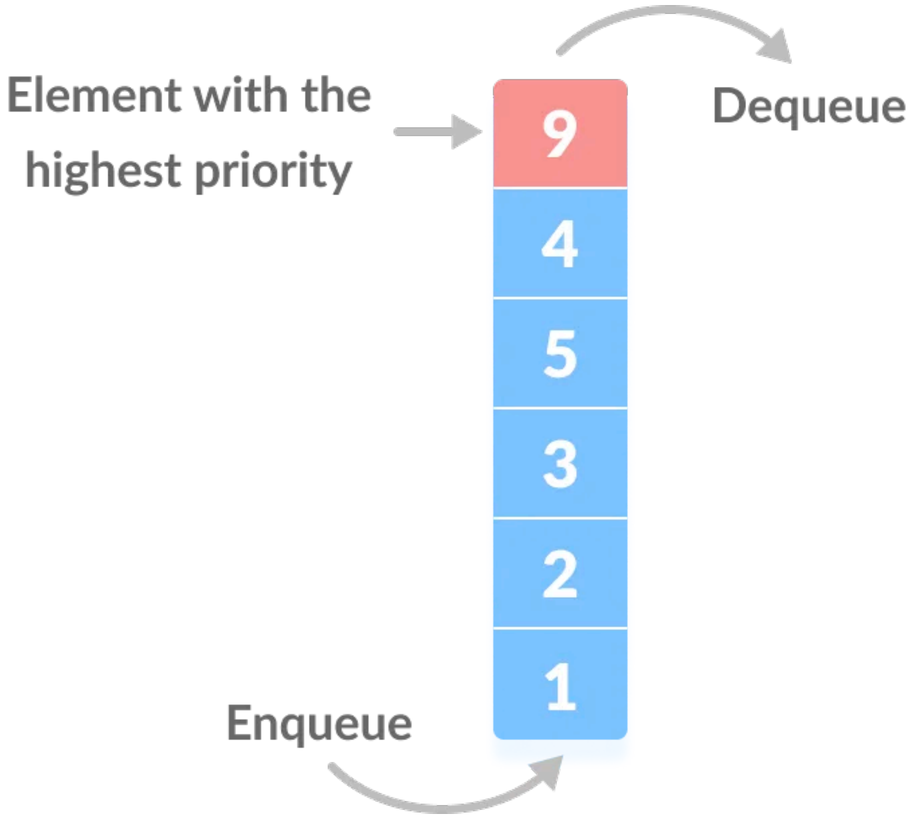
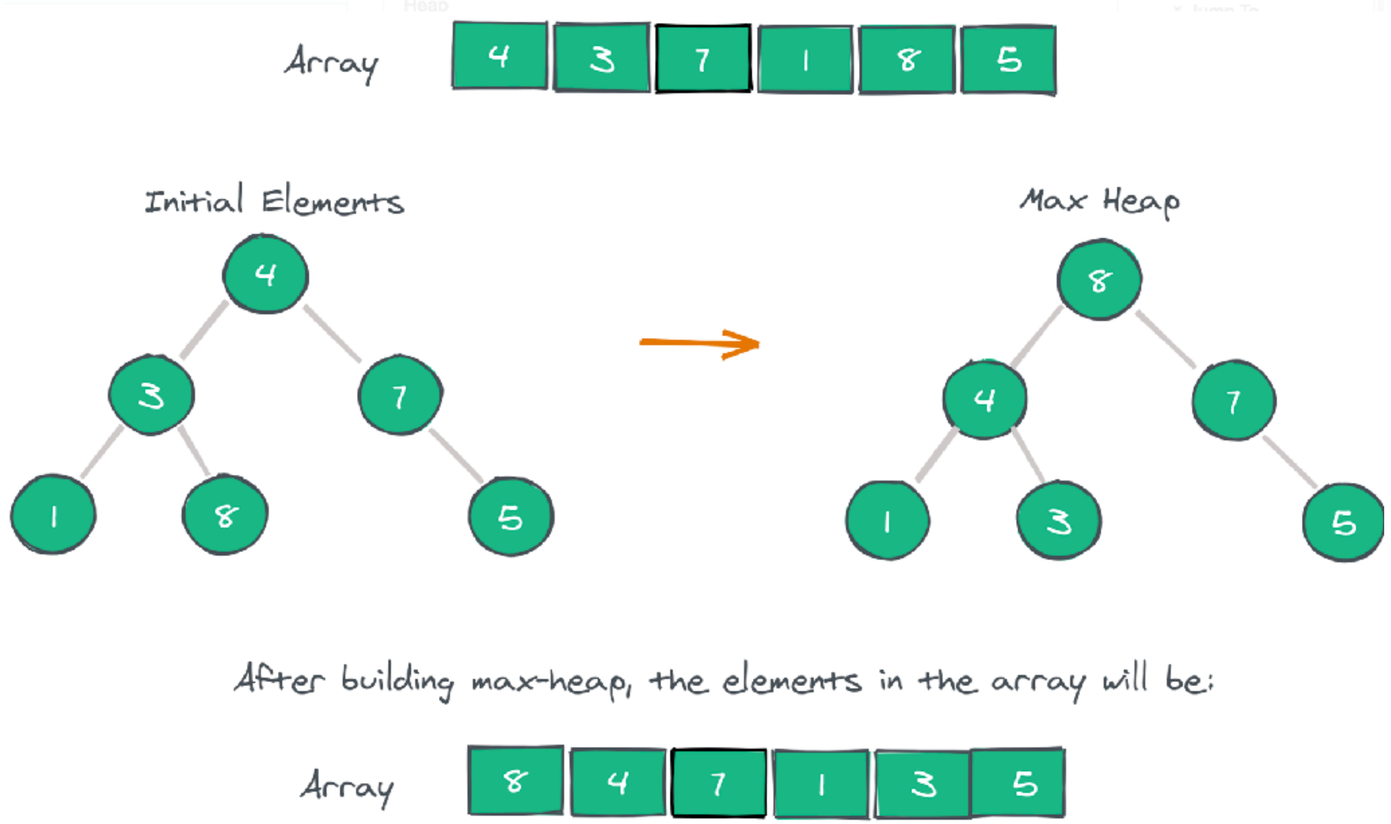
CS62 Class 14: Priority Queues & Heapsort

Sorting



Priority queue: another representation of a binary heap



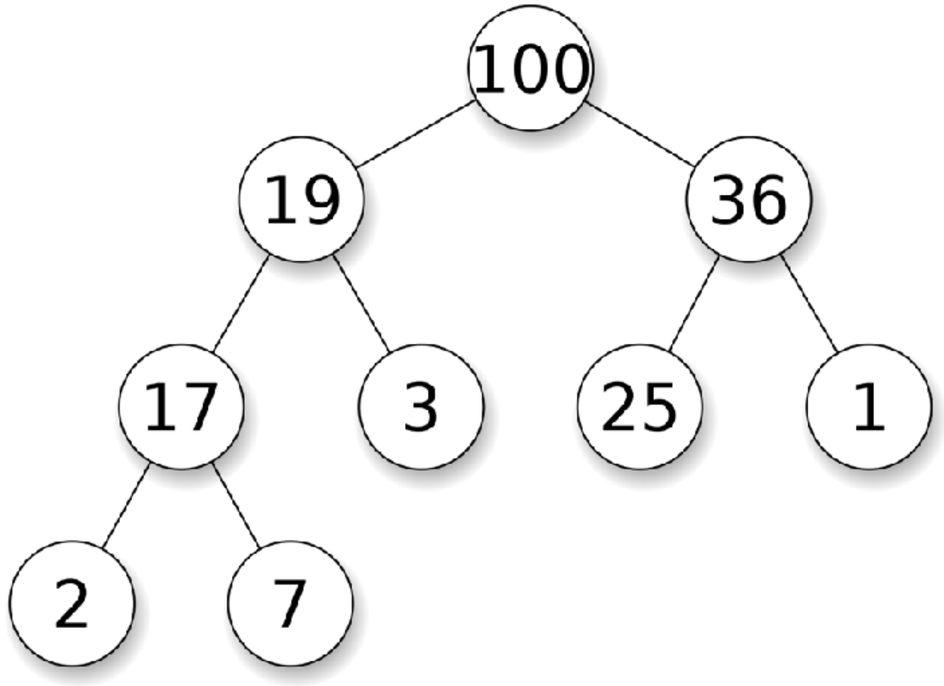
Heapsort: sorting using a binary heap

Agenda

- From last time: Continuing Binary Heaps
- Priority Queues
- Heapsort
- Heapsort Analysis

Heap-ordered binary trees aka binary heaps

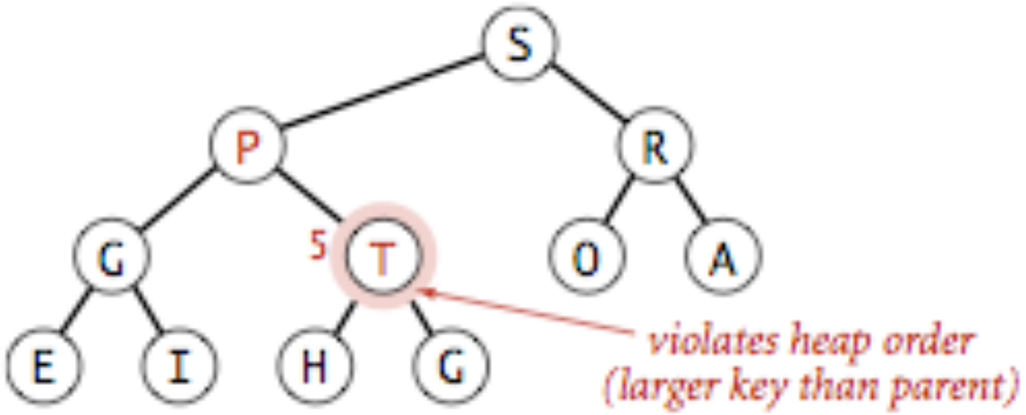
- The largest key in a heap-ordered binary tree is found at the root!



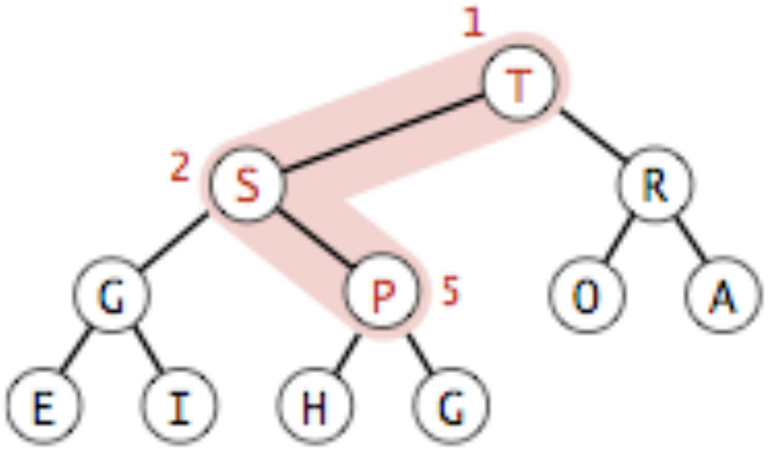
Array rep is in-order traversal: [-, 100, 19, 36, 17, 3, 25, 1, 2, 7]

Swim/promote/percolate up: code

```
private void swim(int k) {
   while (k > 1 && a[k/2].compareTo(a[k])<0) {
      E temp = a[k];
      a[k] = a[k/2];
      a[k/2] = temp;
      k = k/2;
   }
}
```

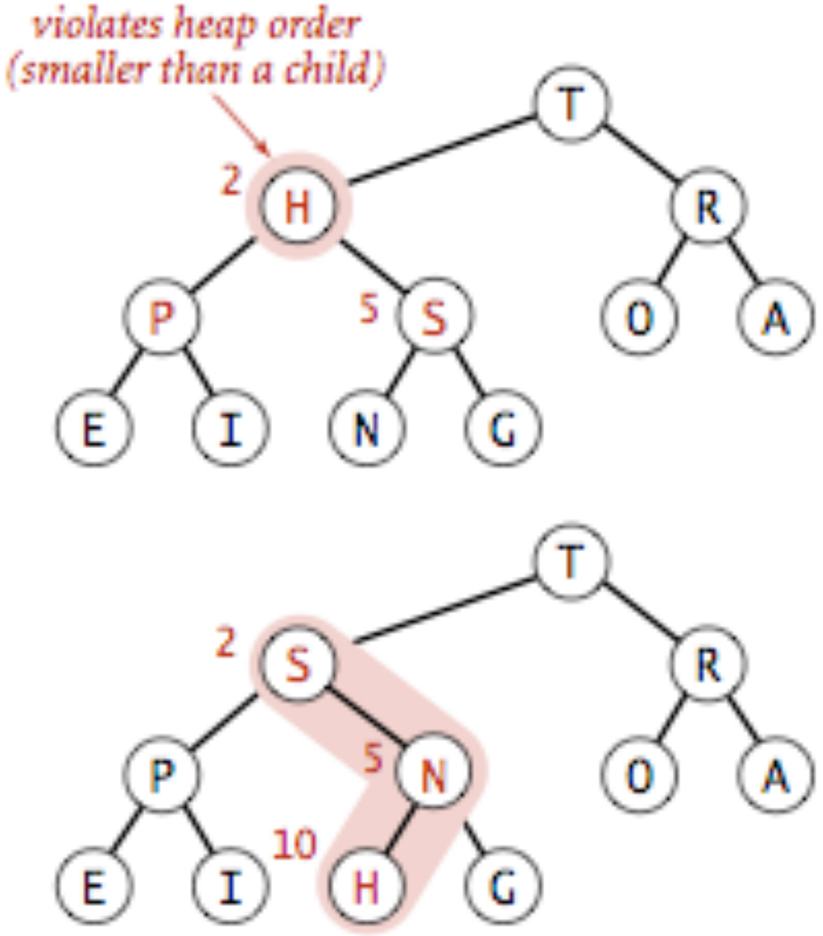


We **swim large nodes** so they become parents
We do this by swapping with the parent if it's larger



Sink/demote/top down heapify code

```
private void sink(int k) {
    while (2*k <= n) {
        int j = 2*k;
        if (j < n && a[j].compareTo(a[j+1])<0))
            j++;
        if (a[k].compareTo(a[j])>=0))
            break;
        E temp = a[k];
        a[k] = a[j];
        a[j] = temp;
        k = j;
    }
}
```

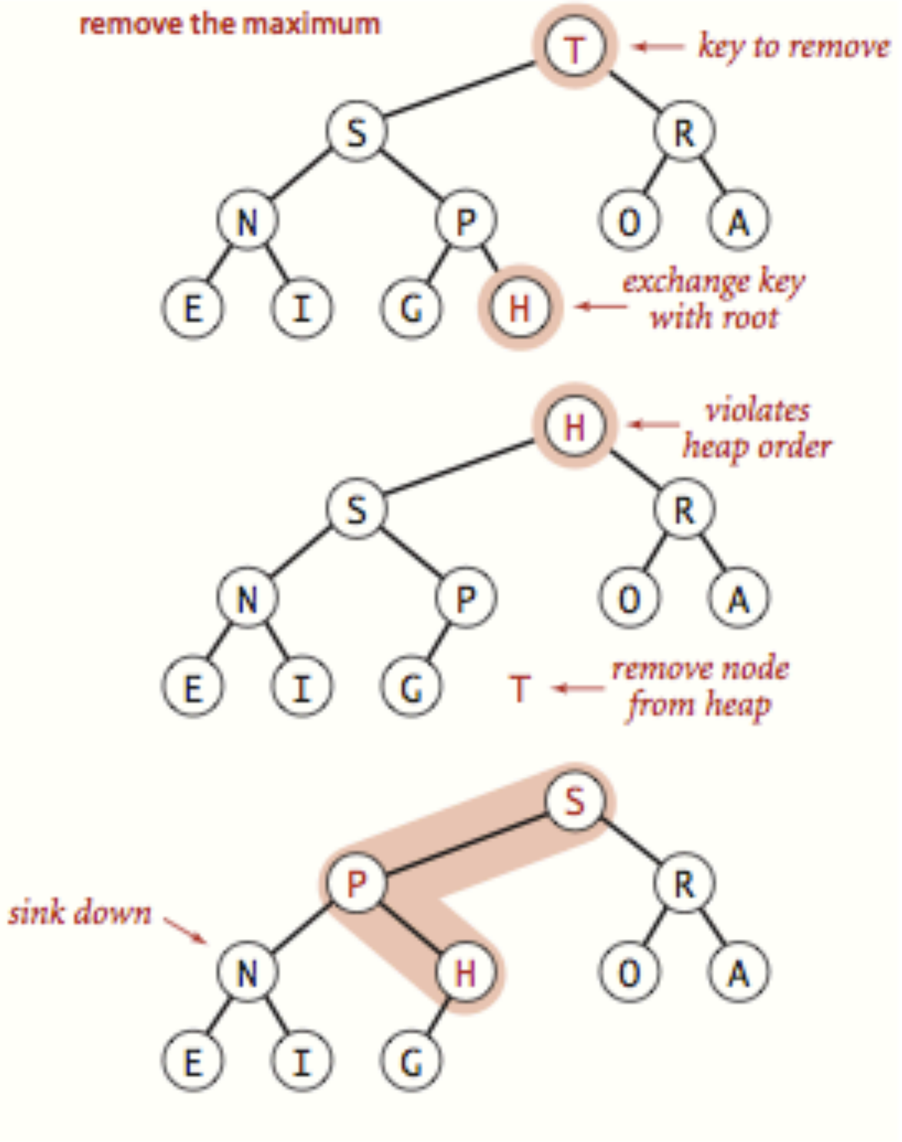


We **sink small nodes** so they become leaves
We do this by swapping with the larger child

So…why sink?

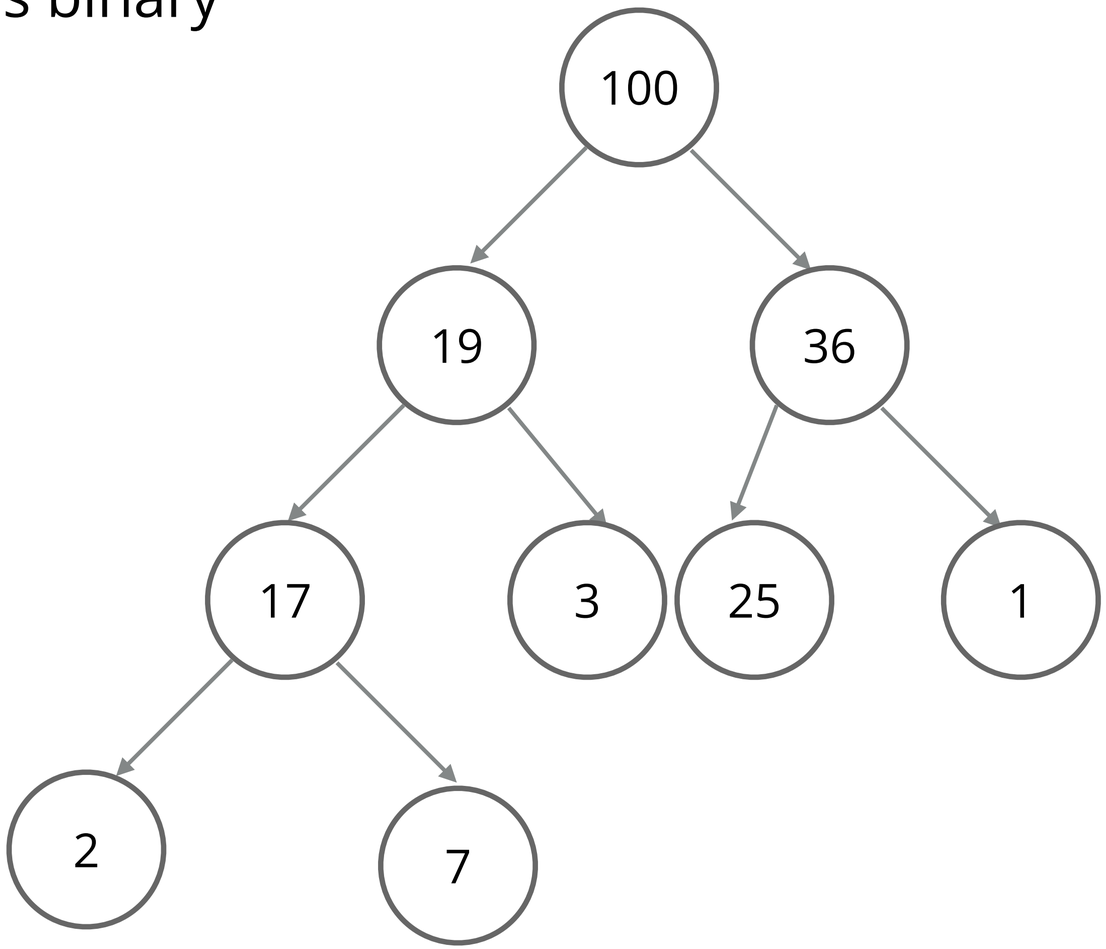
Binary heap: return (and delete) the maximum

- Delete max: Swap the root with the last node (the rightmost child). Return and delete the root. Sink the new root down.
  - Why do we swap with the rightmost child? Only element we can remove without breaking completeness.
- Cost: At most $2 \log n$ compares.



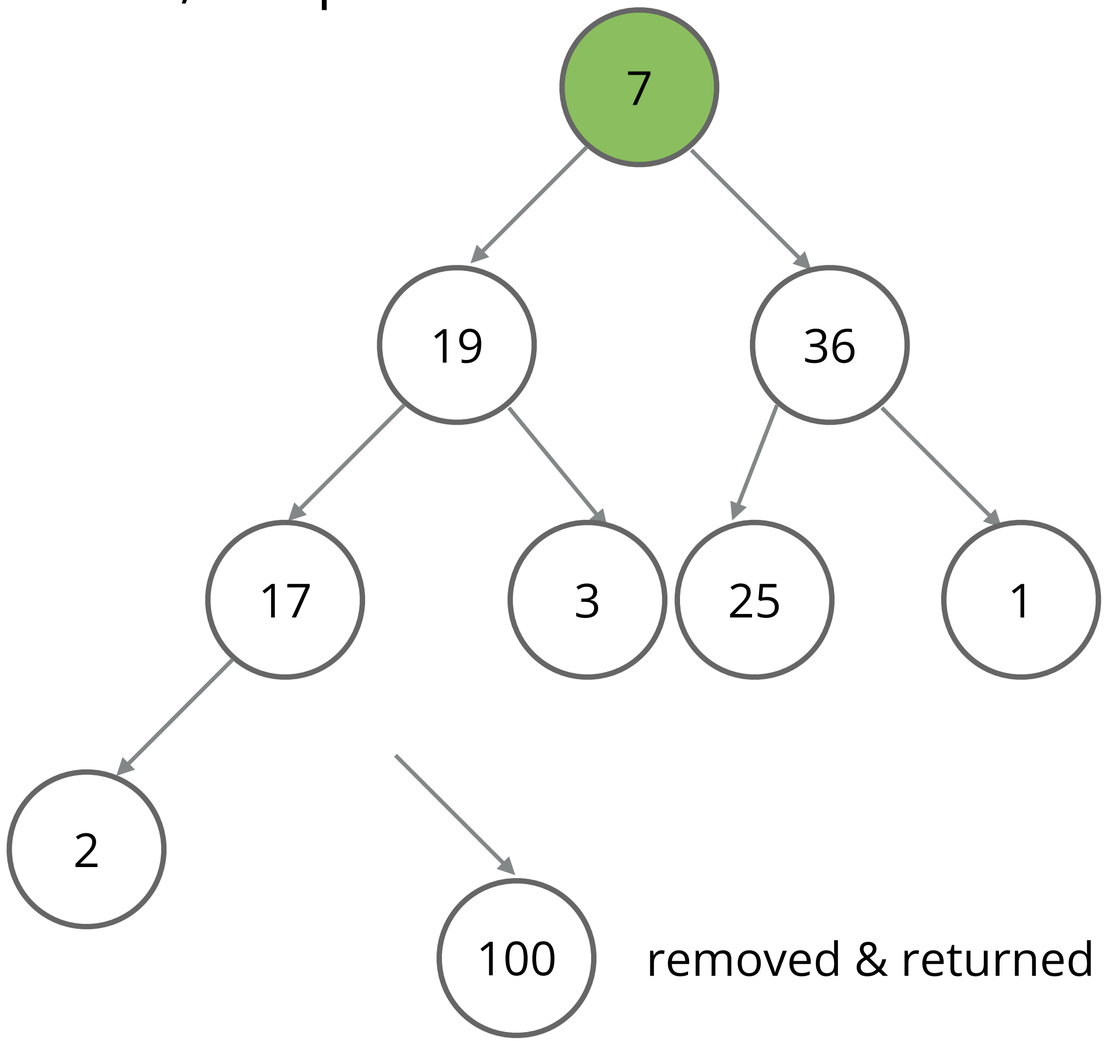
Worksheet time!

- Delete and return the maximum of this binary heap.

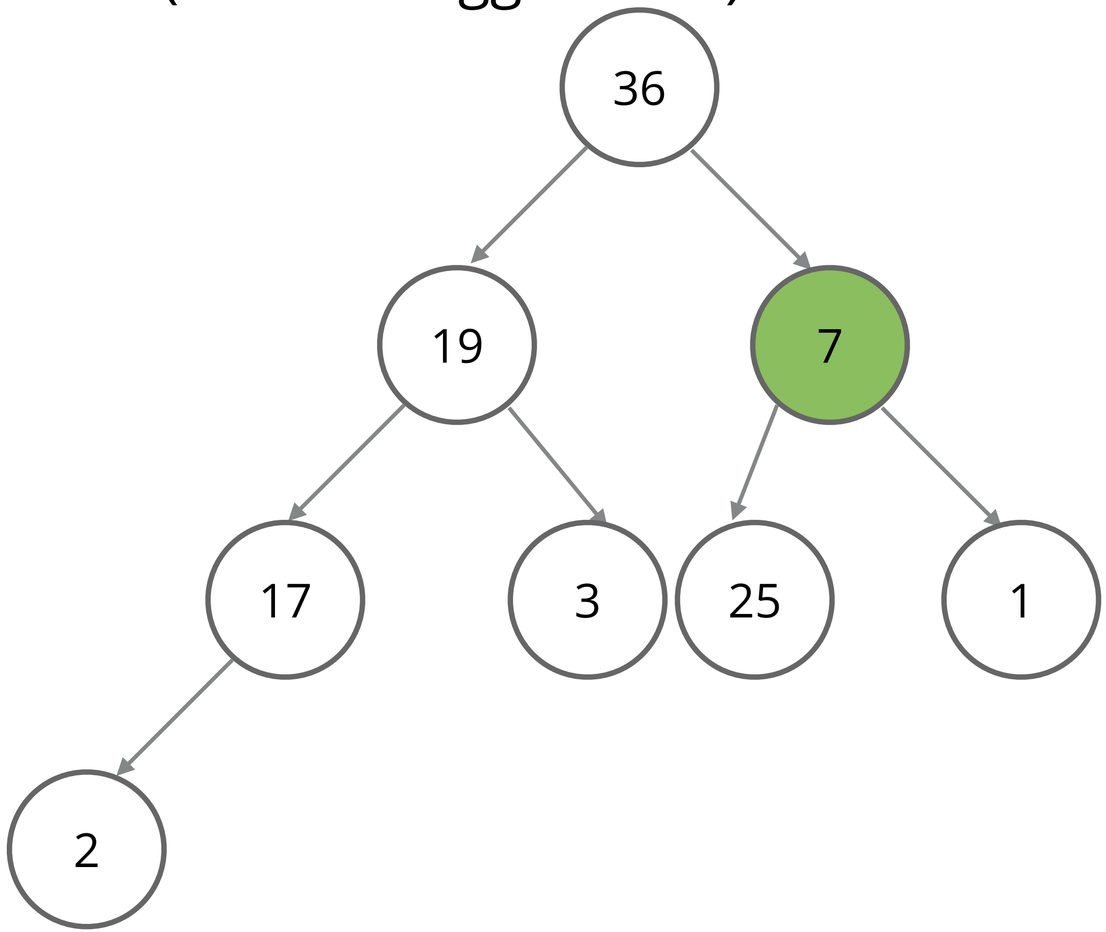


Worksheet answers

- First, swap with 7

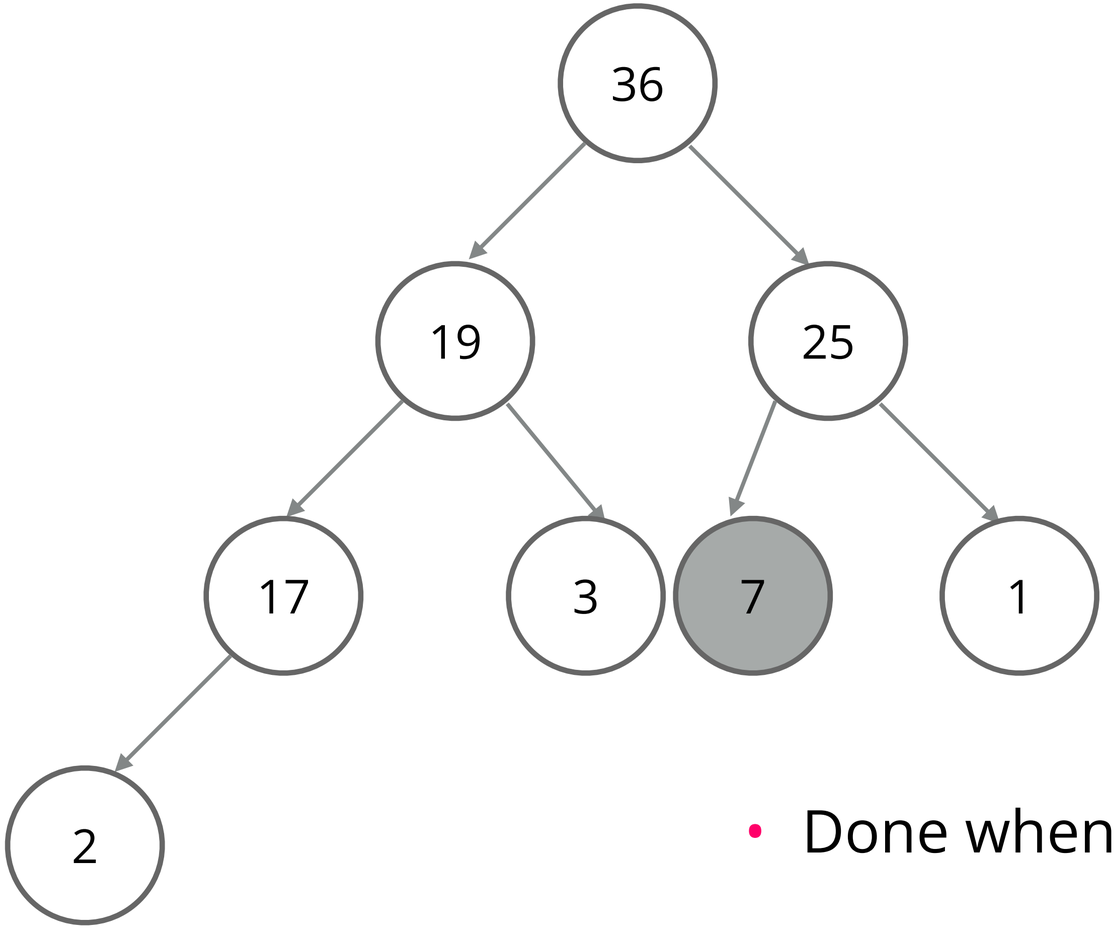


- Then, sink 7 (find the bigger child)



Worksheet answers

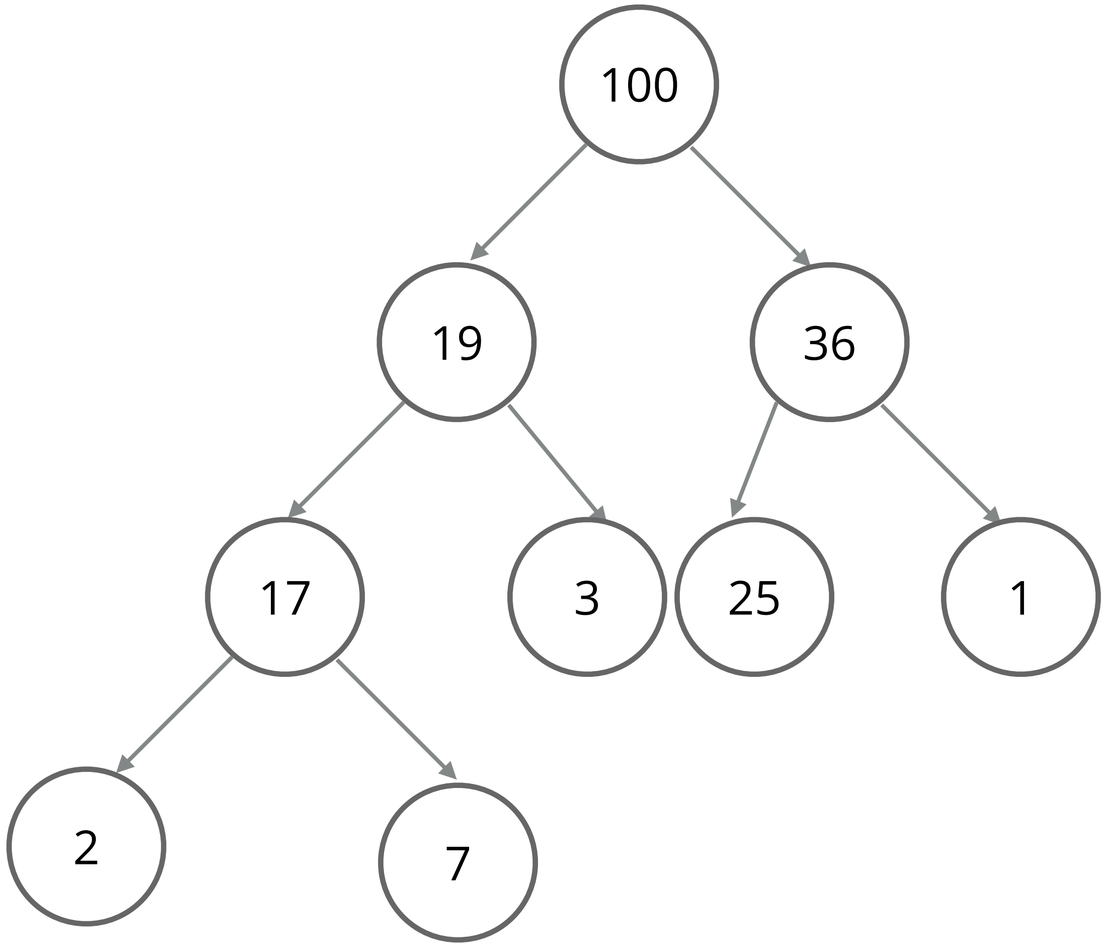
- Then, sink 7 (find the bigger child)



- Done when 7 has no more bigger children

Worksheet time!

- Implement public E deleteMax().
- Assume precondition (n > 0) is true.
- Hint: you can do it in 4 lines of code.
- 1. find max
- 2. ??
- 3. ??
- 4. return max



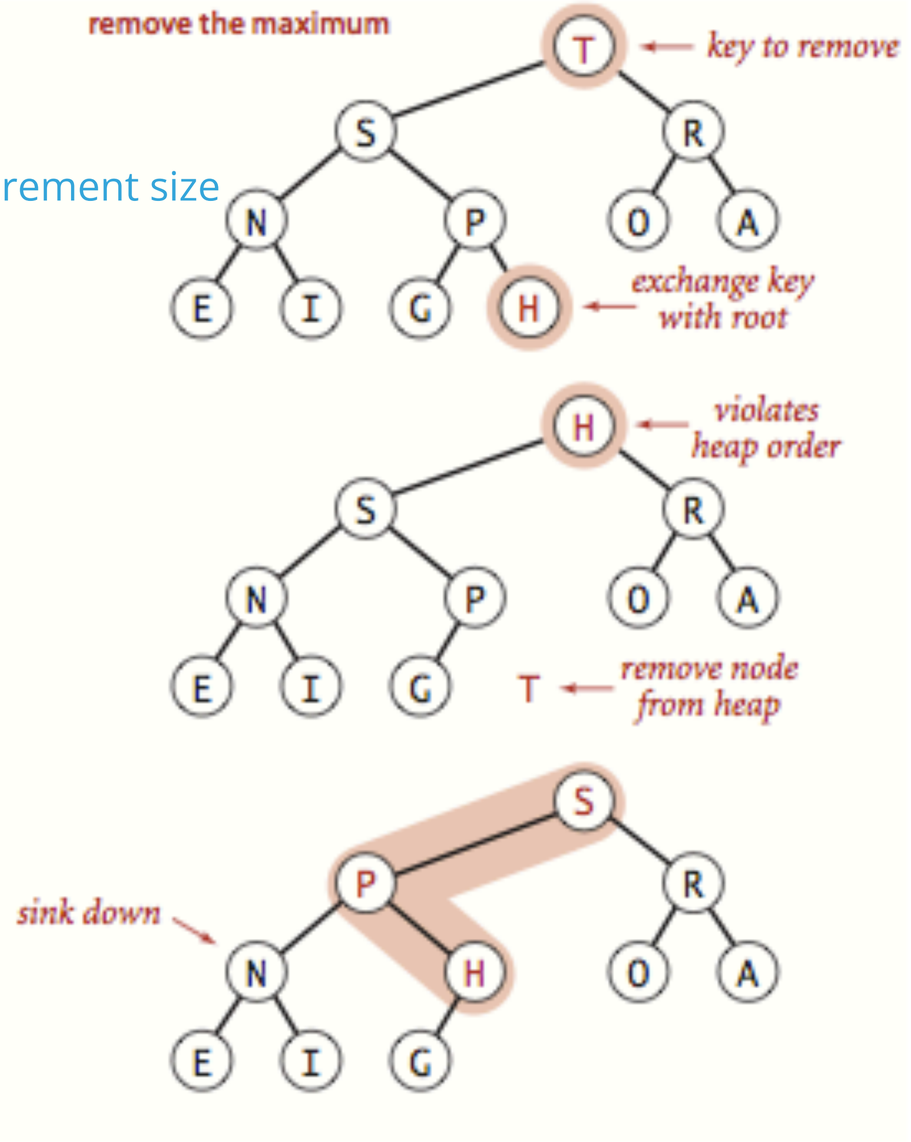
Worksheet answers

```
public E deleteMax() {
    E max = a[1];
    a[1] = a[n--];
    sink(1);
    return max;
}
```

max is always the root

swap root with the last element, decrement size

sink the last element to update tree



Binary heap operation run times

- Insertion is $O(\log n)$ (because O(1) insert at the end, O(log n) swim up to proper place).
- Delete max is $O(\log n)$ (because O(1) swap last node to root, and then O(log n) sink down to proper place).
- Space efficiency is $O(n)$ (because of array representation).

http://algs4.cs.princeton.edu

2.4 BINARY HEAP DEMO

Priority Queues

Priority Queue

- An abstract data type of a queue where each element additionally has a *priority*.
- Two operations:
  - Dequeue, aka delete the maximum
  - Enqueue, aka insert
- How can we implement a priority queue efficiently?

Option 1: Unordered array

- The *lazy* approach where we defer doing work (deleting the maximum) until necessary.
- Insert is $O(1)$ and assumes we have the space in the array.
- Delete maximum is $O(n)$ (have to traverse the entire array to find the maximum element and exchange it with the last element).

Option 2: Ordered array

- The *eager* approach where we do the work (keeping the array sorted) up front to make later operations efficient.
- Insert is $O(n)$ (we have to find the index to insert and shift elements to perform insertion).
- Delete maximum is $O(1)$ (just take the last element which will be the maximum).

Option 3: Binary heap

- Will allow us to both insert and delete max in $O(\log n)$ running time.
- There is no way to implement a priority queue in such a way that insert *and* delete max can be achieved in $O(1)$ running time.
- Priority queues are synonymous to binary heaps.

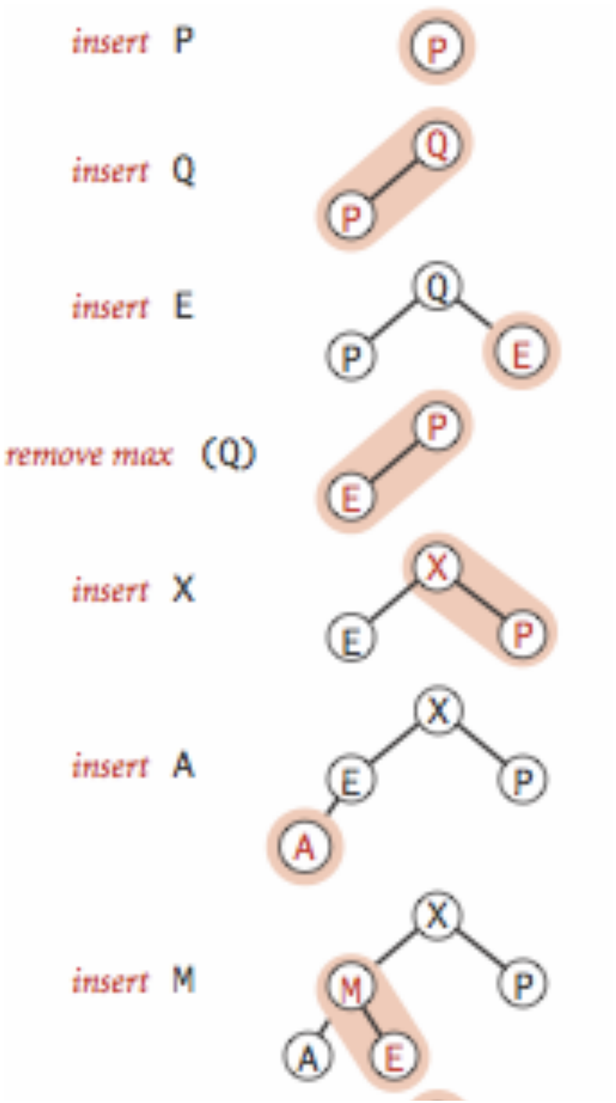
Worksheet time!

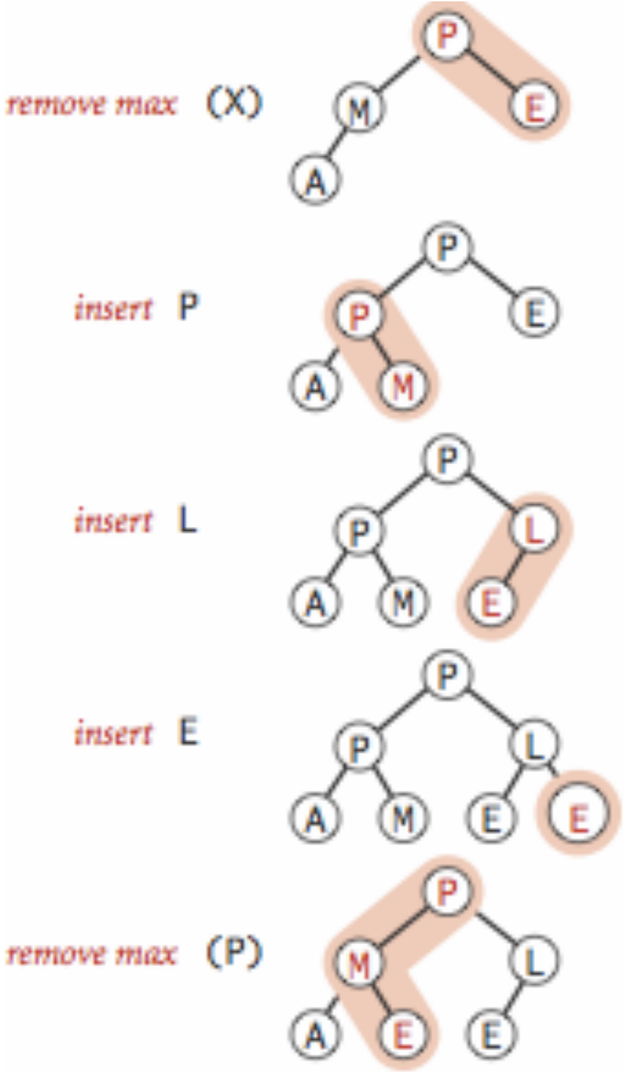
1. Insert P (16)
2. Insert Q (17)
3. Insert E (5)
4. Delete Max
5. Insert X (24)
6. Insert A (1)
7. Insert M (13)
8. Delete Max
9. Insert P (16)
10. Insert L (12)
11. Insert E (5)
12. Delete Max

Given an empty binary heap that represents a priority queue, perform the following operations. Ideally draw the binary tree at each step, but compare with your neighbors what it looks like in the end, and what the 3 delete maxes return.

Worksheet answers

1. Insert P
2. Insert Q
3. Insert E
4. Delete max
5. Insert X
6. Insert A
7. Insert M
8. Delete max
9. Insert P
10. Insert L
11. Insert E
12. Delete max





- Look into MaxPQ class https://algs4.cs.princeton.edu/code/edu/princeton/cs/algs4/MaxPQ.java.html

Heapsort

Basic plan for heap sort

- Given an array to be sorted, use a priority queue to develop a sorting method that works in two steps:
- 1) Heap construction: build a binary heap with all n keys that need to be sorted.
- 2) Sortdown: repeatedly remove and return the maximum key.
- Basically, we sort an array by constructing a binary heap and continually removing the max (root).

$O(n \log n)$ Naïve heap construction

- Insert n elements, one by one, swim up to their appropriate position.
  - Remember that insert() in a binary heap takes O(log n) time because swim takes O(log n) time)
- We can do better!

```
private void swim(int k) {
    while (k > 1 && a[k / 2].compareTo(a[k]) < 0) {
        E temp = a[k];
        a[k] = a[k / 2];
        a[k / 2] = temp;
        k = k / 2;
    }
}

public void insert(E x) {
    a[++n] = x;
    swim(n);
}
```

$O(n)$ Heap construction

- Recall `sink(k)`: small nodes who are parents are sunken down to their proper place (switched with their larger child)
- Key insight: After `sink(k)` completes, the subtree rooted at k is a heap. Basically, performing sink guarantees the subtree at node *k* is a valid binary heap because of the switches.

```
private void sink(int k) {
    while (2 * k <= n) {
        int j = 2 * k;
        if (j < n && a[j].compareTo(a[j + 1]) < 0) j++;
        if (a[k].compareTo(a[j]) >= 0) break;
        E temp = a[k];
        a[k] = a[j];
        a[j] = temp;
        k = j;
    }
}
```

$O(n)$ Heap construction algorithm

- 1. Insert all nodes as is, in indices 1 to n (e.g., starting point is the first element is the root, the second element is the left child, the third is the right child, etc.). This is a binary tree definitely not in heap order.
- 2. Sink each **internal node**, ignoring all the leaves (indices n/2+1,...,n). Remember the leaves will be placed in correct order since they are subtrees of the internal nodes.

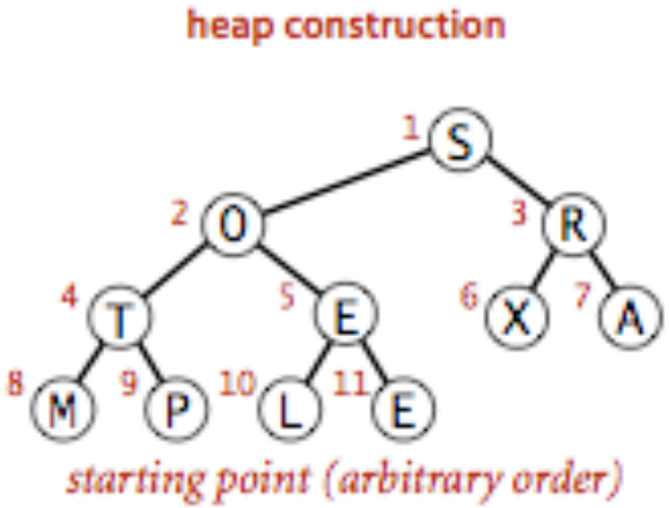
```
3  public class HeapSort {
4      public static <E extends Comparable<E>> void sort(E[] input) {
5          int n = input.length;
6
7          // create a 1-indexed array to make the math cleaner for this demo
8          // (though you shouldn't do this in practice)
9          E[] a = (E[]) new Comparable[n + 1];
10         System.arraycopy(input, 0, a, 1, n);
11
12         // Heap construction in O(n)
13         for (int k = n / 2; k >= 1; k--) {
14             sink(a, k, n);
15         }
```

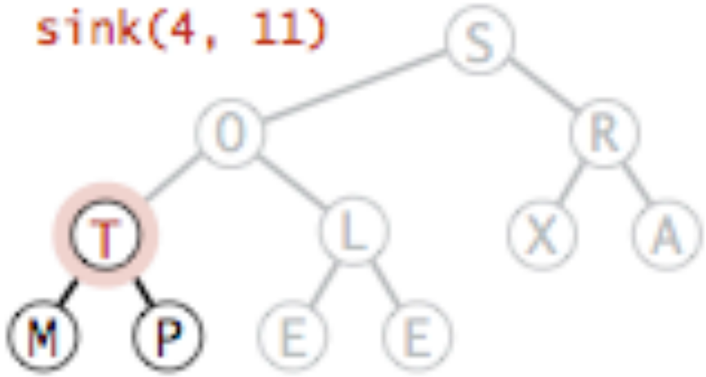
Example: SORTEDEXAMPLE

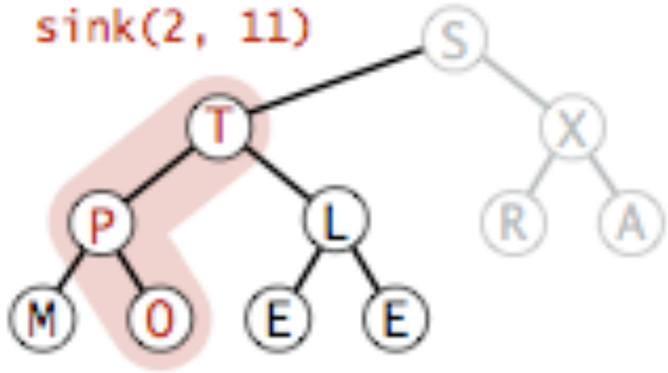
```
for (int k = n / 2; k >= 1; k--) {
```

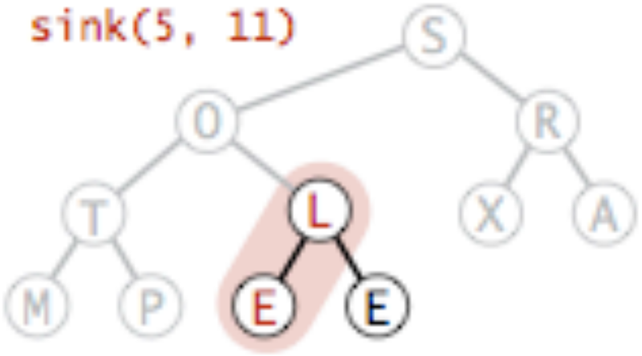
n=11, so k=5 initially

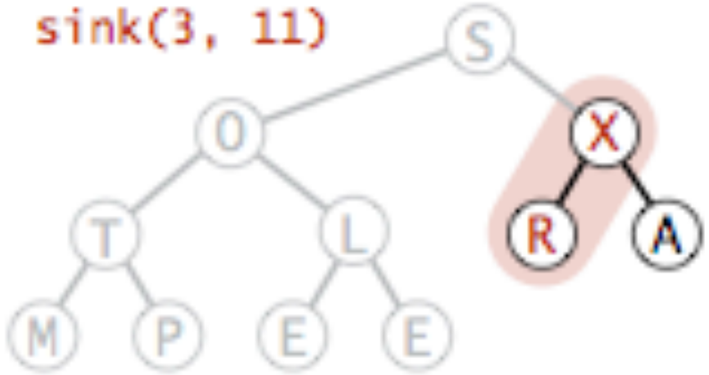
Why “bottom up” and starting with the lowest internal node first?
Because once the smaller subtrees are correct, they’re guaranteed to stay correct when we sink down their parents.

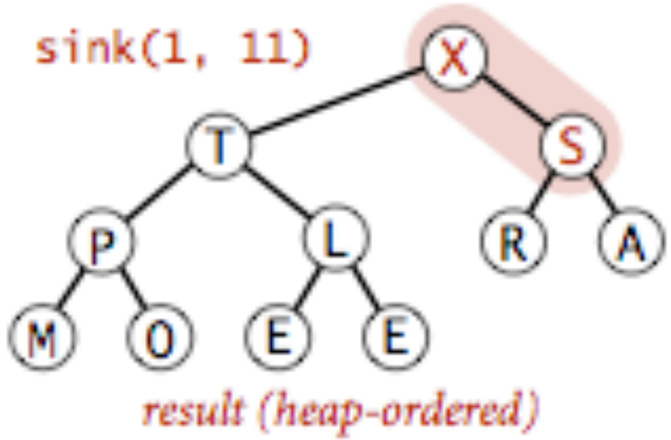








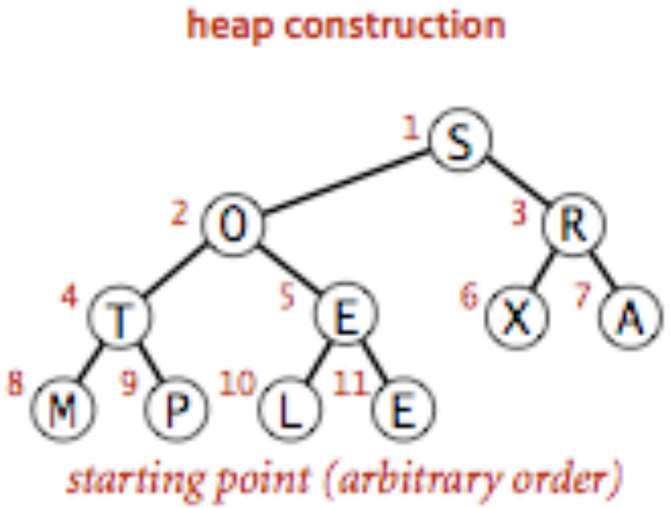


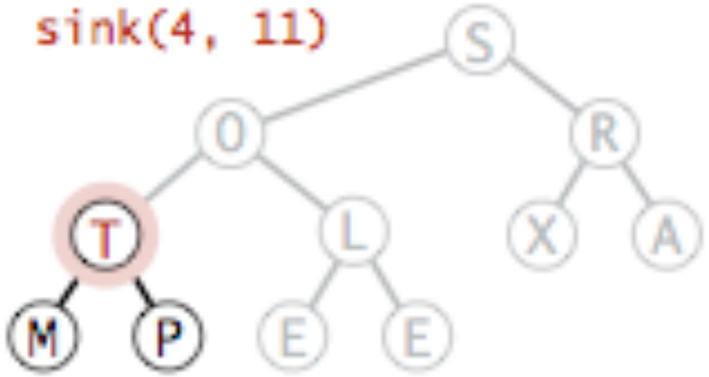


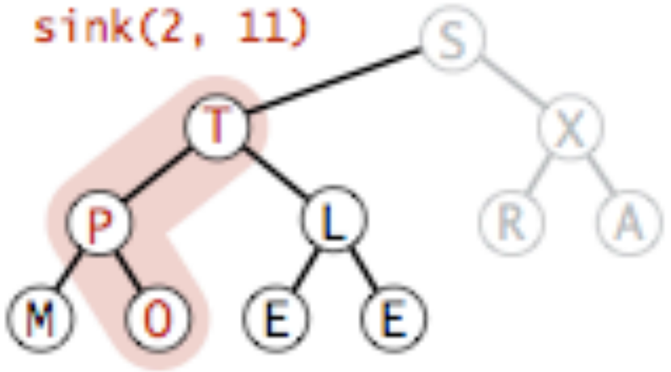
Why O(n)?

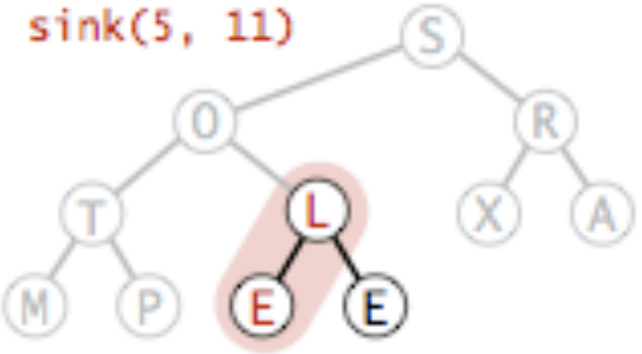
```
for (int k = n / 2; k >= 1; k--) {
```

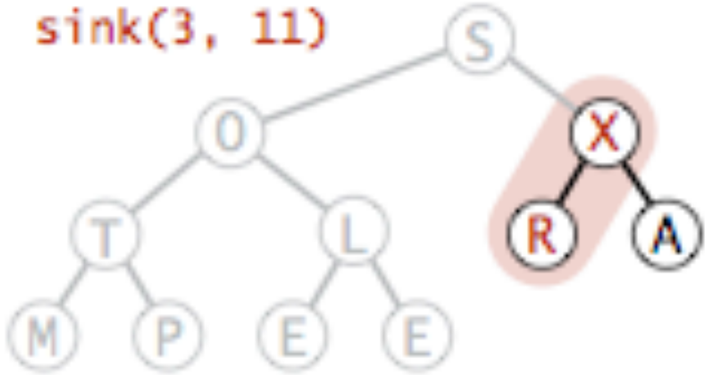
Intuition: A single sink worst case is O(log n). But we don't sink every node. Most internal nodes are near the bottom of the heap, so it is just 1 swap (a constant time operation). In total, we do ~2n swaps and compares maximum.

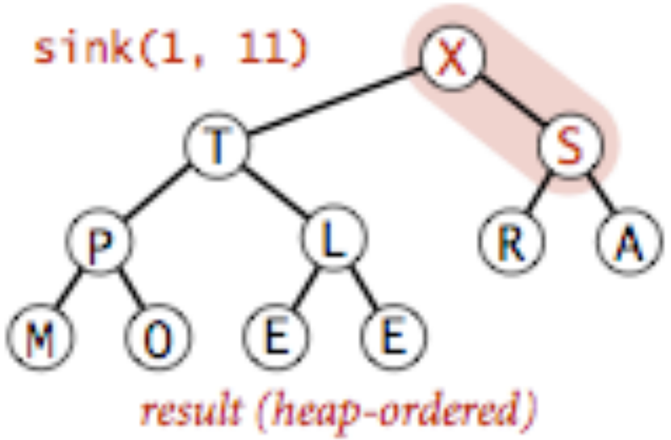










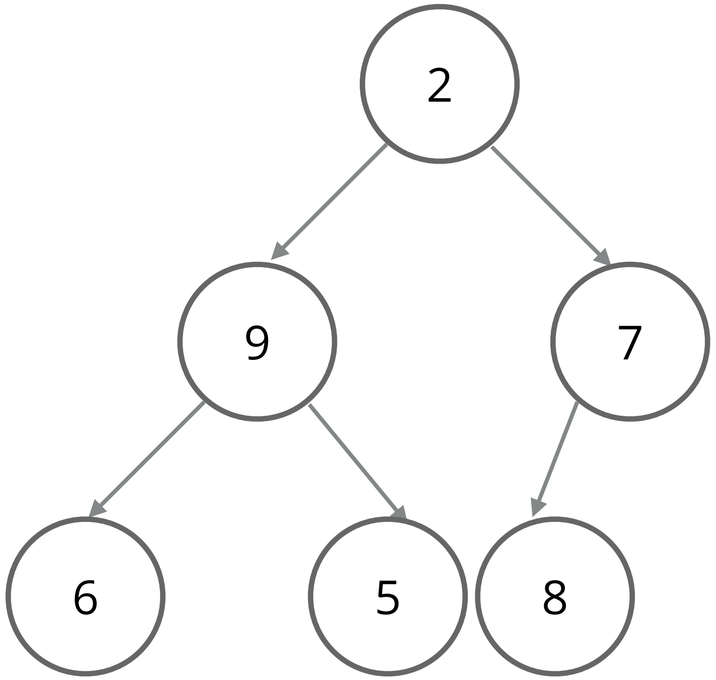


Worksheet time!

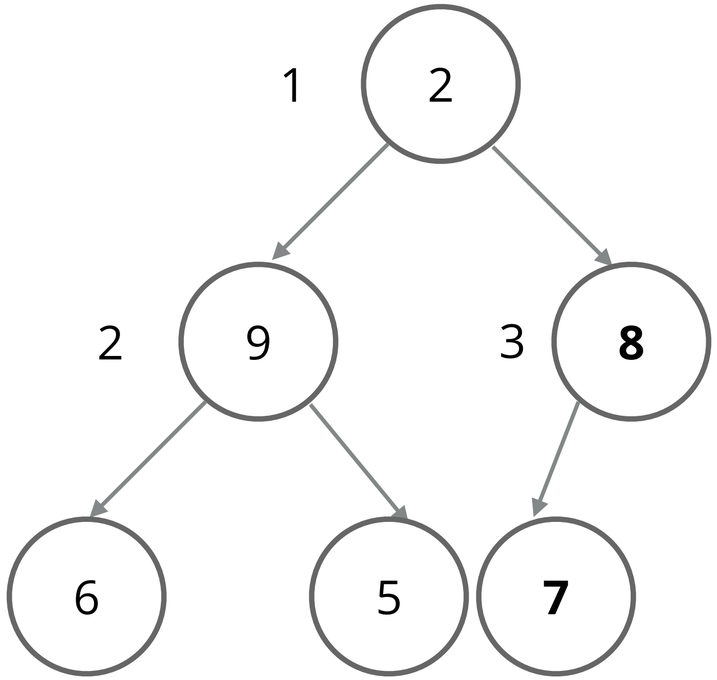
- Run the first step of heapsort, heap construction, on the array [2,9,7,6,5,8]. What is the resultant binary heap?

Worksheet answer

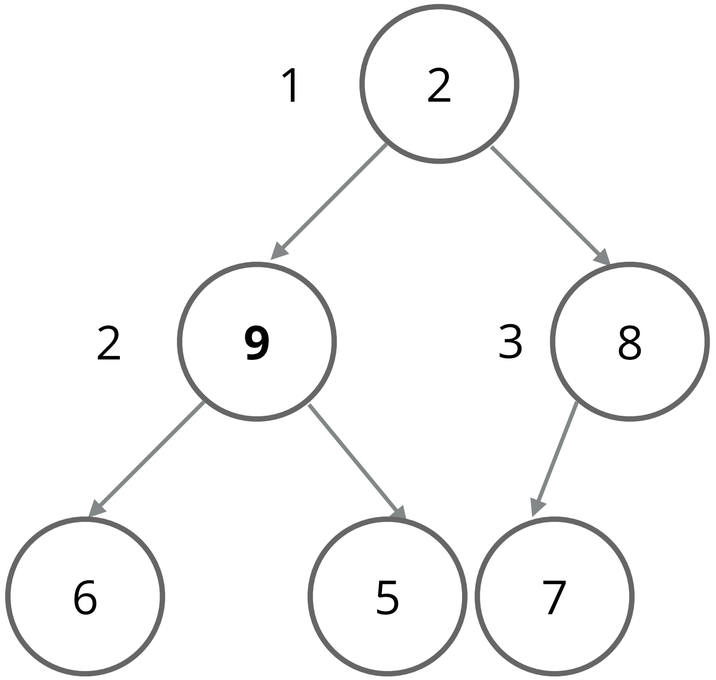
Step one: just in array order



2. sink(3, 6)



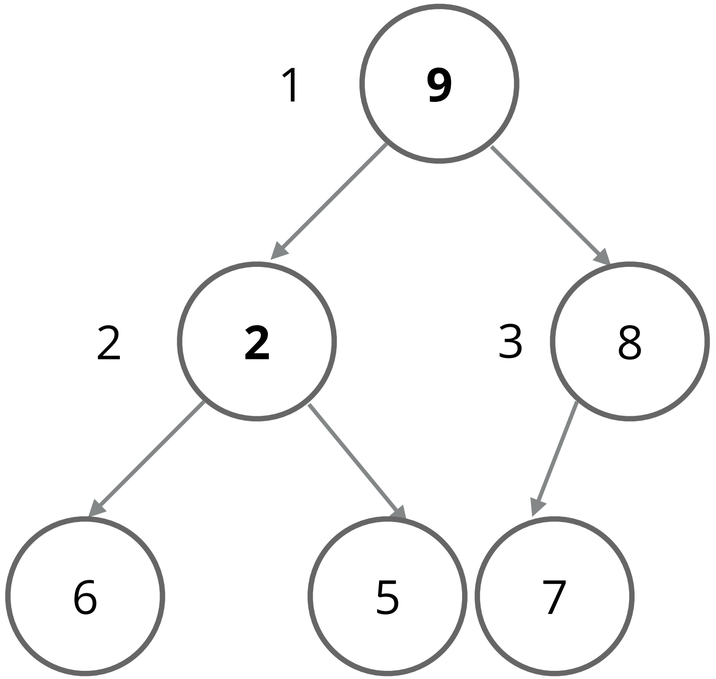
3. sink(2, 6)



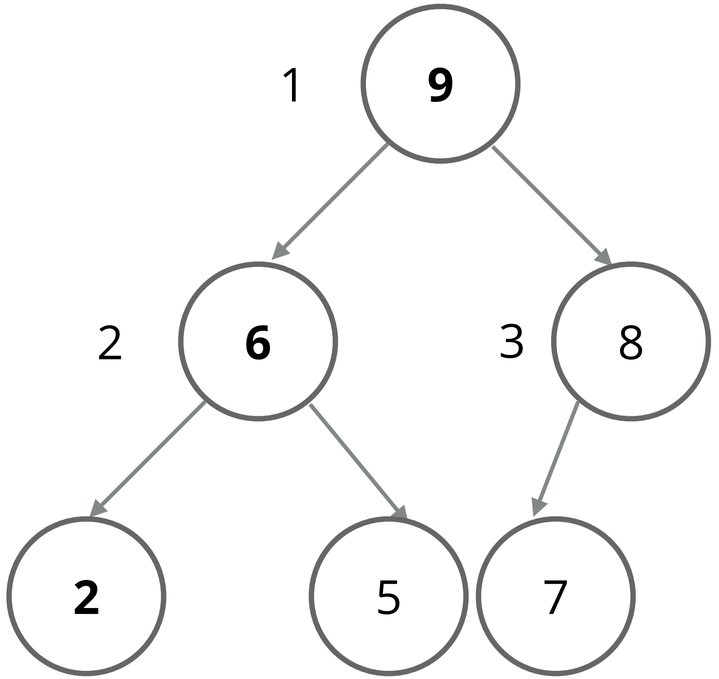
(no action needed)

Worksheet answer

4. sink(1,6)



part 1: swap 2 & 9 (9 > 8)



part 2: swap 6 & 2

Final heap!

Sortdown

- Now that we have an ordered binary heap, all that remains is to pull out the roots (each subsequent max element).
- Recall: deleteMax() in binary heaps swaps the last element to be the new root and sinks that down.
- Key insight: After each iteration of sortDown, the array consists of a heap-ordered subarray of k elements, followed by a sub-array of n-k elements in final order.

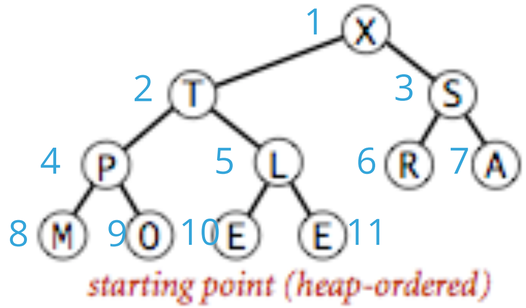
```
// Sorting in O(nlogn)
while (n > 1) {
    swap(a, 1, n--);
    sink(a, 1, n);
}
```

While the heap has > 1 element,

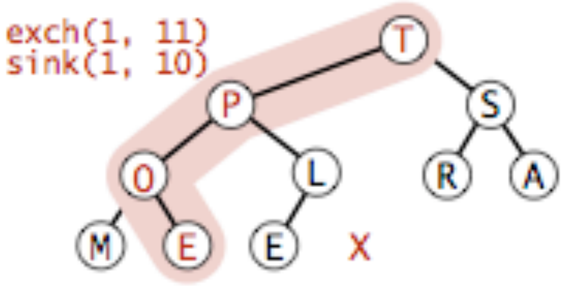
swap the root with the last element

sink the new root appropriately

Sortdown example

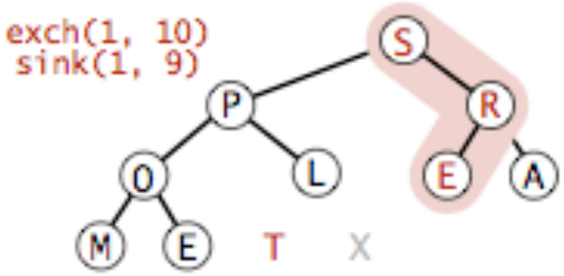



```
while (n > 1) {
    swap(a, 1, n--);
    sink(a, 1, n);
}
```

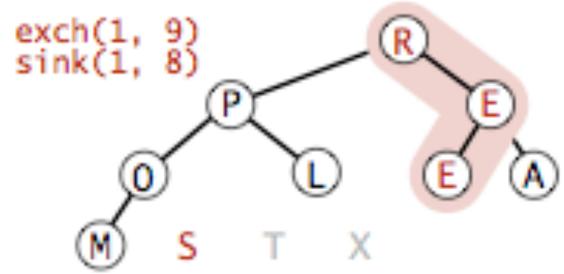


n = 11, so first we call swap
(or “exch”) on (1, 11), then sink(1, 10)

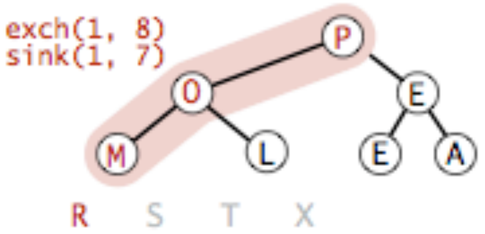
Swap X with E, sink down E -> T is new root
return X



swap T with E, sink down E -> S is new root
return T, X

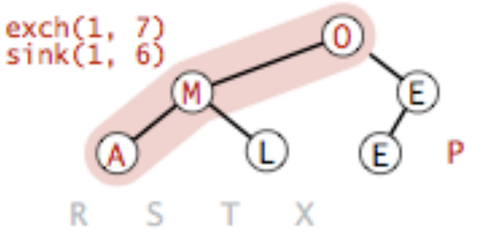


swap S with E, sink down E -> R is new root
return S, T, X



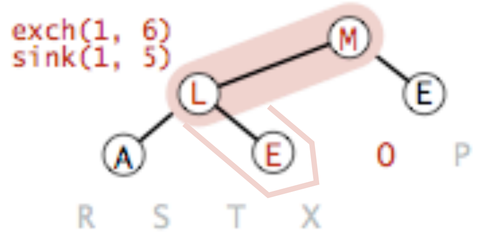
swap R with M, sink M -> P is new root

return R, S, T, X



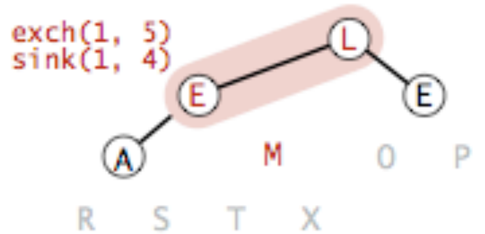
swap P with A, sink A -> O is new root

return P, R, S, T, X



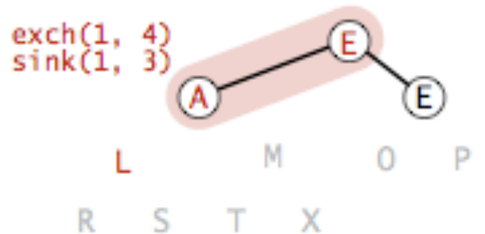
swap O with E, sink E -> M is new root

return O, P, R, S, T, X



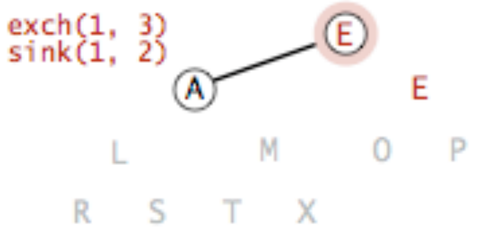
swap M with E, sink E -> L is new root

return M, O, P, R, S, T, X



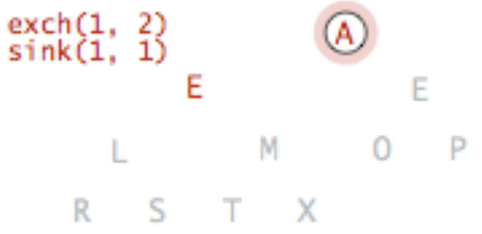
swap L with A, sink A -> E is new root

return L, M, O, P, R, S, T, X



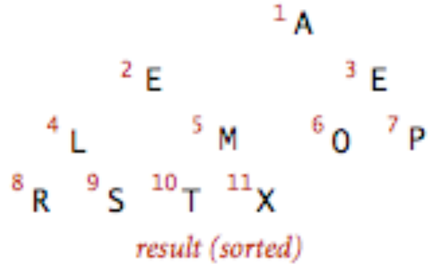
swap E with E, sink E (no action) -> E is new root

return E, L, M, O, P, R, S, T, X



swap E with A, sink A (A is just a single node, nothing to sink)

return E, E, L, M, O, P, R, S, T, X



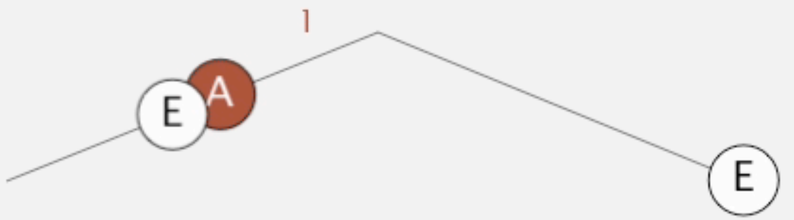
because n = 1, we're done

return A, E, E, L, M, O, P, R, S, T, X

Heapsort demo

Sortdown. Repeatedly delete the largest remaining item.

sink 1



L M O P

R S T X

~~A~~	~~E~~	E	L	M	O	P	R	S	T	X
1										

Worksheet time!

- Given the heap you constructed before, run the second step of heapsort, sortdown, to sort the array [2,9,7,6,5,8].

Worksheet answer

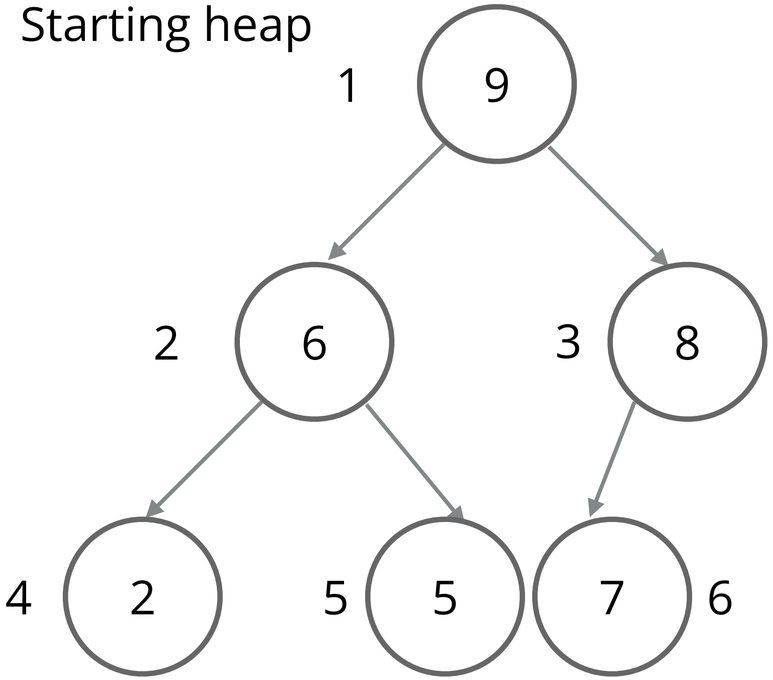
Return: 9

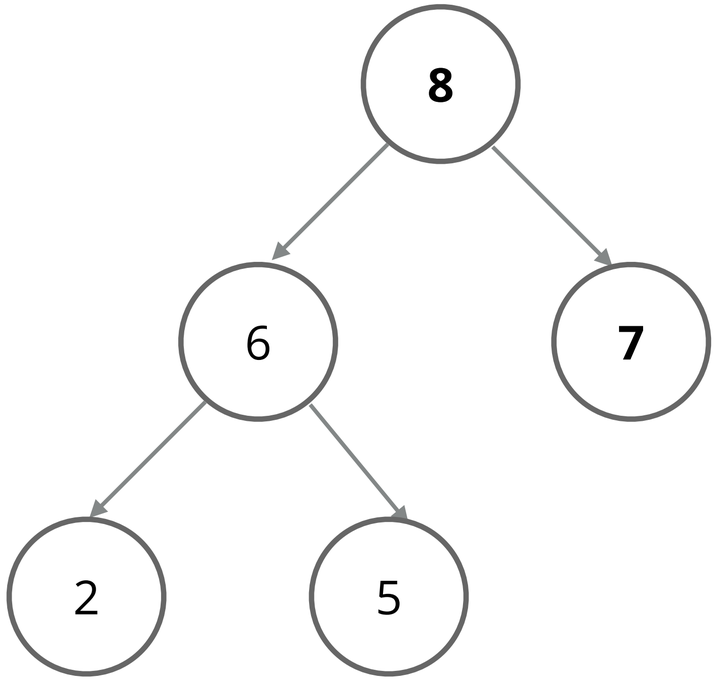
Return: 8, 9

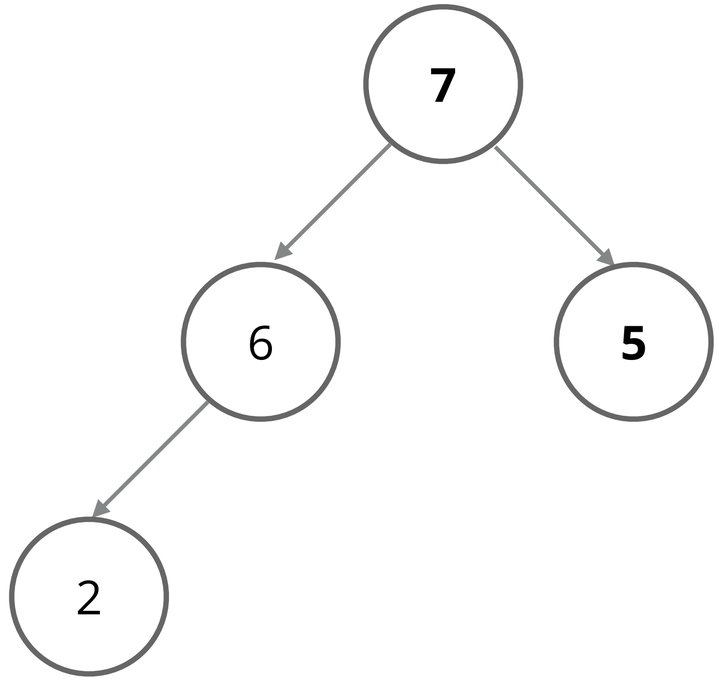
1. swap(1,6) sink(1,5) means swap 9 & 7 and sink 7

2. swap(1,5) sink(1,4) means swap 8 & 5 and sink 5

Starting heap



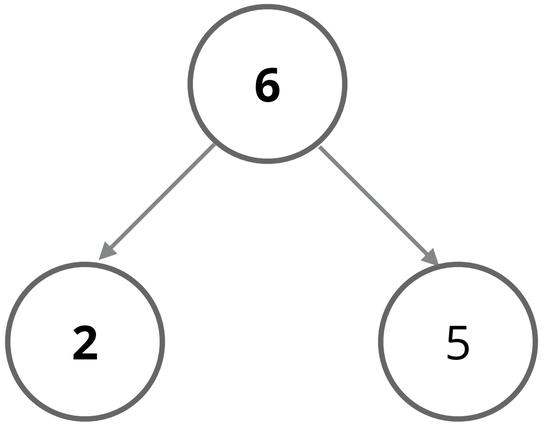




Worksheet answer

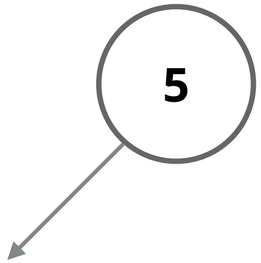
Return: 7, 8, 9

3. swap(1,4) sink(1,3) means
swap 7 & 2 and sink 2



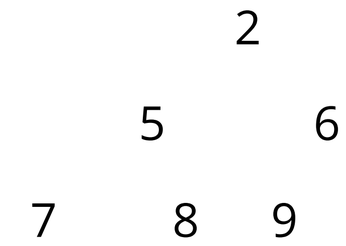
Return: 6, 7, 8, 9

4. swap(1,3) sink(1,2) means
swap 6 & 5 and sink 5 (no sinking needed)



Return: 5, 6, 7, 8, 9

4. swap(1,2) sink(1,1) means
swap 5 & 2 and sink 2 (no sinking needed, single node)



5. done!

Return: 2, 5, 6, 7, 8, 9

Heapsort analysis

Heapsort analysis

- Summary: heapsort has two steps, heap construction and sort down.
- Heap construction (the fast version) makes $O(n)$ exchanges and $O(n)$ compares.
- Sort down and therefore the entire heapsort $O(n \log n)$ exchanges and compares.
  - Each sink() is logn time, and we do n-1 sinks
- $O(n \log n)$ worst case. What about best case? Average case?
  - The same
- In-place (no need to copy anything).
- Not stable (we are swapping elements)

Heapsort analysis

- Review:
  - Mergesort: not in place, requires linear extra space.
  - Quicksort: quadratic time in worst case.
- Heapsort is optimal both for time and space in terms of Big-O, but:
  - Inner loop is longer than quicksort because of sink.
  - Poor use of cache because it accesses memory in non-sequential manner, jumping around the heap/array (more in CS105).
- In general, quicksort is preferred when it comes to speed, and mergesort is preferred when it comes to stability.

Sorting: we're done!

Which Sort	In place	Stable	Best	Average	Worst	Memory	Remarks
Selection	X		$\Omega(n^2)$	$\Theta(n^2)$	$O(n^2)$	$\Theta(1)$	n exchanges
Insertion	X	X	$\Omega(n)$	$\Theta(n^2)$	$O(n^2)$	$\Theta(1)$	Fastest if almost sorted or small
Merge		X	$\Omega(n \log n)$	$\Theta(n \log n)$	$O(n \log n)$	$\Theta(n)$	Guaranteed performance; stable
Quick	X		$\Omega(n \log n)$	$\Theta(n \log n)$	$O(n^2)$	$\Theta(\log n)$	$n \log n$ probabilistic guarantee; fastest in practice
Heap	X		$\Omega(n \log n)$	$\Theta(n \log n)$	$O(n \log n)$	$\Theta(1)$	Guaranteed performance; in place

Lecture 14 wrap-up

- HW7: Autocomplete sort due Tues 11:59pm
- Checkpoint 2 in 2 weeks: **Mon 11/3**. Please schedule SDRC proctoring now. Will cover up to next Mon's lecture on B-Trees. No HW week of checkpoint (next HW, HW8: Hex-a-pawn is about binary search trees)

Resources

- Reading from textbook: 2.5 (336-344)
- Heapsort visualization: https://algostructure.com/sorting/heapsort.php
- More visualization to compare the n and nlogn create heap approaches: https://visualgo.net/en/heap
- Practice problems behind this slide

Practice Problem 1

- Suppose that the sequence 16, 18, 9, 15, *, 18, *, *, 9, *, 20, *, 25, *, *, *, 17, 21, 5, *, *, *, 21, *, 5 (where a number means insert and an asterisk means delete the maximum) is applied to an initially empty priority queue. Give the sequence of numbers returned by the delete maximum operations.

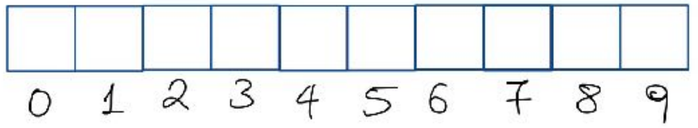
ANSWER 1

- Suppose that the sequence 16, 18, 9, 15, *, 18, *, *, 9, *, 20, *, 25, *, *, *, 17, 21, 5, *, *, *, 21, *, 5 (where a number means insert and an asterisk means delete the maximum) is applied to an initially empty priority queue. Give the sequence of numbers returned by the delete maximum operations.
- 18, 18, 16, 15, 20, 25, 9, 9, 21, 17, 5, 21

Code for priority queue option 1: Unordered array

```
public class UnorderedArrayMaxPQ<Key extends Comparable<Key>> {
    private Key[] pq;      // elements
    private int n;         // number of elements

    // set inititial size of heap to hold size elements
    public UnorderedArrayMaxPQ(int capacity) {
        pq = (Key[]) new Comparable[capacity];
        n = 0;
    }

    public boolean isEmpty()   { return n == 0; }
    public int size()          { return n;      }
    public void insert(Key x)  { pq[n++] = x;   }

    public Key delMax() {
        int max = 0;
        for (int i = 1; i < n; i++){
            if (pq[max].compareTo(pq[i]) < 0) {
                 max = i;
            }
        }
        Key temp = pq[max];
        pq[max] = pq[n-1];
        pq[n-1] = temp;

        return pq[--n];
    }
}
```

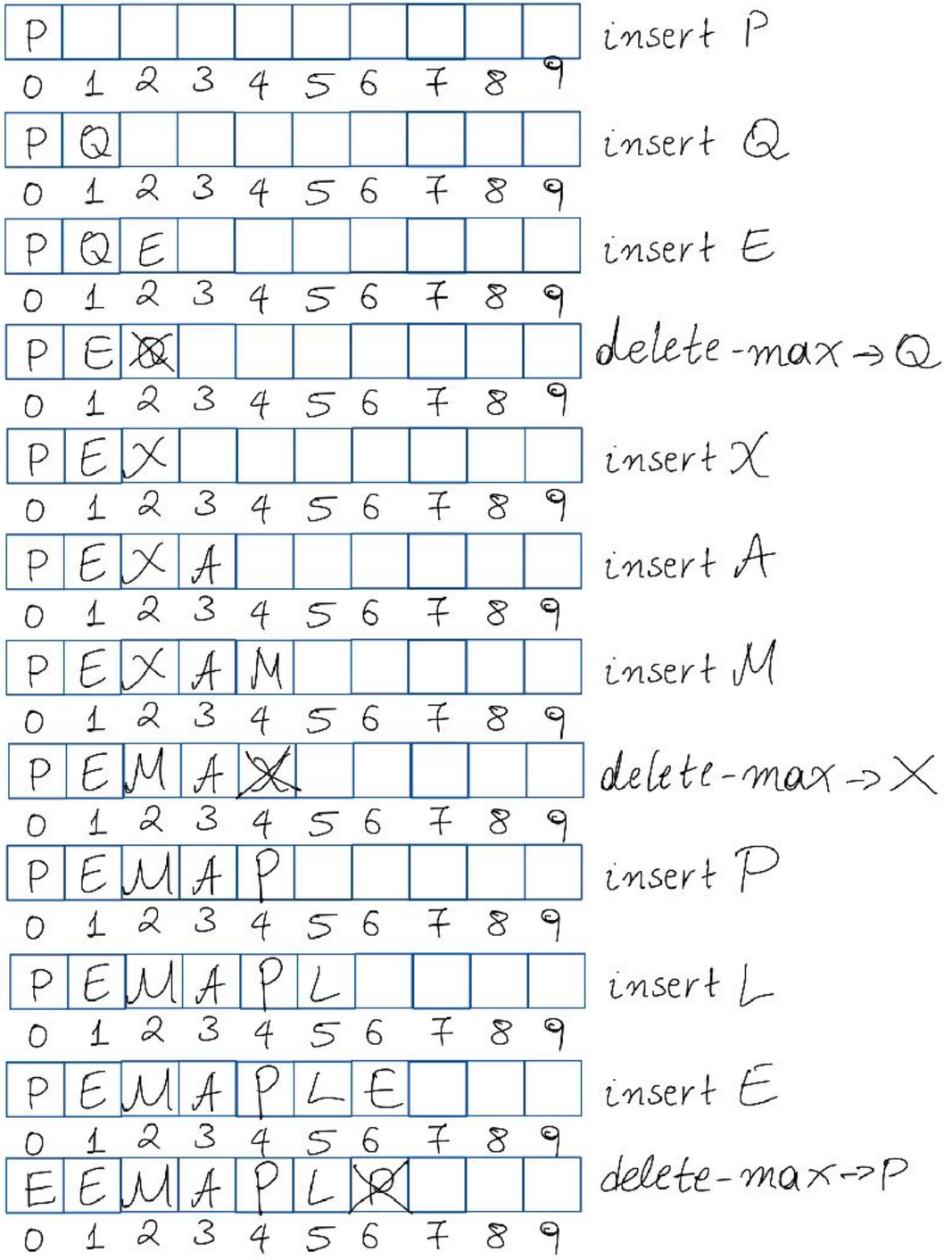
Practice problem 2



1. Insert P
2. Insert Q
3. Insert E
4. Delete max
5. Insert X
6. Insert A
7. Insert M
8. Delete max
9. Insert P
10. Insert L
11. Insert E
12. Delete max

Given an empty array of capacity 10, perform the following operations in a priority queue based on an unordered array (lazy approach):

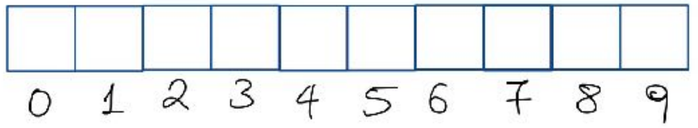
Answer 2


P
0 1 2 3 4 5 6 7 8 9
insert P
P Q
0 1 2 3 4 5 6 7 8 9
insert Q
P Q E
0 1 2 3 4 5 6 7 8 9
insert E
P E Q
0 1 2 3 4 5 6 7 8 9
delete-max → Q
P E X
0 1 2 3 4 5 6 7 8 9
insert X
P E X A
0 1 2 3 4 5 6 7 8 9
insert A
P E X A M
0 1 2 3 4 5 6 7 8 9
insert M
P E M A X
0 1 2 3 4 5 6 7 8 9
delete-max → X
P E M A P
0 1 2 3 4 5 6 7 8 9
insert P
P E M A P L
0 1 2 3 4 5 6 7 8 9
insert L
P E M A P L E
0 1 2 3 4 5 6 7 8 9
insert E
E E M A P L P
0 1 2 3 4 5 6 7 8 9
delete-max → P

Priority queue option 2: Ordered array

```
public class OrderedArrayMaxPQ<Key extends Comparable<Key>> {
    private Key[] pq;          // elements
    private int n;             // number of elements

    // set inititial size of heap to hold size elements
    public OrderedArrayMaxPQ(int capacity) {
        pq = (Key[]) (new Comparable[capacity]);
        n = 0;
    }


    public boolean isEmpty() { return n == 0;  }
    public int size()        { return n;       }
    public Key delMax()      { return pq[--n]; }

    public void insert(Key key) {
        int i = n-1;
        while (i >= 0 && key.compareTo(pq[i]) < 0) {
            pq[i+1] = pq[i];
            i--;
        }
        pq[i+1] = key;
        n++;
    }
}
```

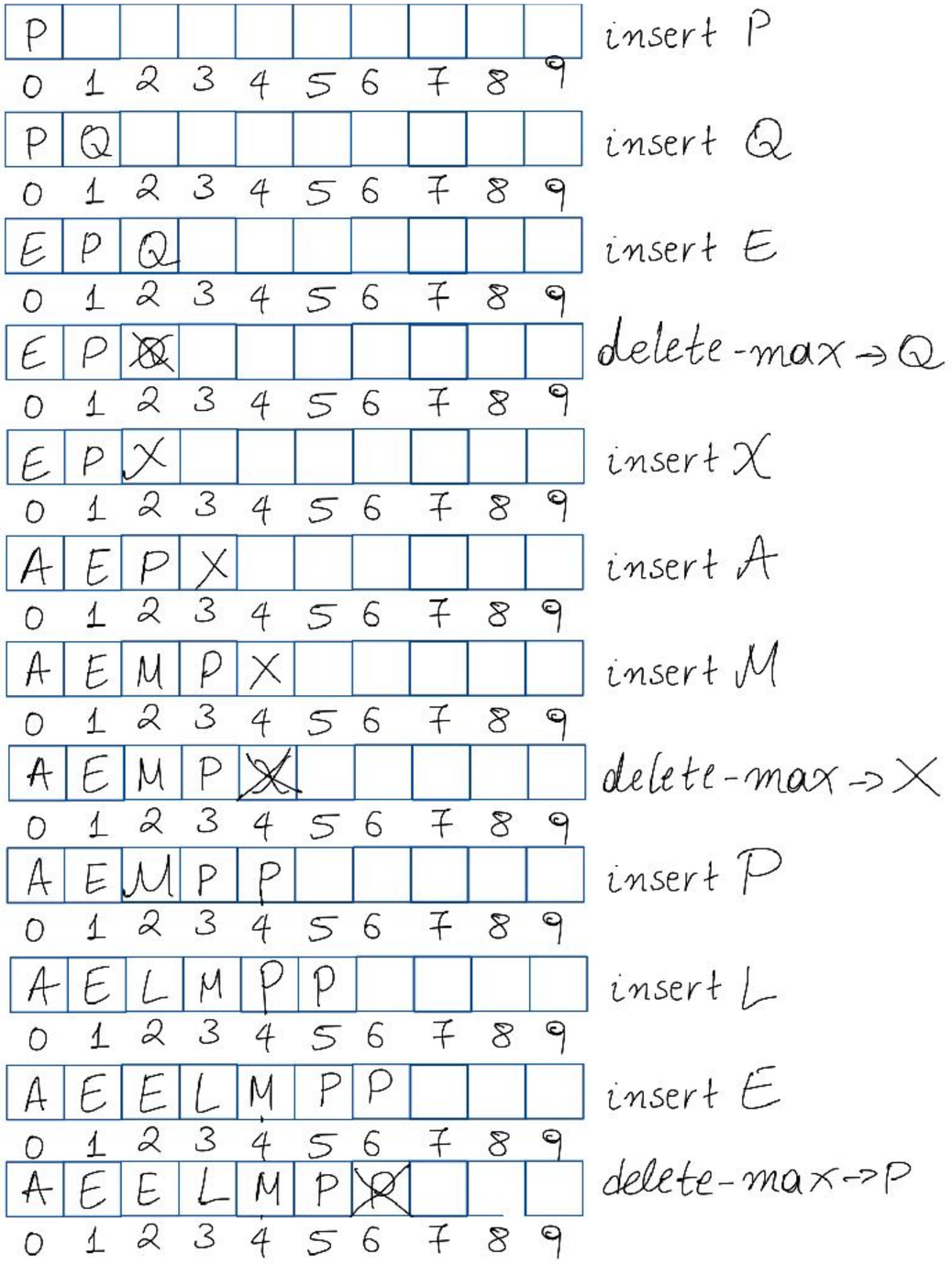
Practice Problem 3



1. Insert P
2. Insert Q
3. Insert E
4. Delete max
5. Insert X
6. Insert A
7. Insert M
8. Delete max
9. Insert P
10. Insert L
11. Insert E
12. Delete max

Given an empty array of capacity 10, perform the following operations in a priority queue based on an ordered array (eager approach):

Answer 3

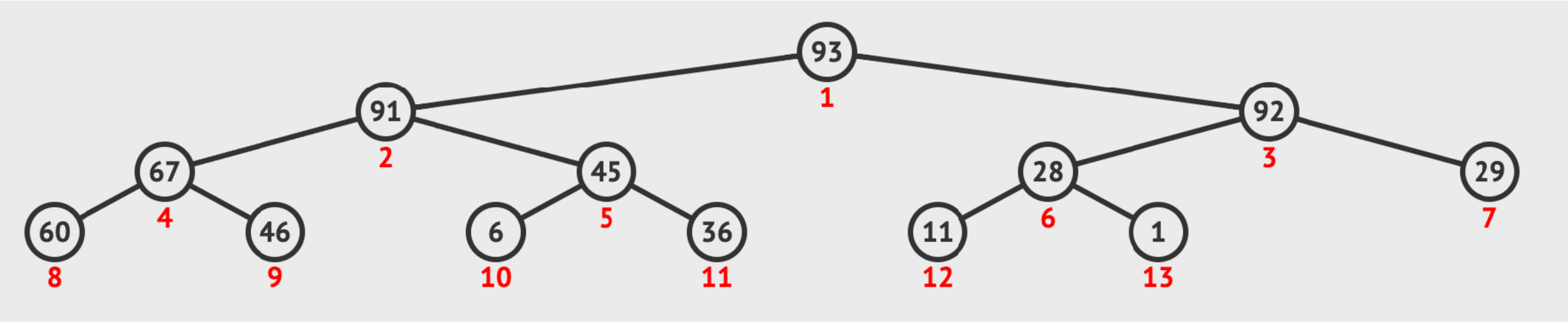

P
0 1 2 3 4 5 6 7 8 9
insert P
P Q
0 1 2 3 4 5 6 7 8 9
insert Q
E P Q
0 1 2 3 4 5 6 7 8 9
insert E
E P Q
0 1 2 3 4 5 6 7 8 9
delete-max → Q
E P X
0 1 2 3 4 5 6 7 8 9
insert X
A E P X
0 1 2 3 4 5 6 7 8 9
insert A
A E M P X
0 1 2 3 4 5 6 7 8 9
insert M
A E M P X
0 1 2 3 4 5 6 7 8 9
delete-max → X
A E M P P
0 1 2 3 4 5 6 7 8 9
insert P
A E L M P P
0 1 2 3 4 5 6 7 8 9
insert L
A E E L M P P
0 1 2 3 4 5 6 7 8 9
insert E
A E E L M P P
0 1 2 3 4 5 6 7 8 9
delete-max → P

Practice Problem 4: Heapsort

- Given the array [93,36,1,46,91,92,29,60,67,6,45,11,28], apply heap sort. Visualize what the heap will initially look like (apply the O(n) heap construction algorithm) and visualize it during sortdown as well.

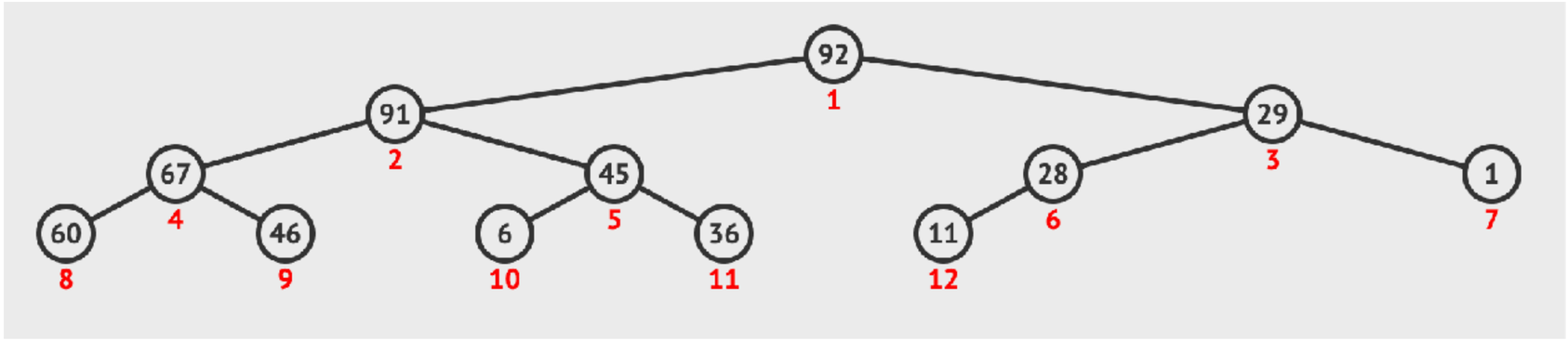
ANSWER 4

- Given the array [93,36,1,46,91,92,29,60,67,6,45,11,28], apply heap sort. Visualize what the heap will initially look like (apply the O(n) heap construction algorithm) and visualize it during sortdown as well.
- Heap construction step:

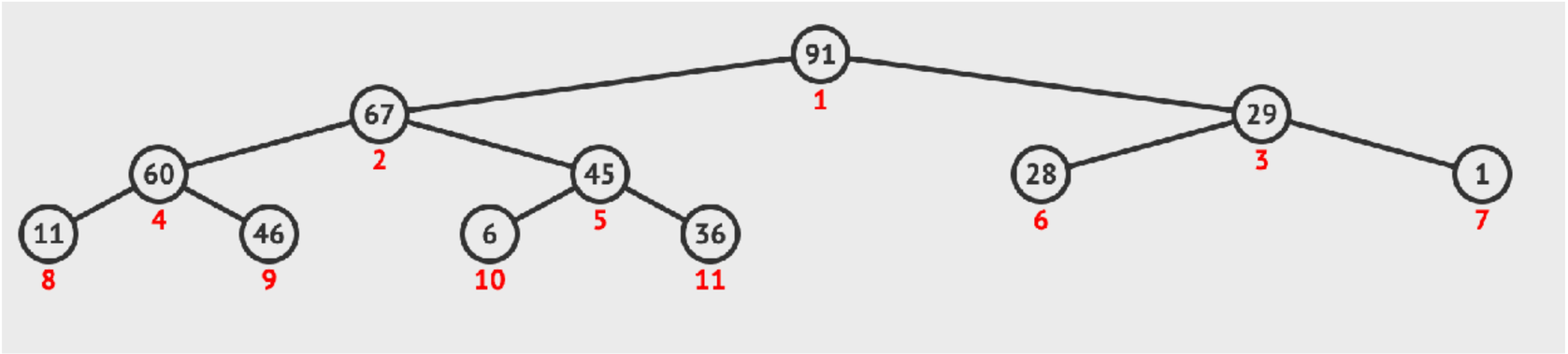


ANSWER 4: sortdown

- Extract max (93)

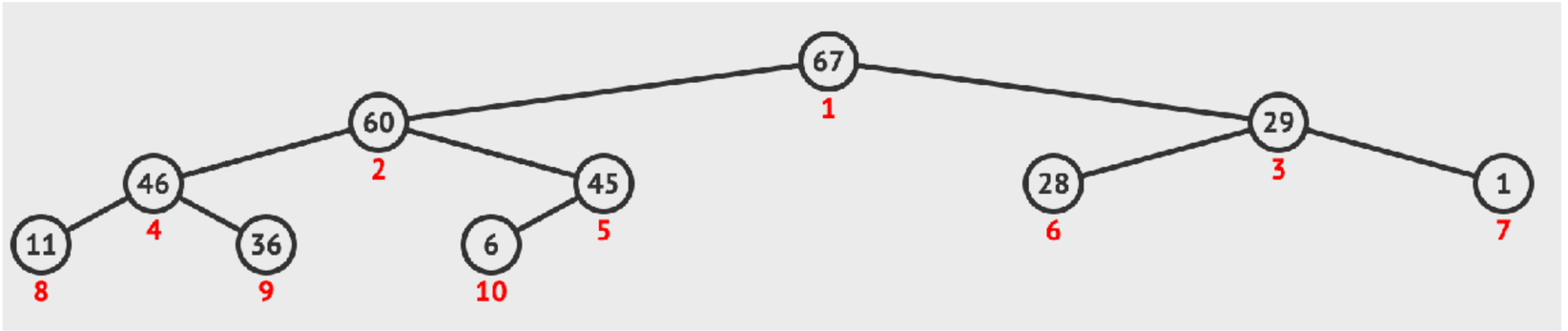


- Extract max (92)

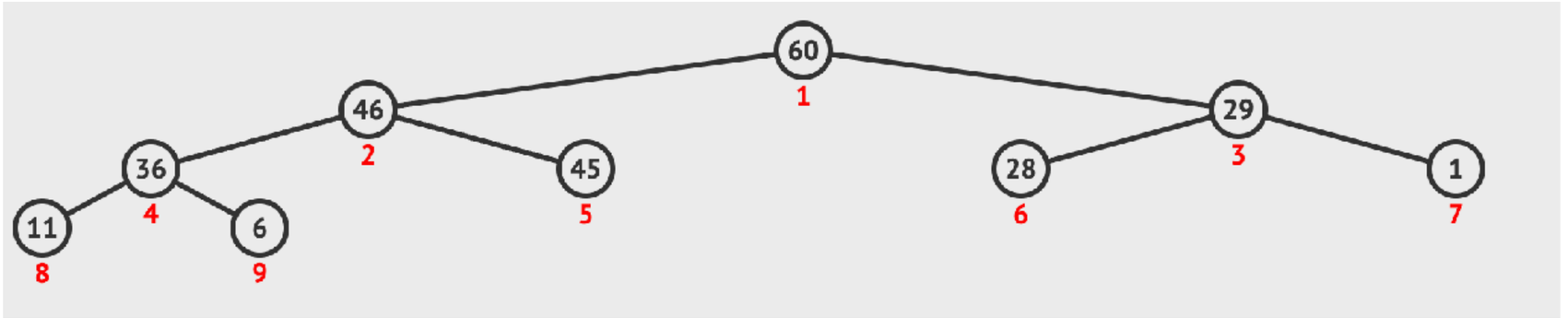


ANSWER 4

- Extract max (91)

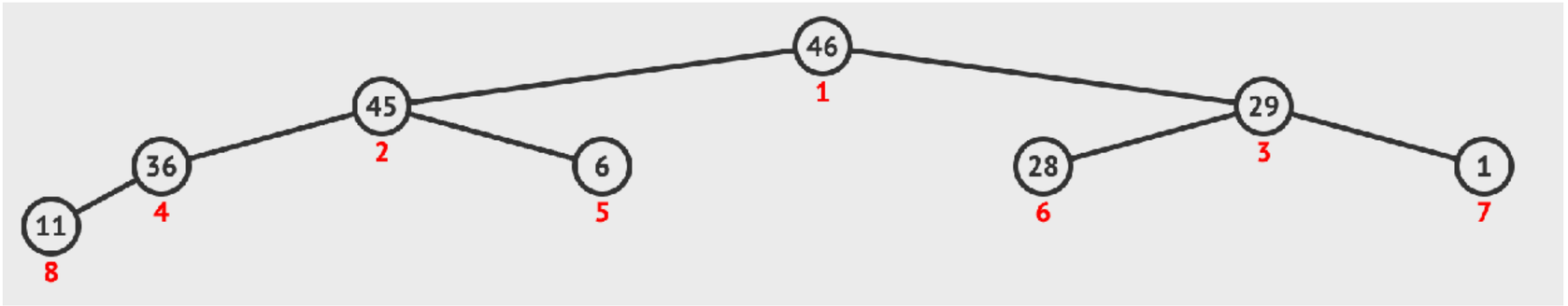


- Extract max (67)

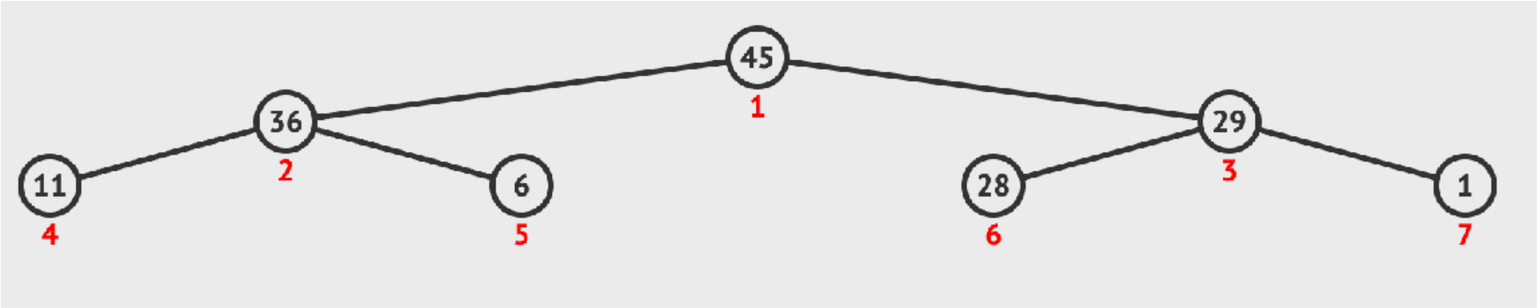


ANSWER 4

- Extract max (60)

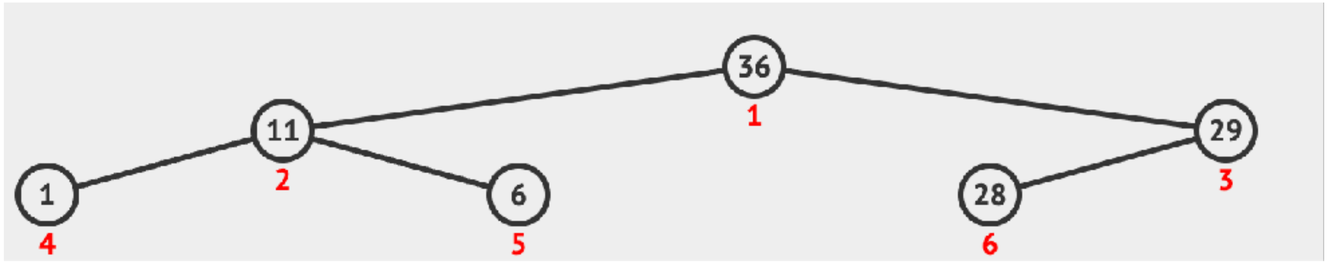


- Extract max (46)

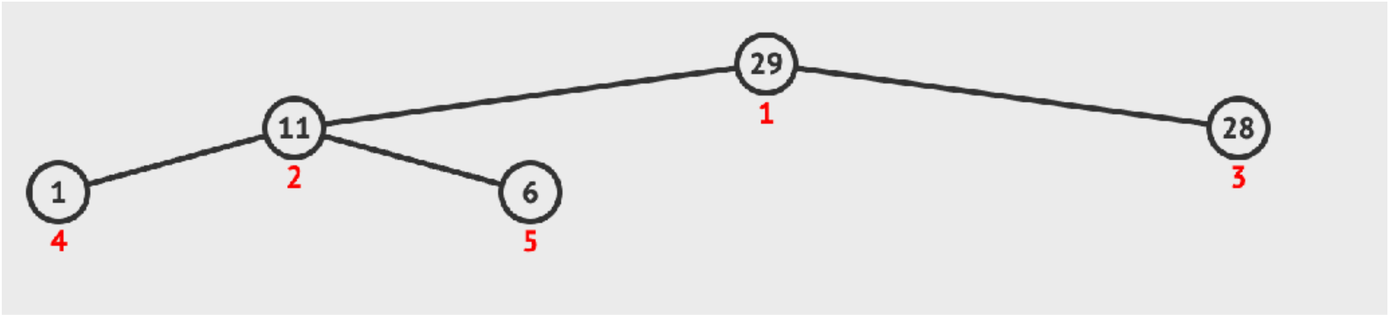


ANSWER 4

- Extract max (45)

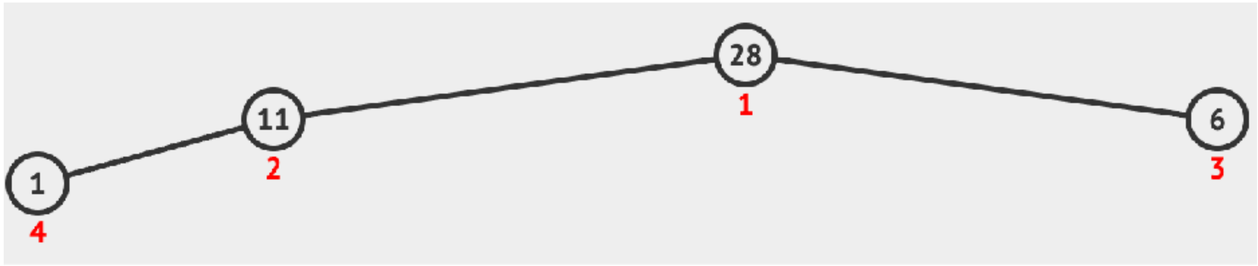


- Extract max (36)

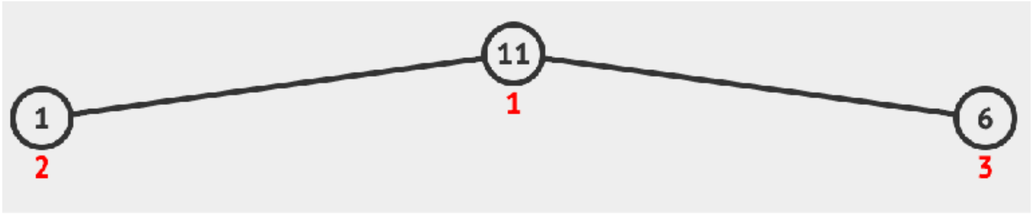


ANSWER 4

- Extract max (29)



- Extract max (28)



ANSWER 4

- Extract max (11)

- Extract max (6)

- Extract max (1)