CS062

DATA STRUCTURES AND ADVANCED PROGRAMMING

8: Analysis of Algorithms

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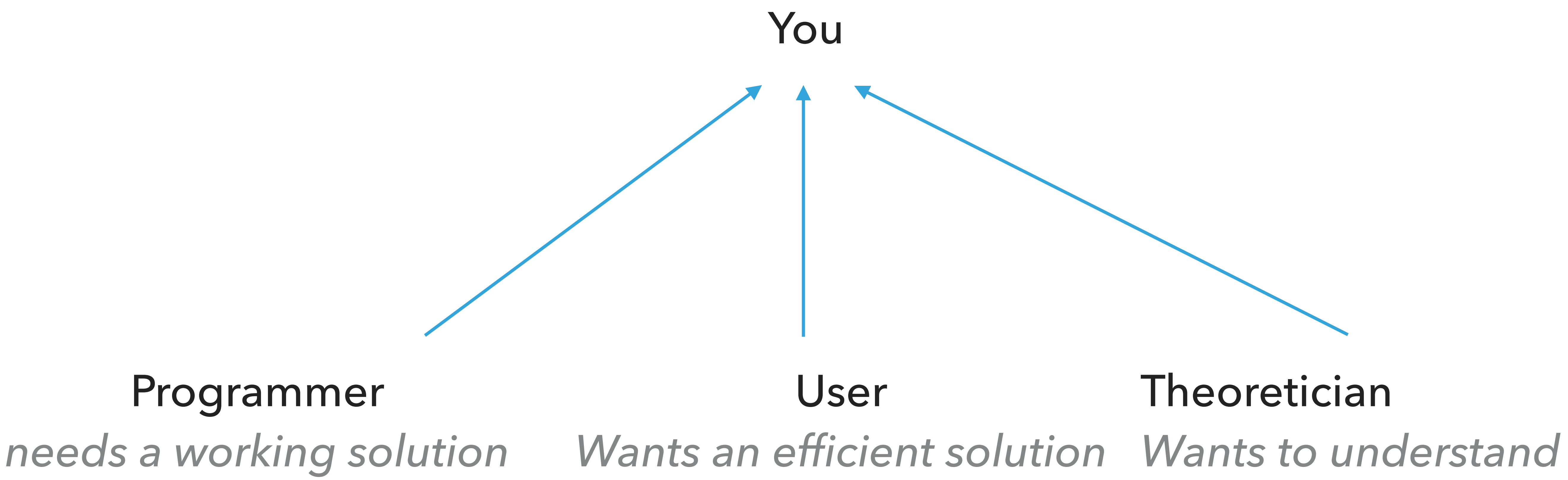
she/her/hers

Lecture 8: Analysis of Algorithms

- Experimental Analysis of Running Time
- Mathematical Models of Running Time
- Order of Growth Classification
- Analysis of ArrayList operations

Some slides adopted from Algorithms 4th Edition or COS226

Different Roles



You

Programmer

needs a working solution

User

Wants an efficient solution

Theoretician

Wants to understand

3-SUM: Given n distinct numbers, how many unordered triplets sum to 0?

- Input: 30 -40 -20 -10 40 0 10 5
- Output: 4
  - 30 -40 10
  - 30 -20 -10
  - -40 40 0
  - -10 0 10

3-SUM: Brute force algorithm

```
public class ThreeSum {

public static int count(int[] a) {
        int n = a.length;
        int count = 0;
        for (int i = 0; i < n; i++) {
            for (int j = i+1; j < n; j++) {
                for (int k = j+1; k < n; k++) {
                    if (a[i] + a[j] + a[k] == 0) {
                        count++;
                    }
                }
            }
        }
        return count;
    }
```

```
public static void main(String[] args)  {
        String filename = args[0];
        int fileSize = Integer.parseInt(args[1]);
        try {
            Scanner scanner = new Scanner(new File(filename));
            int intList[] = new int[fileSize];
            int i=0;
            while(scanner.hasNextInt()){
                intList[i]=scanner.nextInt();
                 i++;
            }
            Stopwatch timer = new Stopwatch();
            int count = count(intList);
            System.out.println("elapsed time = " + timer.elapsedTime());
            System.out.println(count);
        }
        catch (IOException e) {
            throw new IllegalArgumentException("Could not open " + filename, e);
        }
    }
```

Empirical Analysis

- Input: 8ints.txt
- Output: 4 and 0

- Input: 1Kints.txt
- Output: 70 and 0.081

- Input: 2Kints.txt
- Output: 528 and 0.38

- Input: 2Kints.txt
- Output: 528 and 0.371

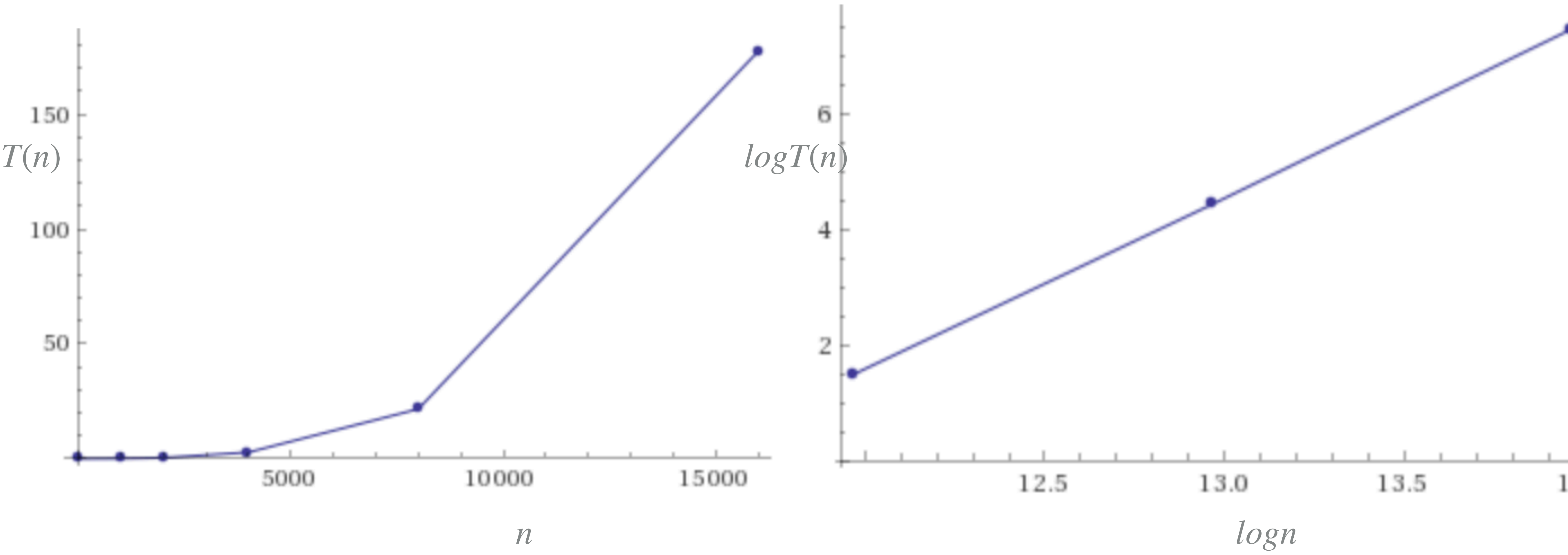
- Input: 4Kints.txt
- Output: 4039 and 2.792

- Input: 8Kints.txt
- Output: 32074 and 21.623

- Input: 16Kints.txt
- Output: 255181 and 177.344

Input size	Time
8	0
1000	0.081
2000	0.38
2000	0.371
4000	2.792
8000	21.623
16000	177.344

Plots and log-log plots



Straight line of slope 3

- Regression: $T(n) = an^b$ (power-law).
- $\log T(n) = b \log n + \log a$, where b is slope.
- Experimentally: ~$0.42 \times 10^{-10} n^3$, in our example for `ThreeSum`.

Input size	Time
8	0
1000	0.081
2000	0.38
4000	2.792
8000	21.623
16000	177.344

Doubling Hypothesis

- Doubling input size increases running time by a factor of $\dfrac{T(n)}{T(n/2)}$
- Run program doubling the size of input. Estimate factor of growth:
  - $\dfrac{T(n)}{T(n/2)} = \dfrac{an^b}{a(\frac{n}{2})^b} = 2^b.$
- E.g., in our example, for pair of input sizes 8000 and 16000 the ratio $(\dfrac{177.344}{21.623})$ is 8.2 or ~8 which can be written as 2^3, therefore b is approximately 3.
- Assuming we know b, we can figure out a.
  - E.g., in our example, $T(16000) = 177.34 = a \times 16000^3$.
    - Solving for a we get $a = 0.42 \times 10^{-10}$.

PRACTICE TIME

- Suppose you time your code and you make the following observations. Which function is the closest model of $T(n)$?

A. n^2

B. $6 \times 10^{-4} n$

C. $5 \times 10^{-9} n^2$

D. $7 \times 10^{-9} n^2$

Input size	Time
1000	0
2000	0.0
4000	0.1
8000	0.3
16000	1.3
32000	5.1

ANSWER

- C. $5 \times 10^{-9} n^2$
- T(32000)/T(16000) is approximately 4, therefore $b = 2$.
- $T(32000) = 5.1 = a \times 32000^2$.
- Solving for $a = 4.98 \times 10^{-9}$.s

Input size	Time
1000	0
2000	0.0
4000	0.1
8000	0.3
16000	1.3
32000	5.1

Effects on Performance

- System independent effects: Algorithm + input data
  - Determine b in power law relationships.
- System dependent effects: Hardware (e.g., CPU, memory, cache) + Software (e.g., compiler, garbage collector) + System (E.g., operating system, network, etc).
  - Dependent and independent effects determine a in power law relationships.

- Although it is hard to get precise measurements, experiments in Computer Science are cheap to run.

Lecture 8: Analysis of Algorithms

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Total Running Time

- Popularized by Donald Knuth in the 60s in the four volumes of “The Art of Computer Programming”.
  - Knuth won the Turing Award (The “Nobel” in CS) in 1974.

- In principle, accurate mathematical models for performance of algorithms are available.

- Total running time = sum of cost x frequency for all operations.
- Need to analyze program to determine set of operations.
- Exact cost depends on machine, compiler.
- Frequency depends on algorithm and input data.

Cost of Basic Operations

- Add < integer multiply < integer divide < floating-point add < floating-point multiply < floating-point divide.

Operation	Example	Nanoseconds
Variable declaration	`int a`	c_1
Assignment statement	`a = b`	c_2
Integer comparison	`a < b`	c_3
Array element access	`a[i]`	c_4
Array length	`a.length`	c_5
1D array allocation	`new int[n]`	$c_6 n$
2D array allocation	`new int[n][n]`	$c_7 n^2$
string concatenation	`s+t`	$c_8 n$

Example:1-SUM

- How many operations as a function of n?

```
int count = 0;
 for (int i = 0; i < n; i++) {
     if (a[i] == 0) {
         count++;
     }
 }
```

Operation	Frequency
Variable declaration	2
Assignment	2
Less than	$n+1$
Equal to	n
Array access	n
Increment	n to $2n$

Example: 2-SUM

$$1 + 2 + 3 + \ldots + n = n(n+1)/2$$

- How many operations as a function of n?

```
int count = 0;
 for (int i = 0; i < n; i++) {
     for (int j = i+1; j < n; j++) {
         if (a[i] + a[j] == 0) {
             count++;
         }
     }
 }
```

BECOMING TOO TEDIOUS TO CALCULATE

Operation	Frequency
Variable declaration	$n + 2$
Assignment	$n + 2$
Less than	$(n+1)(n+2)/2$
Equal to	$n(n-1)/2$
Array access	$n(n-1)$
Increment	$n(n+1)/2$ to n^2

Tilde Notation

- Estimate running time (or memory) as a function of input size n.
- Ignore lower order terms.
  - When n is large, lower order terms become negligible.

- Example 1: $\frac{1}{6}n^3 + 10n + 100 \quad \sim n^3$

- Example 2: $\frac{1}{6}n^3 + 100n^2 + 47 \quad \sim n^3$

- Example 3: $\frac{1}{6}n^3 + 100n^{\frac{2}{3}} + \frac{1/2}{n} \quad \sim n^3$

Simplification

- Cost model: Use some basic operation as proxy for running time. E.g., array accesses
- Combine it with tilde notation.

Operation	Frequency	Tilde notation
Variable declaration	$n+2$	$\sim n$
Assignment	$n+2$	$\sim n$
Less than	$(n+1)(n+2)/2$	$\sim n^2$
Equal to	$n(n-1)/2$	$\sim n^2$
Array access	$n(n-1)$	$\sim n^2$
Increment	$n(n+1)/2$ to n^2	$\sim n^2$

- $\sim n^2$ array accesses for the 2-SUM problem

Back to the 3-SUM problem

- Approximately how many array accesses as a function of input size n?

```
int count = 0;
for (int i = 0; i < n; i++) {
    for (int j = i+1; j < n; j++) {
        for (int k = j+1; k < n; k++) {
            if (a[i] + a[j] + a[k] == 0) {
                count++;
            }
        }
    }
}
```

- $\sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n-1} 3 = 1/2n(n^2 - 3n + 2) \sim n^3$ array accesses.

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Types of analysis

- Best case: lower bound on cost.
    - What the goal of all inputs should be.
    - Often not realistic, only applies to “easiest” input.
- Worst case: upper bound on cost.
    - Guarantee on all inputs.
    - Calculated based on the “hardest” input.
- Average case: expected cost for random input.
    - A way to predict performance.
    - Not straightforward how we model random input.

Worst case analysis

- Definition: If $f(n) \sim cg(n)$ for some constant $c > 0$, then the order of growth of $f(n)$ is $g(n)$.
  - Ignore leading coefficients.
  - Ignore lower-order terms.

- We will be using the big-Oh (O) notation. For example:
  - $3n^3 + 2n + 7 = O(n^3)$
  - $2^n + n^2 = O(2^n)$
  - $1000 = O(1)$

- Yes, $3n^3 + 2n + 7 = O(n^6)$, but that's a rather useless bound.

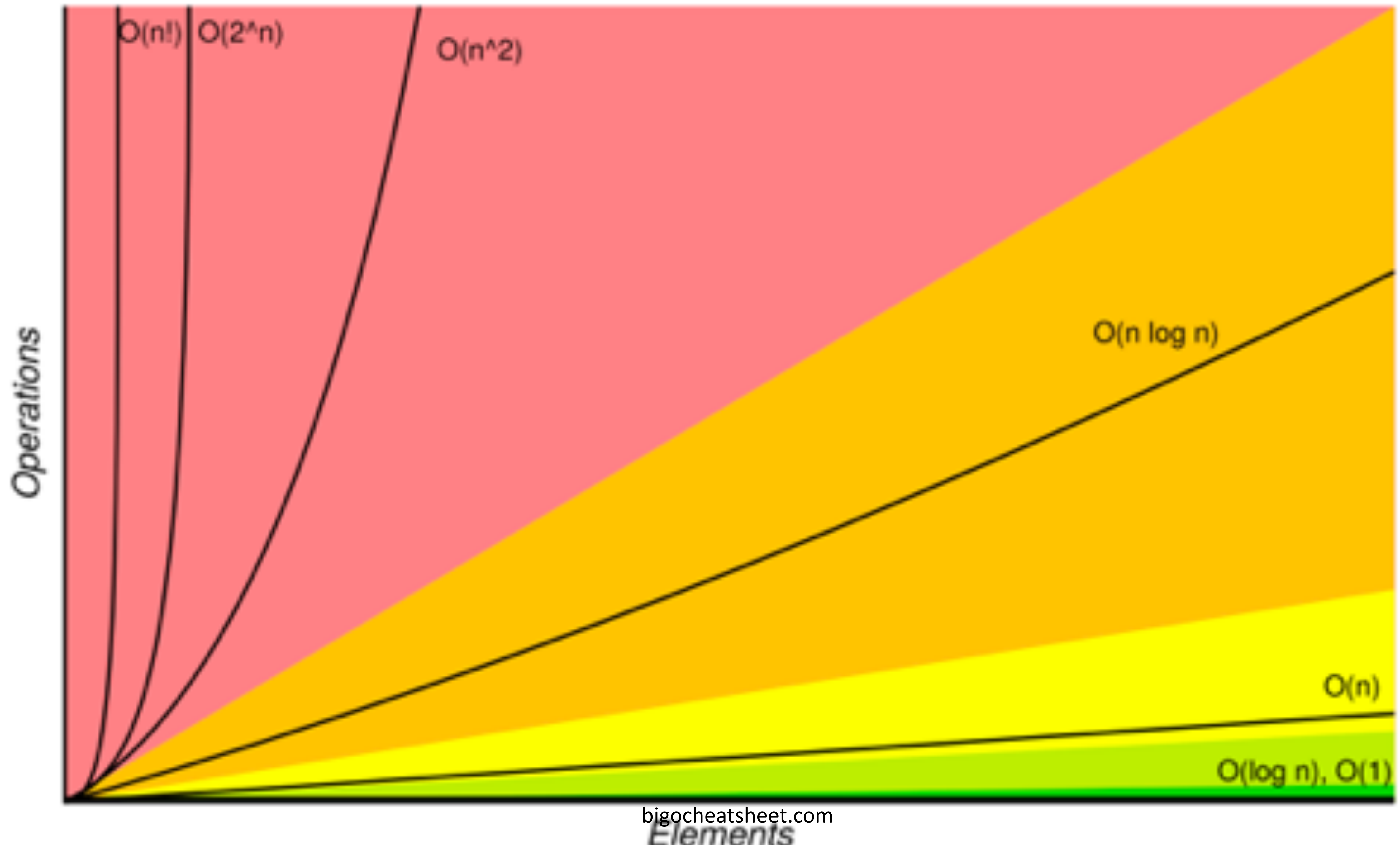
Common order of growth classifications

- Good news: only a small number of function suffice to describe the order-of-growth of typical algorithms.
- 1: constant
  - Doubling the input size won't affect the running time. Holy-grail.
- $\log n$: logarithmic
  - Doubling the input size will increase the running time by a constant.
- n : linear
  - Doubling the input size will result to double the running time.
- $n \log n$: linearithmic
  - Doubling the input size will result to a bit longer than double the running time.
- n^2: quadratic
  - Doubling the input size will result to four times as much running time.
- n^3: cubic
  - Doubling the input size will result to eight times as much running time.
- 2^n: exponential
  - When you increase the input by some constant amount, the time taken is doubled.
- $n!$: factorial
  - Running time grows exponentially with the size of the input.

ORDER OF GROWTH CLASSIFICATION

From slowest growing to fastest growing

- $1 < \log n < n < n \log n < n^2 < n^3 < 2^n < n!$



bigocheatsheet.com

Common order of growth classifications

Order-of-growth	**Name**	**Example code**	$T(n)/T(n/2)$
1	Constant	`a[i]=b+c`	1
$\log n$	Logarithmic	`while(n>1){n=n/2;…}`	~ 1
n	Linear	`for(int i=0; i<n; i++)`	2

$n \log n$	Linearithmic	`for (i = 1; i <= n; i++){`
		`int x = n;`
		`while (x > 0)`
		`x -= i;`
		`}`
			~ 2

n^2	Quadratic	`for(int i=0; i<n; i++) {` `for(int j=0; j<n; j++){`	4

n^3	Cubic	`for(int i=0; i<n; i++) {`
		`for(int j=0; j<n; j++){`
		`for(int k=0; k<n; k++){`
			8

Useful approximations

- Harmonic sum: $H_n = 1 + 1/2 + 1/3 + \ldots + 1/n \quad \sim \ln n$
- Triangular sum: $1 + 2 + 3 + \ldots + n \quad \sim n^2$
- Geometric sum: $1 + 2 + 4 + 8 + \ldots + n = 2n - 1 \quad \sim n$, when n power of 2.
- Binomial coefficients: $\binom{n}{k} \quad \sim \dfrac{n^k}{k!}$ when k is a small constant.

- Use a tool like Wolfram alpha.

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Worst-case performance of `add()` is $O(n)$

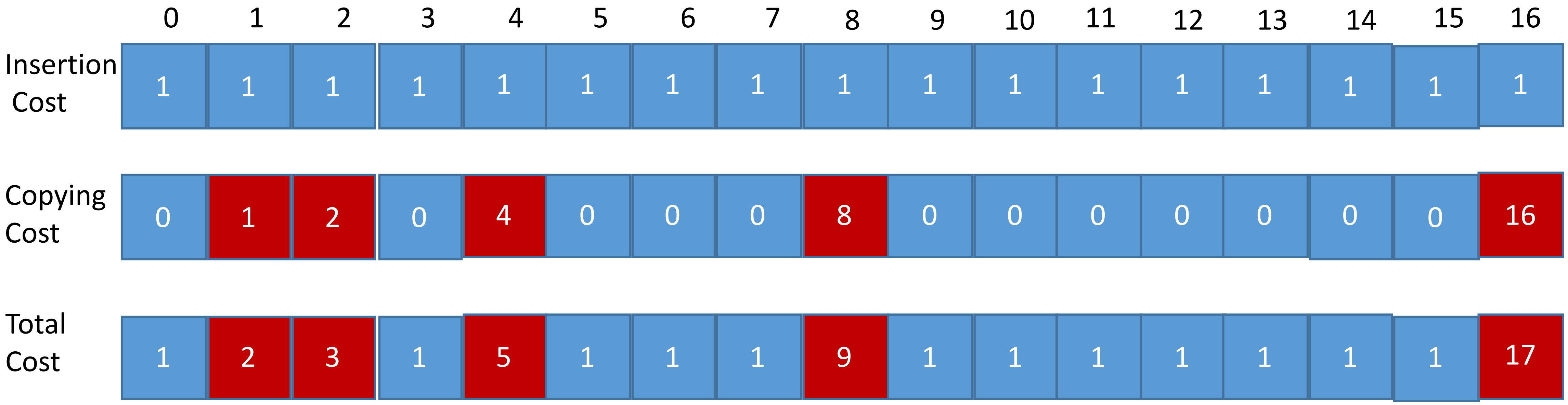
- Cost model: 1 for insertion, n for copying n items to a new array.
- Worst-case: If ArrayList is full, `add()` will need to call `resize` to create a new array of double the size, copy all items, insert new one.
- Total cost: $n + 1 = O(n)$.

- Realistically, this won't be happening often and worst-case analysis can be too strict. We will use amortized time analysis instead.

Amortized analysis

- Amortized cost per operation: for a sequence of n operations, it is the total cost of operations divided by n.

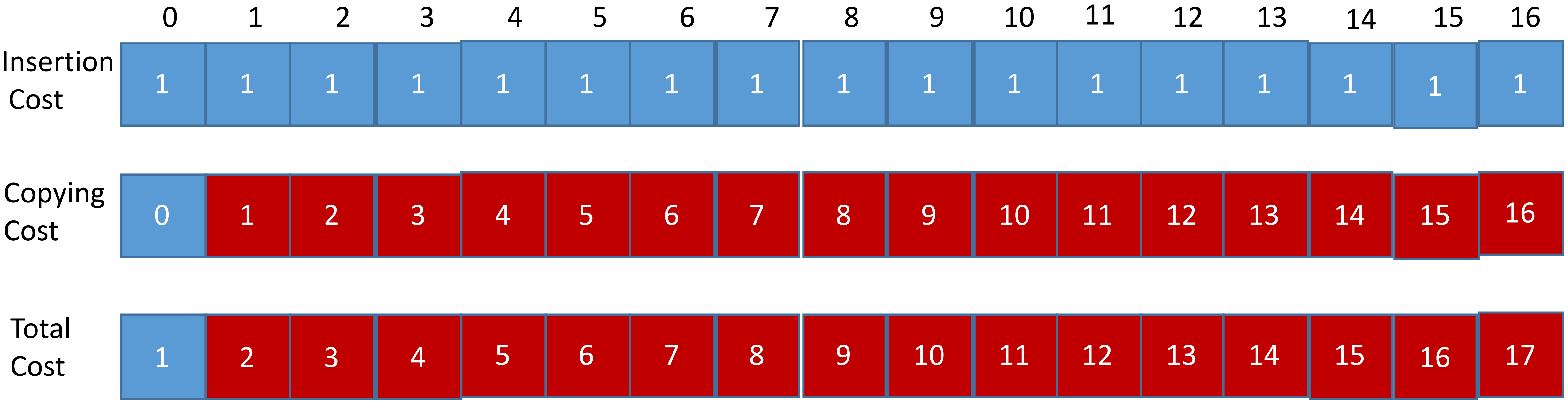
Amortized analysis for n add() operations



	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Insertion Cost	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Copying Cost	0	1	2	0	4	0	0	0	8	0	0	0	0	0	0	0	16
Total Cost	1	2	3	1	5	1	1	1	9	1	1	1	1	1	1	1	17

- As the ArrayList increases, doubling happens *half as often* but *costs twice as much.*
- O(total cost)= $\sum$(“cost of insertions”) + $\sum$(“cost of copying”)
- $\sum$(“cost of insertions”) $= n$.
- $\sum$(“cost of copying”) = $1 + 2 + 2^2 + \dots 2^{\lfloor \log_2 n \rfloor} \leq 2n$.
- O(total cost) $\leq 3n$, therefore amortized cost is $\leq \dfrac{3n}{n} = 3 = O^+(1)$, but “lumpy”.

Amortized analysis for n add() operations when increasing ArrayList by 1.



	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Insertion Cost	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Copying Cost	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Total Cost	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

- $\sum$(“cost of insertions”) $= n$.
- $\sum$(“cost of copying”) = $1 + 2 + 3 + \ldots + n - 1 = n(n-1)/2$.
- O(total cost) $= n + n(n-1)/2 = n(n+1)/2$, therefore amortized cost is $(n+1)/2$ or $O^{+}(n)$.
- Same idea when increasing ArrayList size by a constant.
  - This is why in the lab on Friday, we saw that doubling was the fastest and linear(1) the slowest.

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Readings:

- Recommended Textbook:
  - Chapter 1.4 (pages 172-205)
- Recommended Textbook Website:
  - Resizable arrays (arraylists): https://algs4.cs.princeton.edu/13stacks/
  - Analysis of Algorithms: https://algs4.cs.princeton.edu/14analysis/

Code

- Lecture 8 code

Practice Problems:

- 1.4.1-1.4.9, 1.4.32, 1.4.35-1.4.36