CS062
DATA STRUCTURES AND ADVANCED PROGRAMMING
17: Heaps, Priority Queue, Heap Sort

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she/her/hers

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he/him/his
Recap

- Binary Tree
- Tree Traversal: pre-order, in-order, post-order, and level order:
Answer

- Pre-order: 8, 5, 9, 7, 1, 12, 2, 4, 11, 3
- In-order: 9, 5, 1, 7, 2, 12, 8, 4, 3, 11
- Post-order: 9, 1, 2, 12, 7, 5, 3, 11, 4, 8
- Level-order: 8, 5, 4, 9, 7, 11, 1, 12, 3, 2
Lecture 17: Heaps, Priority Queues and Heapsort

- Binary Heaps
- Priority Queue
- Heapsort
Heap-ordered binary trees

- A binary tree is **heap-ordered** if the key in each node is larger than or equal to the keys in that node’s two children (if any).

- Equivalently, the key in each node of a heap-ordered binary tree is smaller than or equal to the key in that node’s parent (if any).

- No assumption of which child is smaller.

- Moving up from any node, we get a non-decreasing sequence of keys.

- Moving down from any node we get a non-increasing sequence of keys.
Heap-ordered binary trees

- The largest key in a heap-ordered binary tree is found at the root!
Binary heap representation

- We could use a linked representation but we would need three links for every node (one for parent, one for left subtree, one for right subtree).

- If we use complete binary trees, we can use an array instead.

- Compact arrays vs explicit links means memory savings and faster execution!
Binary heaps

- **Binary heap**: the array representation of a complete heap-ordered binary tree.
  - Items are stored in an array such that each key is guaranteed to be larger (or equal to) than the keys at two other specific positions (children).
  - Max-heap but there are min-heaps, too.
Array representation of heaps

- Nothing is placed at index 0.
- Root is placed at index 1.
- Rest of nodes are placed in level order.
- No unnecessary indices and no wasted space because it’s complete.
- What's the relationship between node index and 2 children?
Reuniting immediate family members.

- For every node at index $k$, its parent is at index $\lfloor k/2 \rfloor$.
- Its two children are at indices $2k$ and $2k + 1$.
- We can travel up and down the heap by using this simple arithmetic on array indices.
- Accesses using indices are much faster than using pointers/references.
Swim/promote/percolate up/bottom up reheapify

- Scenario: a key becomes larger than its parent therefore it violates the heap-ordered property.

- To eliminate the violation:
  - Exchange key in child with key in parent.
  - Repeat until heap order restored.
Swim/promote/percolate up

private void swim(int k) {
    while (k > 1 && less(k/2, k)) {
        exch(k, k/2);
        k = k/2;
    }
}
Binary heap: insertion

- **Insert**: Add node at end in bottom level, then swim it up.

- **Cost**: At most $\log n + 1$ compares.

```java
class BinaryHeap {
    public void insert(Key x) {
        pq[++n] = x;
        swim(n);
    }
}
```
Practice Time

- Insert 47 in this binary heap.
Answer
Sink/demote/top down heapify

- Scenario: a key becomes smaller than one (or both) of its children’s keys.

- To eliminate the violation:
  - Exchange key in parent with key in larger child.

- Repeat until heap order is restored.
BINARY HEAP

Sink/demote/top down heapify

private void sink(int k) {
    while (2*k <= n) {
        int j = 2*k;
        if (j < n && less(j, j+1))
            j++;
        if (!less(k, j))
            break;
        exch(k, j);
        k = j;
    }
}
Practice Time

- Sink 7 to its appropriate place in this binary heap.
Answer
Binary heap: return (and delete) the maximum

- **Delete max**: Exchange root with node at end. Return it and delete it. Sink the new root down.

- **Cost**: At most \(2 \log n\) compares.

```java
public Key delMax() {
    Key max = pq[1];
    exch(1, n--);
    sink(1);
    pq[n+1] = null;
    return max;
}
```
Binary heap: delete and return maximum
Practice Time

- Delete max (and return it!)
Answer
Things to remember about runtime complexity of heaps

- Insertion is $O(\log n)$.
- Delete max is $O(\log n)$.
- Space efficiency is $O(n)$. 
Lecture 17: Heaps, Priority Queues and Heapsort

- Binary Heaps
- Priority Queue
- Heapsort
Priority Queue ADT

- Two operations:
  - Delete (return) the maximum
  - Insert

- Applications: load balancing and interruption handling in OS, Huffman codes for compression, A* search for AI, Dijkstra’s and Prim's algorithm for graph search, etc.

- How can we implement a priority queue efficiently?
  - Unordered array
  - Ordered array
  - Binary Heap
Option 1: Unordered array

- The *lazy* approach where we defer doing work (deleting the maximum) until necessary.
- Insert is $O(1)$ (will be implemented as push in stacks).
- Delete maximum is $O(n)$ (have to traverse the entire array to find the maximum element).
public class UnorderedArrayMaxPQ<Key extends Comparable<Key>> {
    private Key[] pq;     // elements
    private int n;        // number of elements

    // set initial size of heap to hold size elements
    public UnorderedArrayMaxPQ(int capacity) {
        pq = (Key[]) new Comparable[capacity];
        n = 0;
    }

    public boolean isEmpty() { return n == 0; }
    public int size() { return n; }
    public void insert(Key x) { pq[n++] = x; }

    public Key delMax() {
        int max = 0;
        for (int i = 1; i < n; i++)
            if (less(max, i)) max = i;
        exch(max, n-1);

        return pq[--n];
    }

    private boolean less(int i, int j) {
        return pq[i].compareTo(pq[j]) < 0;
    }

    private void exch(int i, int j) {
        Key swap = pq[i];
        pq[i] = pq[j];
        pq[j] = swap;
    }
}
Practice Time

Given an empty array of capacity 10, perform the following operations in a priority queue based on an unordered array (lazy approach):

1. Insert P
2. Insert Q
3. Insert E
4. Delete max
5. Insert X
6. Insert A
7. Insert M
8. Delete max
9. Insert P
10. Insert L
11. Insert E
12. Delete max
Answer
Option 2: Ordered array

- The *eager* approach where we do the work (keeping the list sorted) up front to make later operations efficient.
- Insert is $O(n)$ (we have to find the index to insert and shift elements to perform insertion).
- Delete maximum is $O(1)$ (just take the last element which will be the maximum).
public class OrderedArrayMaxPQ<Key extends Comparable<Key>> {
    private Key[] pq; // elements
    private int n; // number of elements

    // set initial size of heap to hold size elements
    public OrderedArrayMaxPQ(int capacity) {
        pq = (Key[]) (new Comparable[capacity]);
        n = 0;
    }

    public boolean isEmpty() { return n == 0; }
    public int size() { return n; }
    public Key delMax() { return pq[--n]; }

    public void insert(Key key) {
        int i = n-1;
        while (i >= 0 && less(key, pq[i])) {
            pq[i+1] = pq[i]; // Empty element is at index i
            i--;
        }
        pq[i+1] = key; // I+1 to get to the empty element
        n++;
    }

    private boolean less(Key v, Key w) {
        return v.compareTo(w) < 0;
    }
}
Practice Time

Given an empty array of capacity 10, perform the following operations in a priority queue based on an ordered array (eager approach):

1. Insert P 7. Insert M
2. Insert Q 8. Delete max
3. Insert E 9. Insert P
4. Delete max 10. Insert L
5. Insert X 11. Insert E
6. Insert A 12. Delete max
Answer

PRIORITY QUEUE

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insert P

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insert Q

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insert E

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delete-max → Q

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insert A

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insert M

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delete-max → X

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insert P

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delete-max → P

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</table>
Option 3: Binary heap

- Will allow us to both insert and delete max in $O(\log n)$ running time.
- There is no way to implement a priority queue in such a way that insert and delete max can be achieved in $O(1)$ running time.
- Priority queues are synonyms to binary heaps.
Practice Time

- Given an empty binary heap that represents a priority queue, perform the following operations:

1. Insert P
2. Insert Q
3. Insert E
4. Delete max
5. Insert X
6. Insert A
7. Insert M
8. Delete max
9. Insert P
10. Insert L
11. Insert E
12. Delete max
Answer
Lecture 22: Priority Queues and Heapsort

- Priority Queue
- Heapsort
Basic plan for heap sort

- Use a priority queue to develop a sorting method that works in two steps:

  1) **Heap construction**: build a binary heap with all $n$ keys that need to be sorted.

  2) **Sortdown**: repeatedly remove and return the maximum key.
**O(n)** Heap construction

- Ignore all leaves (indices n/2+1,...,n).
- **for** (int k = n/2; k >= 1; k--)
  
  `sink(a, k, n);`

- **Key insight:** After `sink(a, k, n)` completes, the subtree rooted at `k` is a heap.
Practice Time

- Run the first step of heapsort, heap construction, on the array [2,9,7,6,5,8].
Answer: Heap construction

Starting point (arbitrary order):

1. Initial heap:

2. First extraction:
   - 9 is removed and replaced with 8
   - 8 becomes the new root
   - New heap:

3. Second extraction:
   - 6 is removed and replaced with 7
   - 7 becomes the new root
   - New heap:

4. Third extraction:
   - 5 is removed and replaced with 3
   - 3 becomes the new root
   - New heap:

5. Final extraction:
   - 1 is removed and replaced with 8
   - 8 becomes the new root
   - New heap (sorted):

Result (heap-ordered):
Sortdown

- Remove the maximum, one at a time, but leave in array instead of nulling out.

- **while** (n > 1) {
  - exch(a, 1, n--);
  - sink(a, 1, n);
}

**Key insight**: After each iteration the array consists of a heap-ordered subarray followed by a sub-array in final order.
HEAPSORT

Sortdown

- while (n > 1) {
  exch(a, 1, n--);
  sink(a, 1, n);
}

```plaintext
while (n > 1) {
  exch(a, 1, n--);
  sink(a, 1, n);
}
```
Heapsort demo

Sortdown. Repeatedly delete the largest remaining item.
Practice Time

- Given the heap you constructed before, run the second step of heapsort, sortdown, to sort the array [2, 9, 7, 6, 5, 8].
Answer: Sortdown

Starting point (heap-ordered)

exch(1,5) sink(1,4)

exch(1,6) sink(1,5)

exch(13) sink(1,2)

Result (sorted)
Heapsort analysis

- Heap construction makes $O(n)$ exchanges and $O(n)$ compares.
- Sortdown and therefore the entire heap sort $O(n \log n)$ exchanges and compares.
- In-place sorting algorithm with $O(n \log n)$ worst-case!
- Remember:
  - mergesort: not in place, requires linear extra space.
  - quicksort: quadratic time in worst case.
- Heapsort is optimal both for time and space in terms of Big-O, but:
  - Inner loop longer than quick sort.
  - Poor use of cache. Why?
  - Not stable.
## Sorting: Everything you need to remember about it!

<table>
<thead>
<tr>
<th>Which Sort</th>
<th>In place</th>
<th>Stable</th>
<th>Best</th>
<th>Average</th>
<th>Worst</th>
<th>Remarks</th>
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<tr>
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<td></td>
<td>$O(n^2)$</td>
<td>$O(n^2)$</td>
<td>$O(n^2)$</td>
<td>$n$ exchanges</td>
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<tr>
<td>Insertion</td>
<td>X</td>
<td>X</td>
<td>$O(n)$</td>
<td>$O(n^2)$</td>
<td>$O(n^2)$</td>
<td>Use for small arrays or partially ordered</td>
</tr>
<tr>
<td>Merge</td>
<td></td>
<td>X</td>
<td>$O(n \log n)$</td>
<td>$O(n \log n)$</td>
<td>$O(n \log n)$</td>
<td>Guaranteed performance; stable</td>
</tr>
<tr>
<td>Quick</td>
<td>X</td>
<td></td>
<td>$O(n \log n)$</td>
<td>$O(n \log n)$</td>
<td>$O(n^2)$</td>
<td>$n \log n$ probabilistic guarantee; fastest!</td>
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<tr>
<td>Heap</td>
<td>X</td>
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<td>$O(n \log n)$</td>
<td>$O(n \log n)$</td>
<td>$O(n \log n)$</td>
<td>Guaranteed performance; in place</td>
</tr>
</tbody>
</table>
Lecture 22: Priority Queues and Heapsort

- Priority Queue
- Heapsort
Assignments and Practice Problems

Readings:

- Textbook:
  - Chapter 2.4 (Pages 308-327), 2.5 (336-344)

- Website:
  - Priority Queues: [https://algs4.cs.princeton.edu/24pq/](https://algs4.cs.princeton.edu/24pq/)

- Visualization:
  - Create (nlogn) and heapsort: [https://visualgo.net/en/heap](https://visualgo.net/en/heap)

Practice Problems:

- 2.4.1-2.4.11. Also try some creative problems.
Readings:

- Textbook:
  - Chapter 2.4 (Pages 308-327)

- Website:
  - Priority Queues: https://algs4.cs.princeton.edu/24pq/

- Visualization:
  - Insert and ExtractMax: https://visualgo.net/en/heap

Practice Problems:

- Practice with traversals of trees and insertions and deletions in binary heaps