allocate a new node z
        left[z] ← x ← EXTRACTMIN(Q)
        right[z] ← y ← EXTRACTMIN(Q)
        f[z] ← f[x] + f[y]
        INSERT(Q, z)
return EXTRACTMIN(Q)
```

1 call to MakeHeap

2(n-1) calls ExtractMin

n-1 calls Insert

O(n log n)

Non-optimal greedy algorithms

All the greedy algorithms we've looked at so far give the optimal answer

Some of the most common greedy algorithms generate good, but non-optimal solutions

- set cover
- clustering
- hill-climbing
- relaxation

Knapsack problems: Greedy or not?

0-1 Knapsack – A thief robbing a store finds n items worth $v_1, v_2, .., v_n$ dollars and weight $w_1, w_2, \ldots, w_n$ pounds, where v_i and w_i are integers. The thief can carry at most W pounds in the knapsack. Which items should the thief take if he wants to maximize value.

Fractional knapsack problem – Same as above, but the thief happens to be at the bulk section of the store and can carry fractional portions of the items. For example, the thief could take 20% of item i for a weight of $0.2w_i$ and a value of $0.2v_i$.