GRAPHS

CS062

DATA STRUCTURES AND ADVANCED PROGRAMMING

24: Minimum Spanning Trees

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she/her/hers

Last lecture for the semester and we are ready to finish our unit on graphs!

Lecture 24: Minimum Spanning Trees

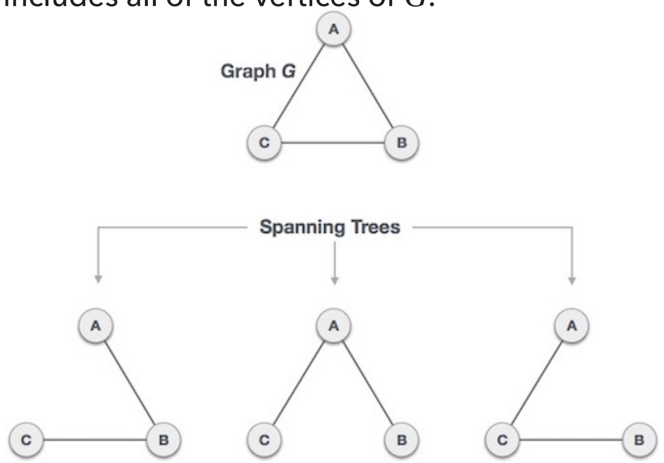
- Introduction
- Kruskal's Algorithm
- Prim's Algorithm

Some slides adopted from Algorithms 4th Edition or COS226

Today, we will talk about minimum spanning trees and will see two famous algorithms to calculate them.

Spanning Trees

- Given an edge weighted graph G (not digraph!), a spanning tree of G is a subgraph T that is:
  - A tree: connected and acyclic.
  - Spanning: includes all of the vertices of G.



https://www.tutorialspoint.com/data_structures_algorithms/spanning_tree.htm

Given an edge weighted graph (not a digraph), a spanning tree will be a subgraph that is a tree (that is connected and acyclic), and spanning (that is it includes all the vertices of the original graph). For example, on the graph above, you can see three possible spanning trees. They all are subgraphs that contain all three vertices, and form a connected and acyclic graph.

Properties

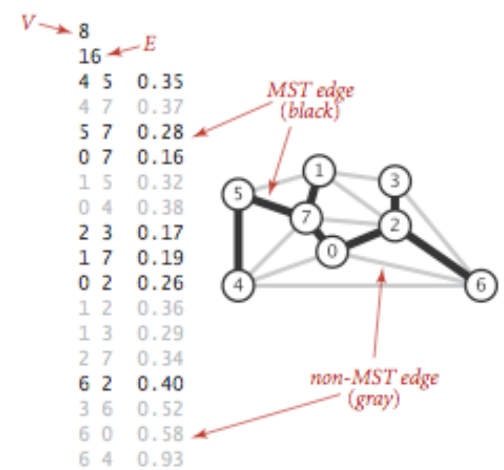
- A connected graph G can have more than one spanning tree.
- All possible spanning trees of G have the same number of vertices and edges.
- A spanning tree has $|V| - 1$ edges.
- A spanning tree by definition cannot have any cycle.
- Adding one edge to the spanning tree would create a cycle (i.e. spanning trees are maximally acyclic).
- Removing one edge from the spanning tree would make the graph disconnected (i.e. spanning trees are minimally connected).

https://www.tutorialspoint.com/data_structures_algorithms/spanning_tree.htm

Let's talk about some graph properties in relation to spanning trees. As we already saw, a graph can have more than one spanning tree and all spanning trees of a graph have the same number of vertices and edges. A spanning tree has n-1 edges, where n is the number of nodes (vertices). By definition, it cannot have a cycle. And adding an edge to it would create a cycle (that is known as maximally acyclic). Also, removing one edge from the spanning tree would disconnect it (that is known as minimally connected).

Minimum spanning tree (MST) problem

- Given a connected edge-weighted undirected graph find a spanning tree of minimum weight.



An edge-weighted graph and its MST

The minimum spanning tree problem states that given a connected edge-weighted undirected graph, we have to find a spanning tree of minimum weight, e.g., as in the example above.

Minimum spanning tree applications

- Network design
- Cluster analysis
- Cancer imaging
- Many others
  - https://www.ics.uci.edu/~eppstein/gina/mst.html
  - https://personal.utdallas.edu/~besp/teaching/mst-applications.pdf

There are a lot of scenarios that finding a minimum spanning tree is useful and I have a couple of links here but in general it makes sense when you want fully connected system at the lowest possible cost.

Lecture 24: Minimum Spanning Trees

- Introduction
- Kruskal's Algorithm
- Prim's Algorithm

Let's see how to calculate MSTs with our first such algorithm, Kruskal's.

Kruskal's algorithm

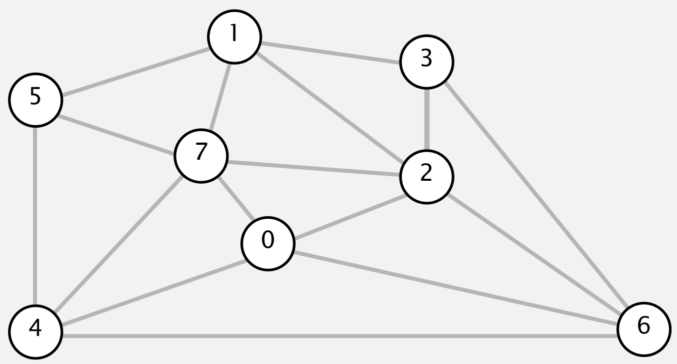
- Sort edges in ascending order of weight.
- Starting from the one with the smallest weight, add it to the MST unless doing so would create a cycle.
- Running time of $|E|\log|V|$ in worst case.
- Uses union-find, a data structure we haven't covered.

It's a very simple algorithm. You sort edges in ascending order of weight. you star from then one with the smallest weight, add it to the MST unless doing so would create a cycle. That can be done in ElogV time in worst case. Kruskal's algorithm uses union-find, a data structure we haven't covered.

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



an edge-weighted graph

graph edges sorted by weight

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

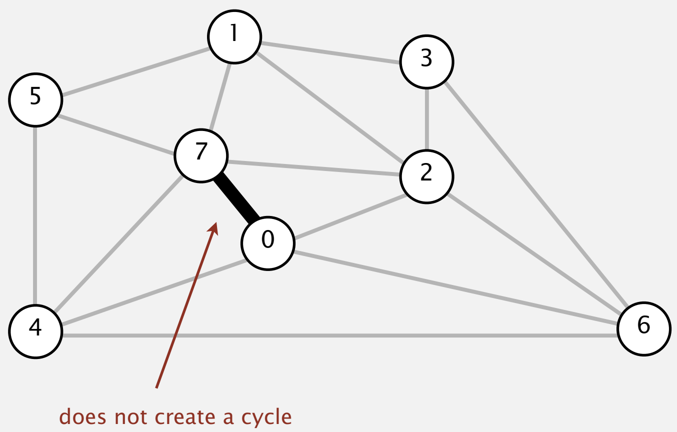
Let's assume we have this edge-weighted graph and the weights on the right that are in ascending order of weight.

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



in MST ⟶ 0-7 0.16

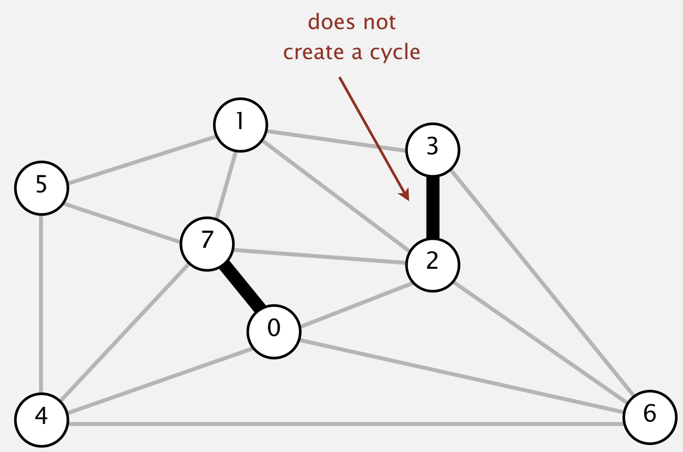
We start at 0-7 and add the edge to the MST.

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



in MST →

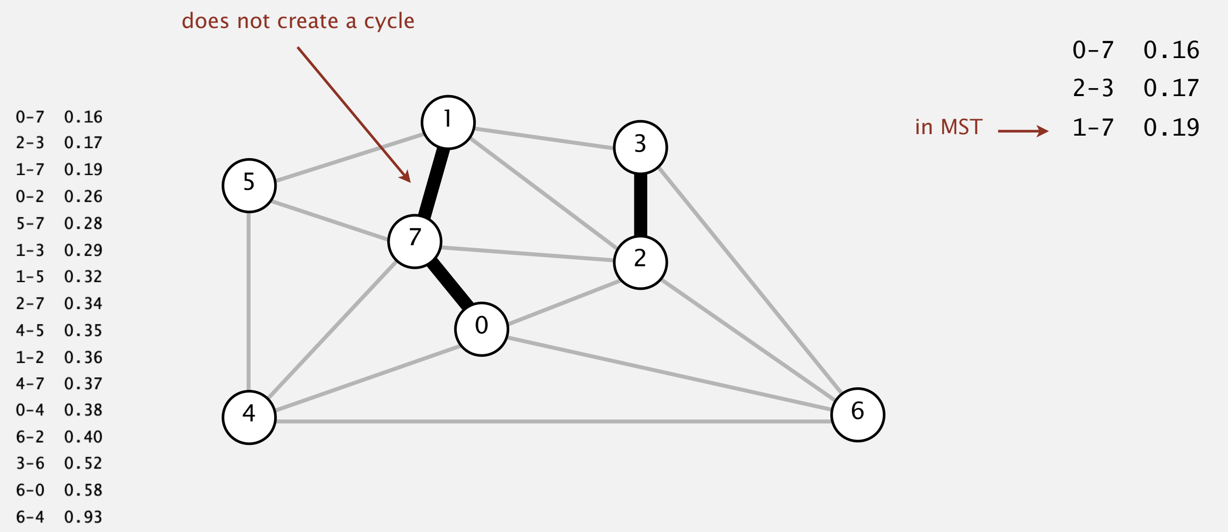
0-7	0.16
2-3	0.17

next comes 2-3 which does not create a cycle so it gets added to the MST.

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



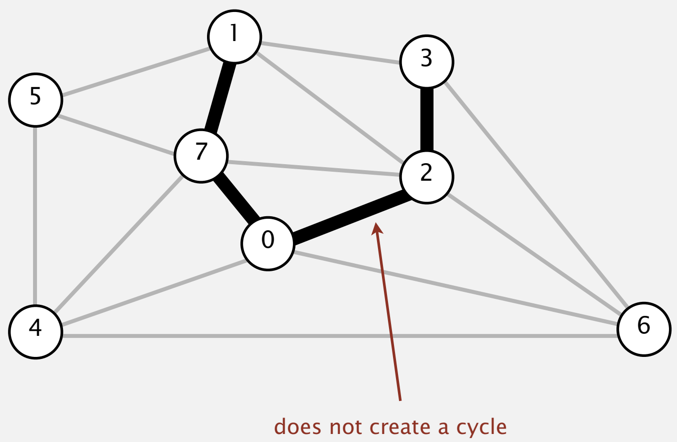
Same for 1-7

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



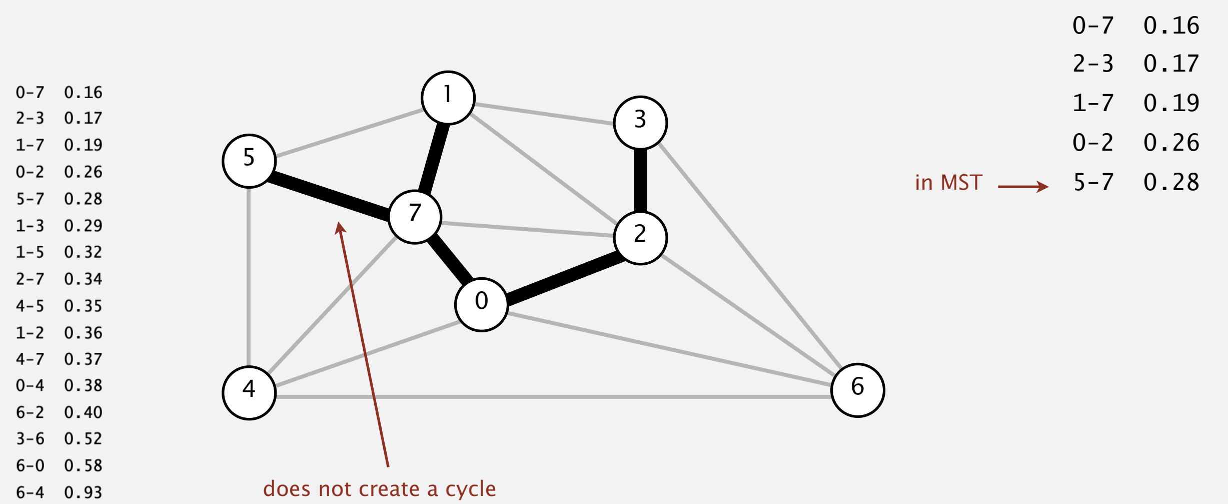
0-7	0.16
2-3	0.17
1-7	0.19
in MST → 0-2	0.26

and 0-2

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



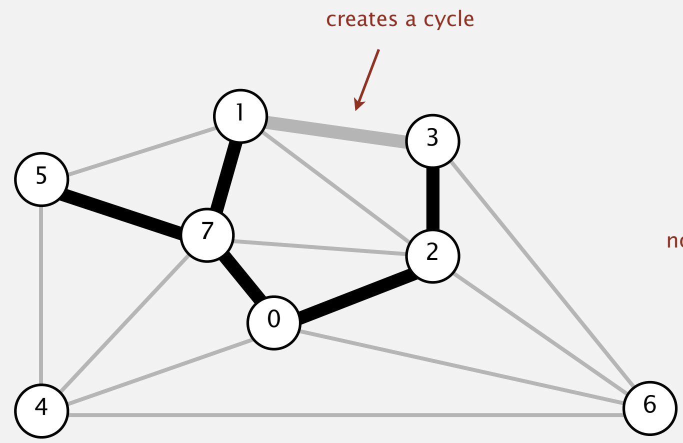
and 5-7

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



	0-7	0.16
	2-3	0.17
	1-7	0.19
	0-2	0.26
	5-7	0.28
not in MST →	1-3	0.29

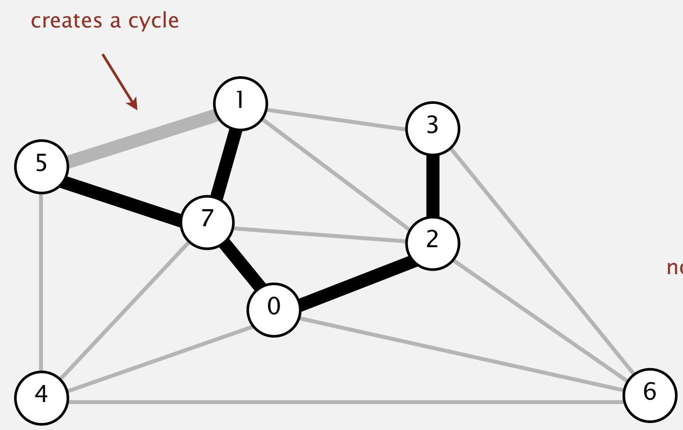
But not 1-3 since it would create a cycle.

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32

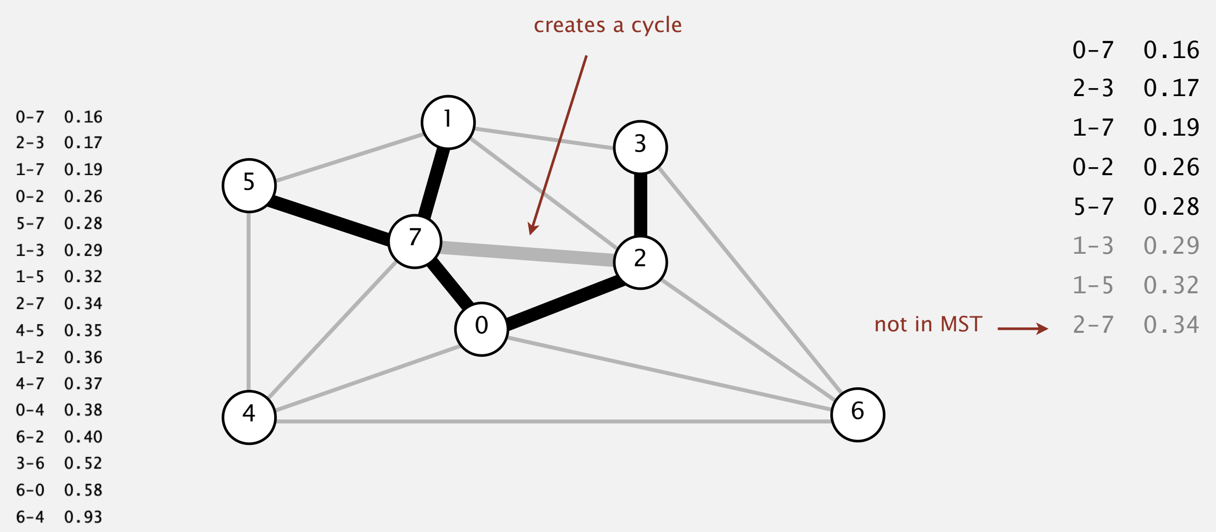
not in MST

Nor 1-5

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



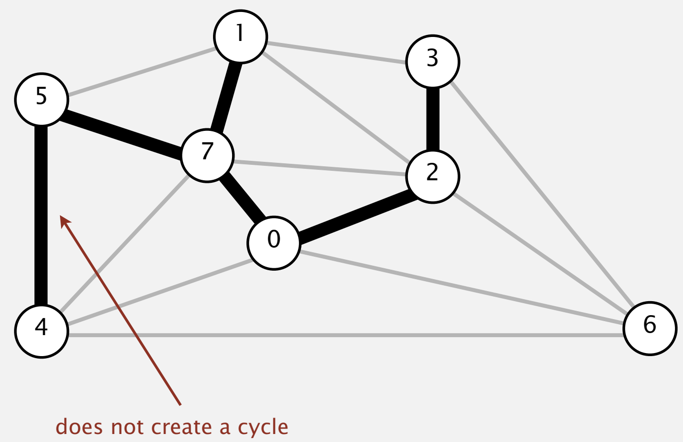
Nor 2-7

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
in MST → 4-5	0.35

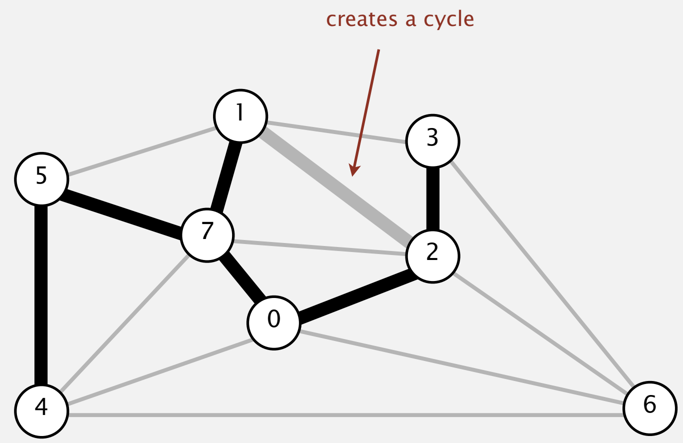
But 4-5 would be OK

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36

not in MST →

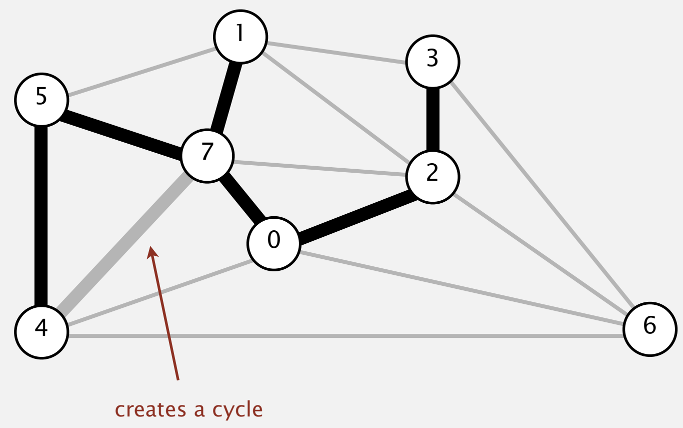
But not 1-2

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37

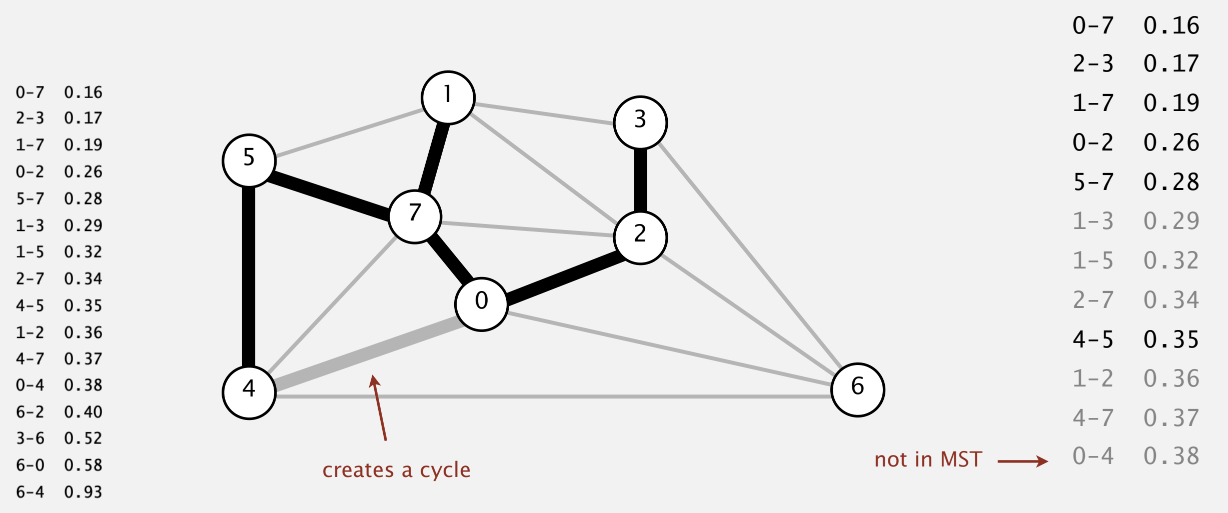
not in MST → 4-7 0.37

nor 4-7

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



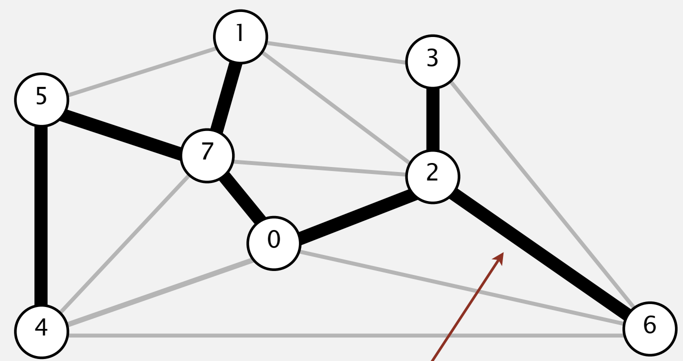
Nor 0-4

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



does not create a cycle

	0-7	0.16
	2-3	0.17
	1-7	0.19
	0-2	0.26
	5-7	0.28
	1-3	0.29
	1-5	0.32
	2-7	0.34
	4-5	0.35
	1-2	0.36
	4-7	0.37
	0-4	0.38
in MST →	6-2	0.40

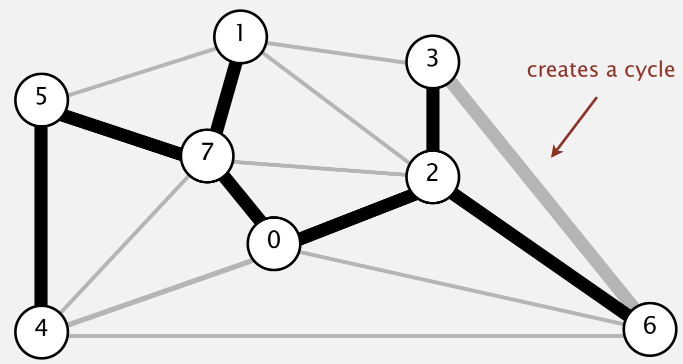
We can add 6-2

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



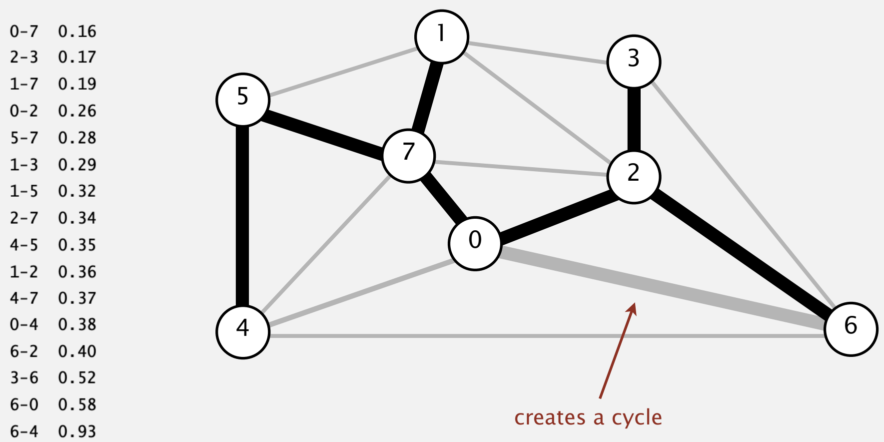
	0-7	0.16
	2-3	0.17
	1-7	0.19
	0-2	0.26
	5-7	0.28
	1-3	0.29
	1-5	0.32
	2-7	0.34
	4-5	0.35
	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-2	0.40
not in MST →	3-6	0.52

The rest all would create cycles

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



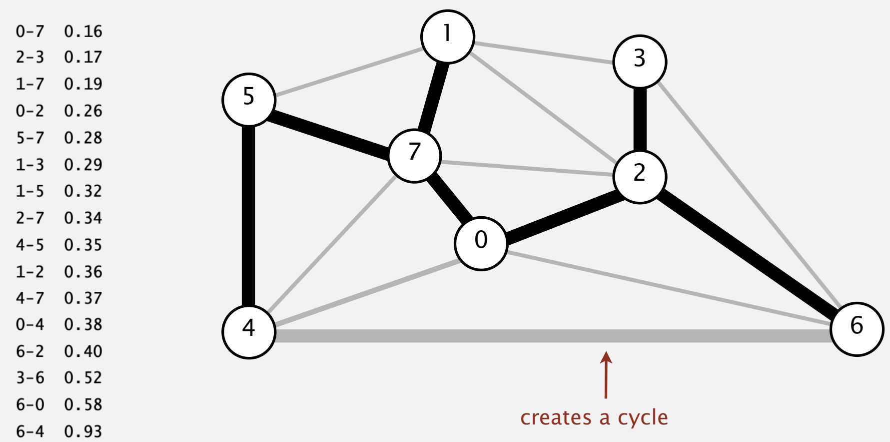
0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

not in MST →

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

not in MST →

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
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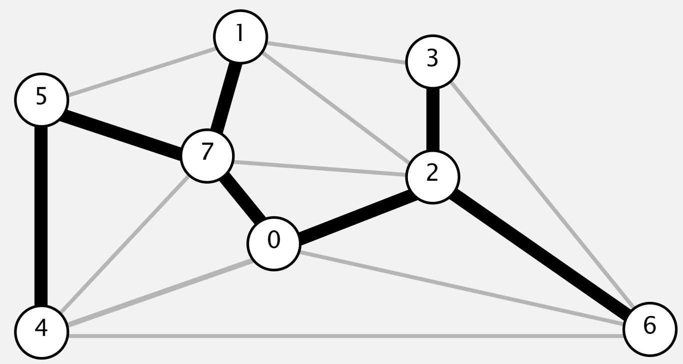
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

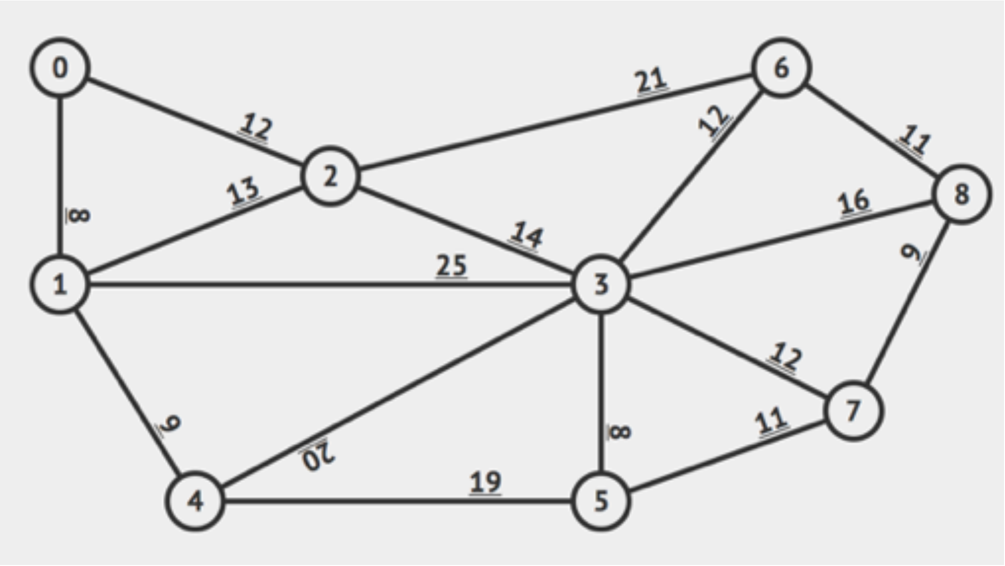


a minimum spanning tree

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

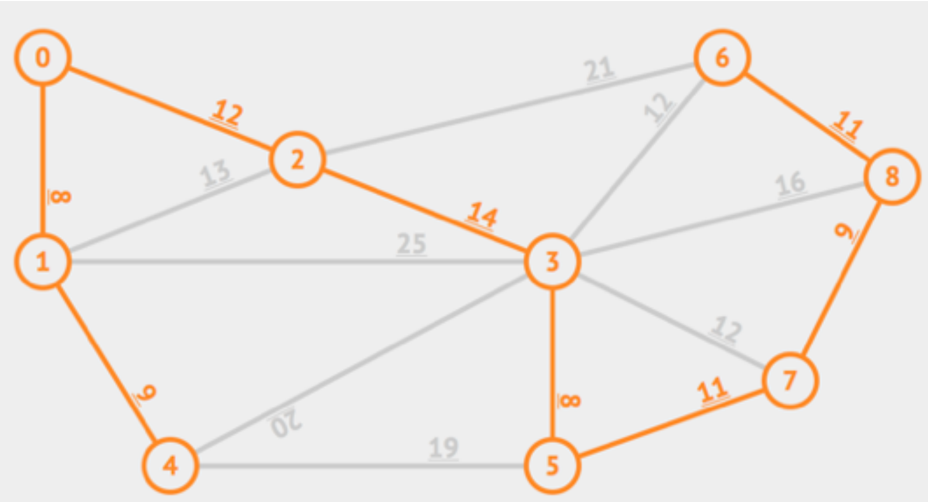
Tada! we have a minimum spanning tree!

Practice Time



Try to run Kruskal's algorithm on this graph.

Answer



This is the MST you should have gotten.

Lecture 24: Minimum Spanning Trees

- Introduction
- Kruskal's Algorithm
- Prim's Algorithm

Let's see how Prim's algorithm calculates the MST.

Prim's algorithm

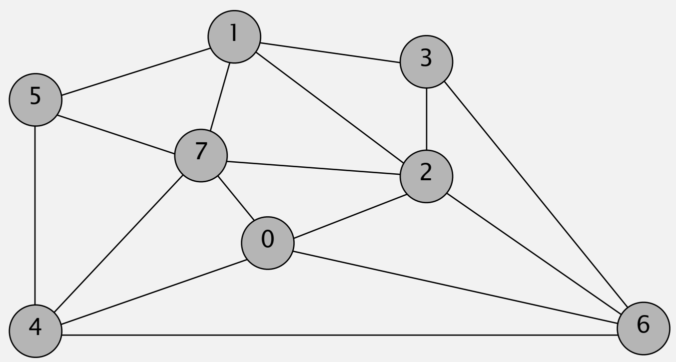
- Start with a random vertex (here, 0) and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $|V|-1$ edges.

- Two versions, lazy and eager. We will see lazy, here…
- Uses min-priority queue.
- Running time of $|E|\log|V|$ in worst case, as well.

We will start with a random vertex say 0, and greetingly grow the MST by adding the min weight edge with exactly one endpoint in the MST and repeat it n-1 times. There are two versions, a lazy and eager and we will see the lazy approach here. The algorithm uses a min-priority queue and its running time is also E log V.

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



an edge-weighted graph

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

Let's see the algorithm on the graph above.

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
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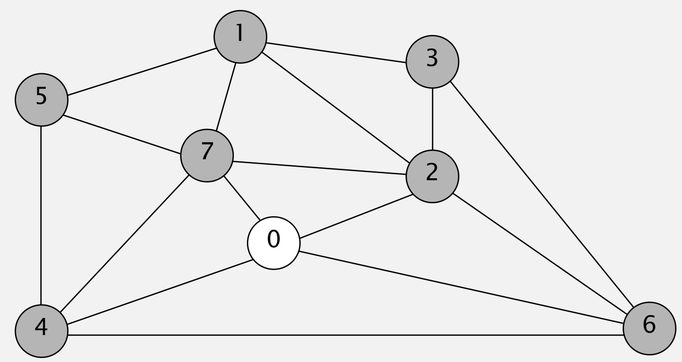
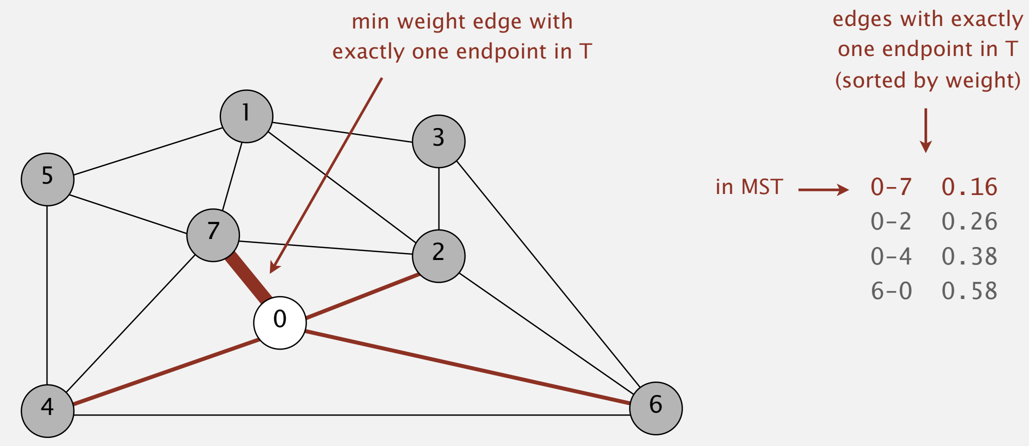
6-0	0.58
6-4	0.93



We'll start with vertex 0

Prim's algorithm demo

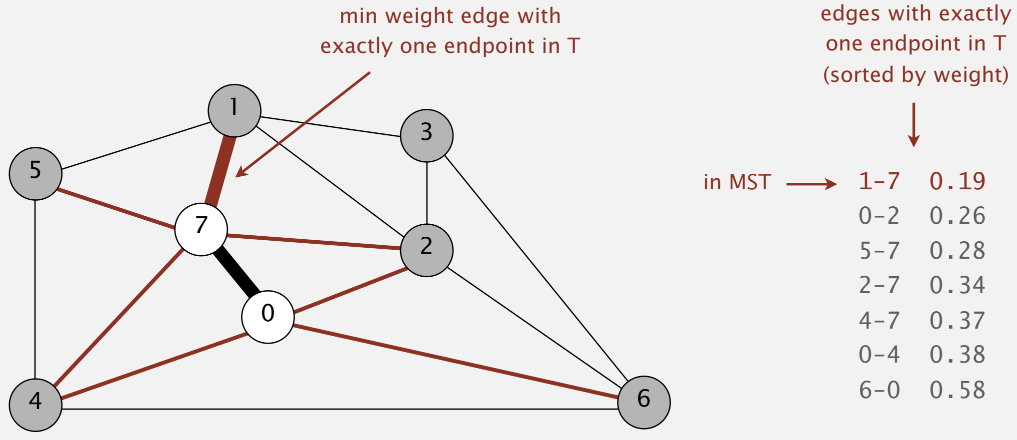
- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



and consider all the edges that have an endpoint to our MST which right now only contains 0. That would be edges 0-2, 0-4, 6-0, and 0-7. The one with minimum weight is edge 0-7 so we will add it to the MST which now includes edge 0-7.

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



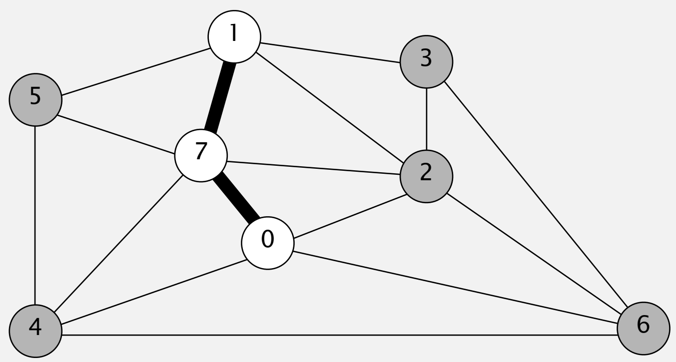
MST edges

0-7

The edges with one endpoint in T would be 1-7, 0-2, 5-7, 2-7, 4-7, 0-4, and 6-0. The minimum is 1-7

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



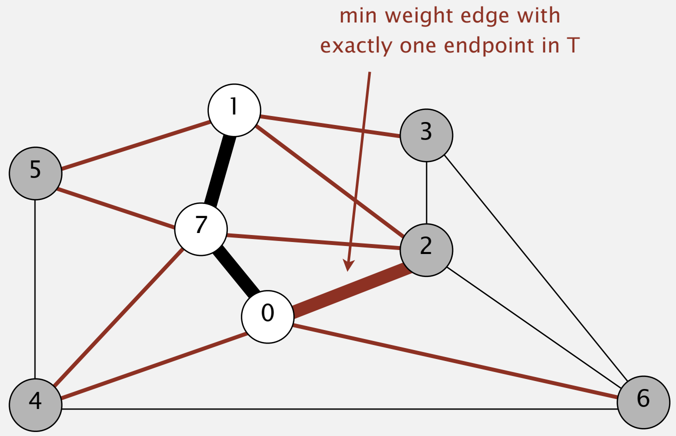
MST edges

0-7 1-7

Now MST contains the edges 0-7 and 1-7

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



edges with exactly one endpoint in T (sorted by weight)

in MST →	0-2	0.26
	5-7	0.28
	1-3	0.29
	1-5	0.32
	2-7	0.34
	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-0	0.58

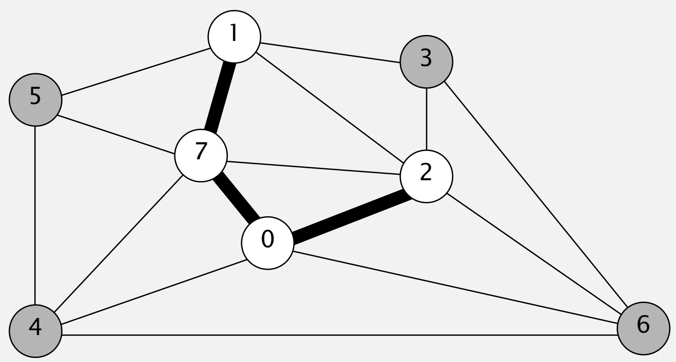
MST edges

0-7 1-7

We proceed similarly with the rest which adds the 0-2 edge.

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



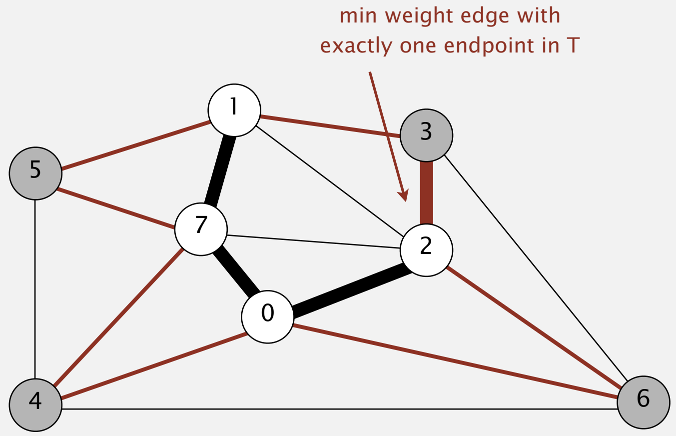
MST edges

0-7 1-7 0-2

so now the MST contains edges 0-7, 1-7, and 0-2.

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



edges with exactly one endpoint in T (sorted by weight)

in MST →	2-3	0.17
	5-7	0.28
	1-3	0.29
	1-5	0.32
	4-7	0.37
	0-4	0.38
	6-2	0.40
	6-0	0.58

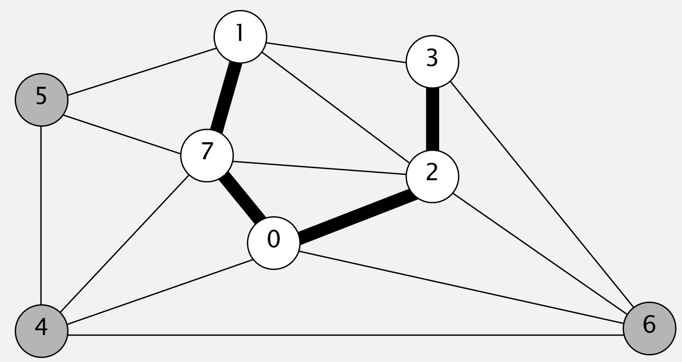
MST edges

0-7 1-7 0-2

Net one would be 2-3

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

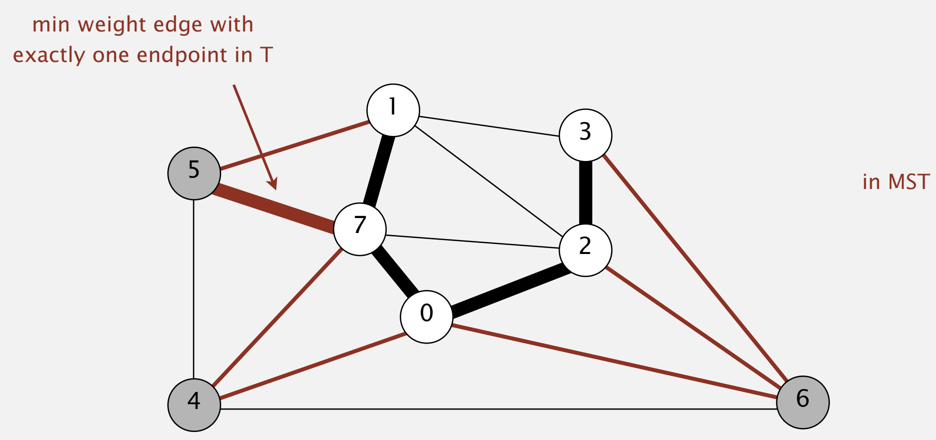


MST edges

0-7 1-7 0-2 2-3

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



5-7	0.28
1-5	0.32
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

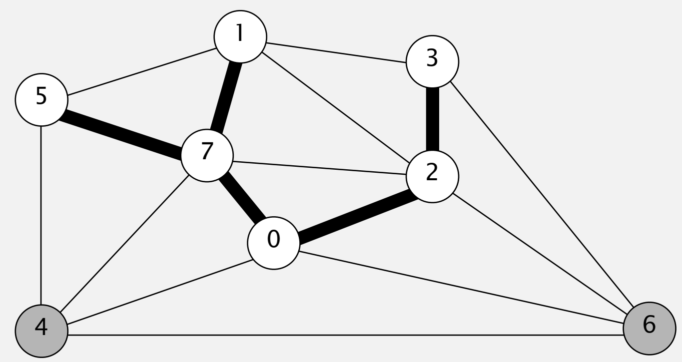
MST edges

0-7 1-7 0-2 2-3

followed by 5-7

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



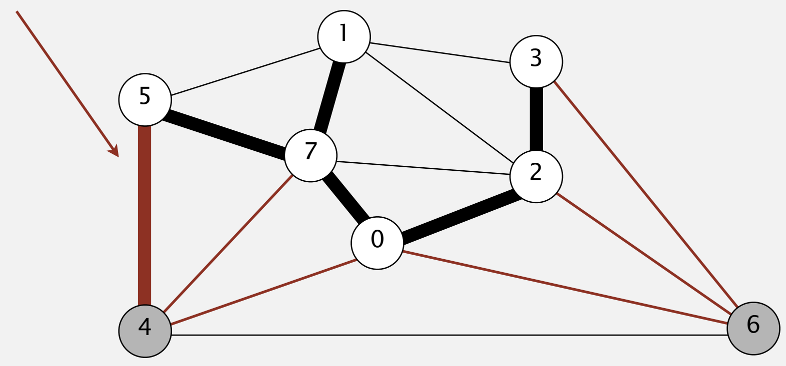
MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

min weight edge with exactly one endpoint in T



edges with exactly one endpoint in T (sorted by weight)

in MST → 4-5 0.35
4-7 0.37
0-4 0.38
6-2 0.40
3-6 0.52
6-0 0.58

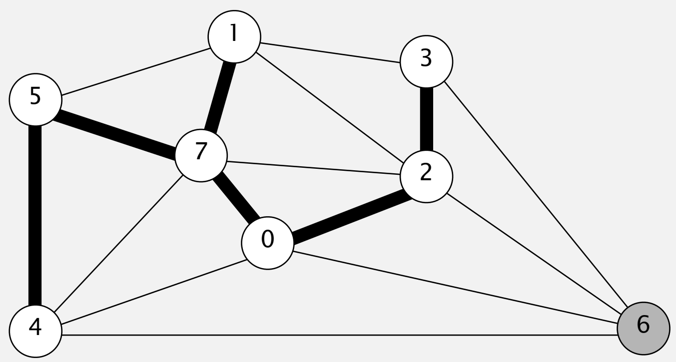
MST edges

0-7 1-7 0-2 2-3 5-7

4-5

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

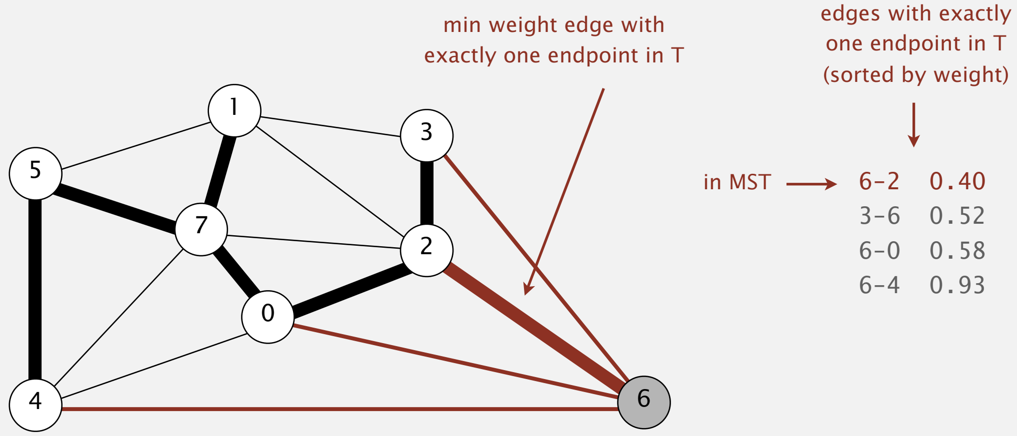


MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



MST edges

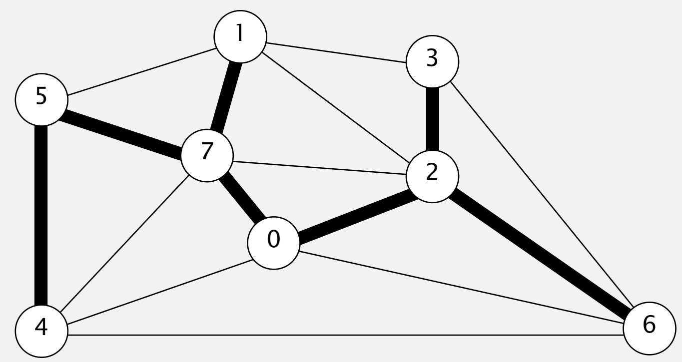
0-7 1-7 0-2 2-3 5-7 4-5

and 6-2

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

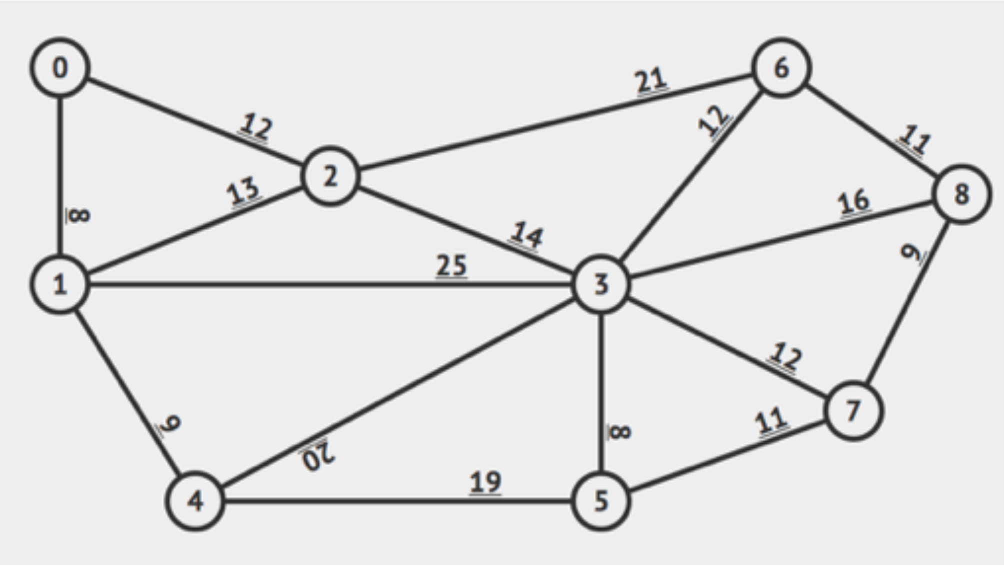


MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2

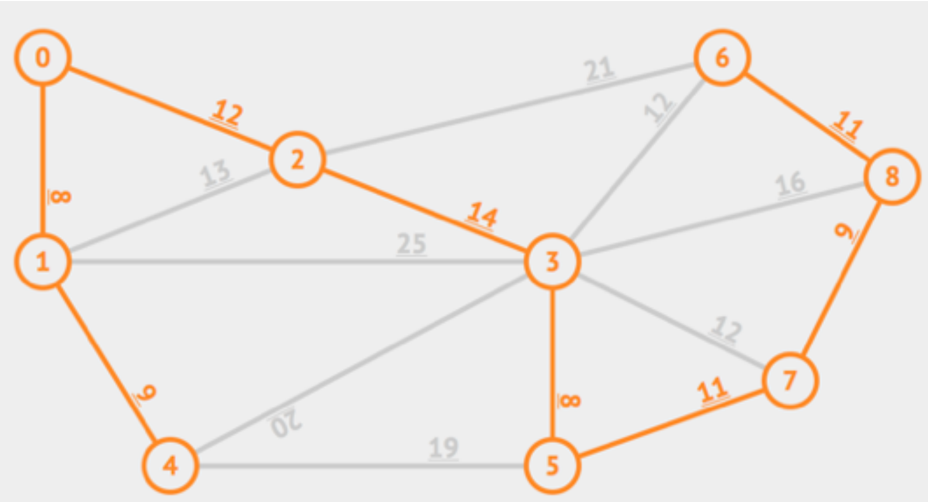
We have repeated this process V-1 times and this is the MST we ended up with

Practice Time



Let's apply Prim's algorithm starting at index 0.

Answer



and here's the answer!

Lecture 24: Minimum Spanning Trees

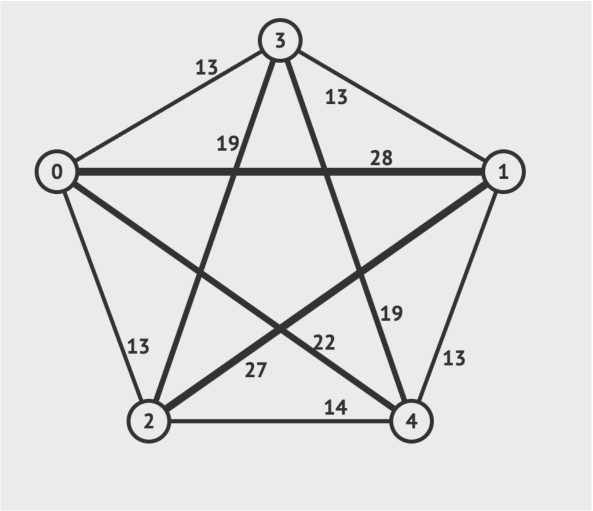
- Introduction
- Kruskal's Algorithm
- Prim's Algorithm

Readings:

- Recommended Textbook: Chapter 4.3 (Pages 604-629)
- Website:
    - https://algs4.cs.princeton.edu/43mst/
- Visualization:
    - https://visualgo.net/en/mst

Problem

- Run Kruskal's and Prim's algorithm (starting at index 0) on the following graph:



Problem

- Run Kruskal's and Prim's algorithm (starting at index 0) on the following graph.
- Both will provide the same MST: