CS062

DATA STRUCTURES AND ADVANCED PROGRAMMING

24: Minimum Spanning Trees

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she/her/hers

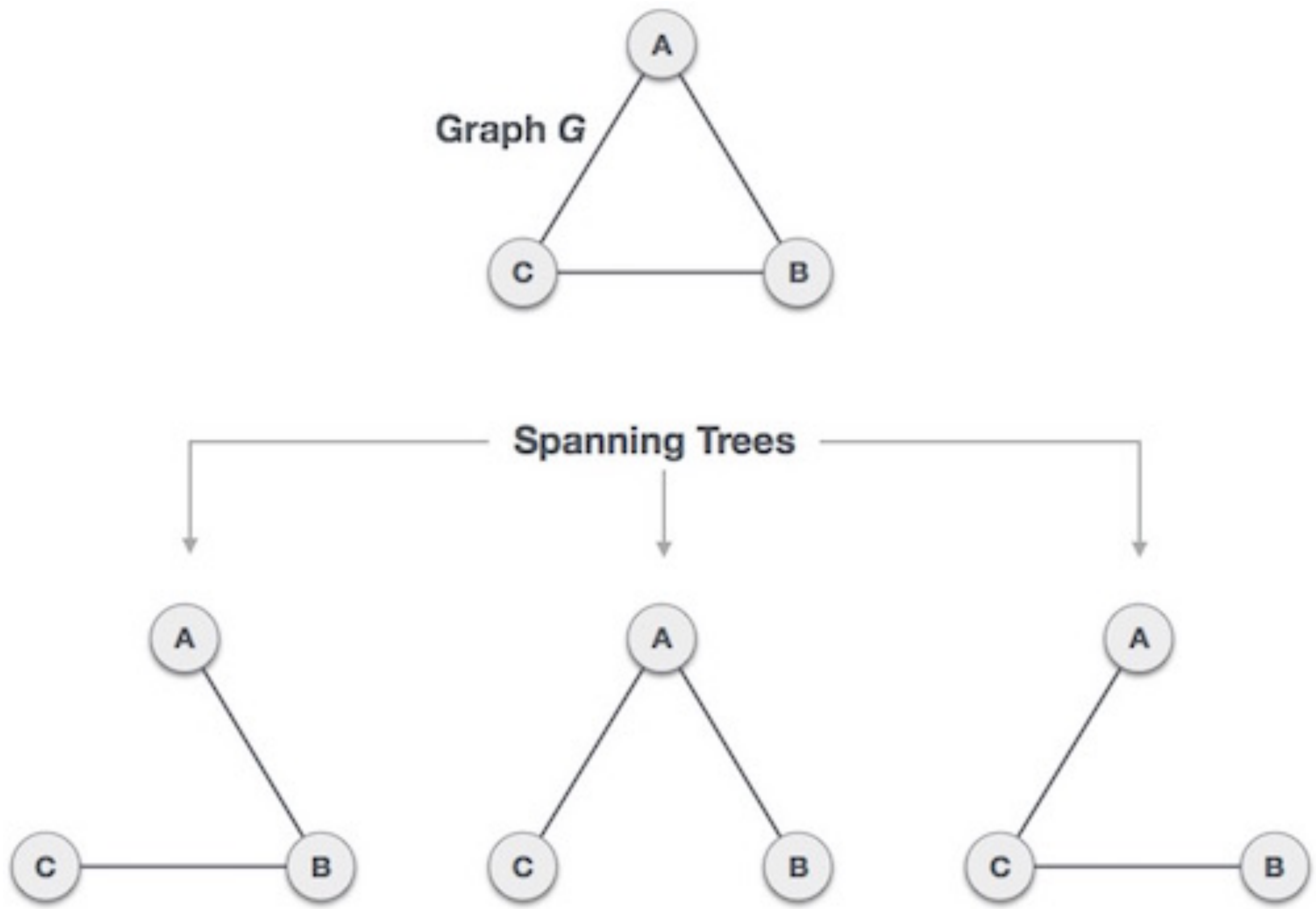
Lecture 24: Minimum Spanning Trees

- Introduction
- Kruskal's Algorithm
- Prim's Algorithm

Some slides adopted from Algorithms 4th Edition or COS226

Spanning Trees

- Given an edge weighted graph G (not digraph!), a spanning tree of G is a subgraph T that is:
  - A tree: connected and acyclic.
  - Spanning: includes all of the vertices of G.



https://www.tutorialspoint.com/data_structures_algorithms/spanning_tree.htm

Properties

- A connected graph G can have more than one spanning tree.
- All possible spanning trees of G have the same number of vertices and edges.
- A spanning tree has $|V| - 1$ edges.
- A spanning tree by definition cannot have any cycle.
- Adding one edge to the spanning tree would create a cycle (i.e. spanning trees are maximally acyclic).
- Removing one edge from the spanning tree would make the graph disconnected (i.e. spanning trees are minimally connected).

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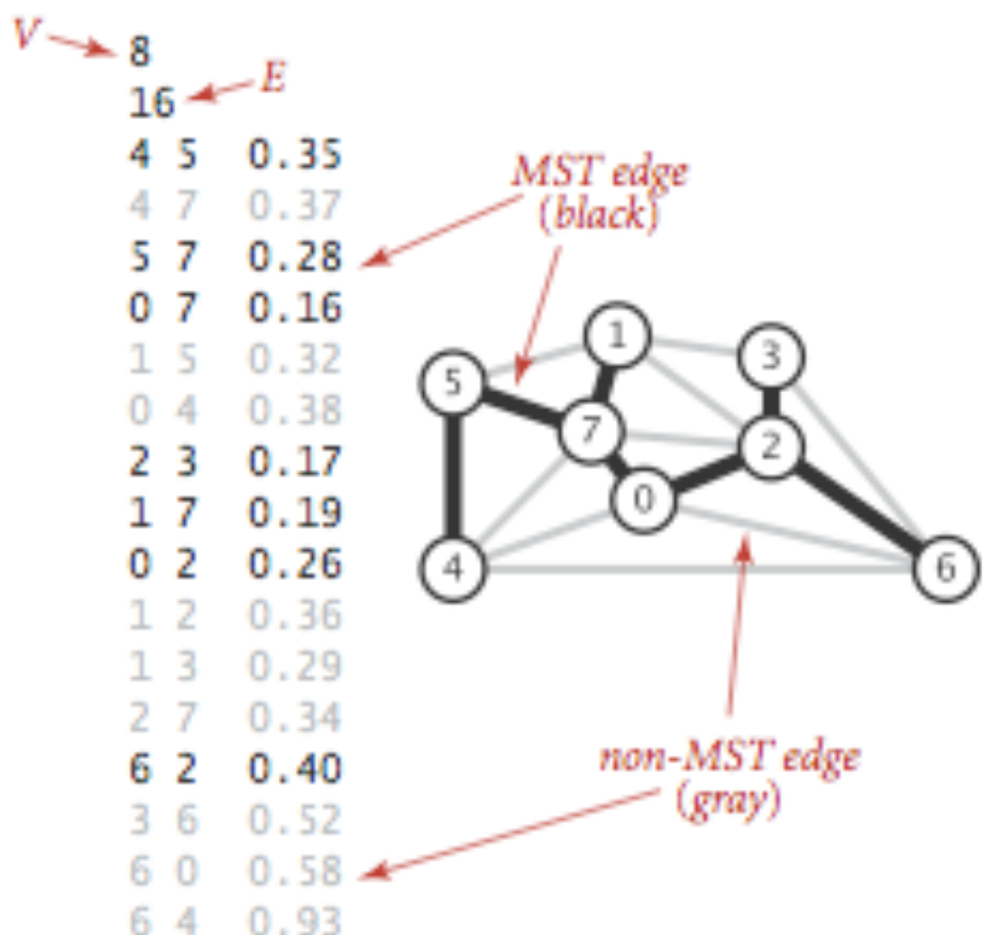
Minimum spanning tree (MST) problem

- Given a connected edge-weighted undirected graph find a spanning tree of minimum weight.



An edge-weighted graph and its MST

Minimum spanning tree applications

- Network design
- Cluster analysis
- Cancer imaging
- Many others
  - https://www.ics.uci.edu/~eppstein/gina/mst.html
  - https://personal.utdallas.edu/~besp/teaching/mst-applications.pdf

Lecture 24: Minimum Spanning Trees

- Introduction
- Kruskal's Algorithm
- Prim's Algorithm

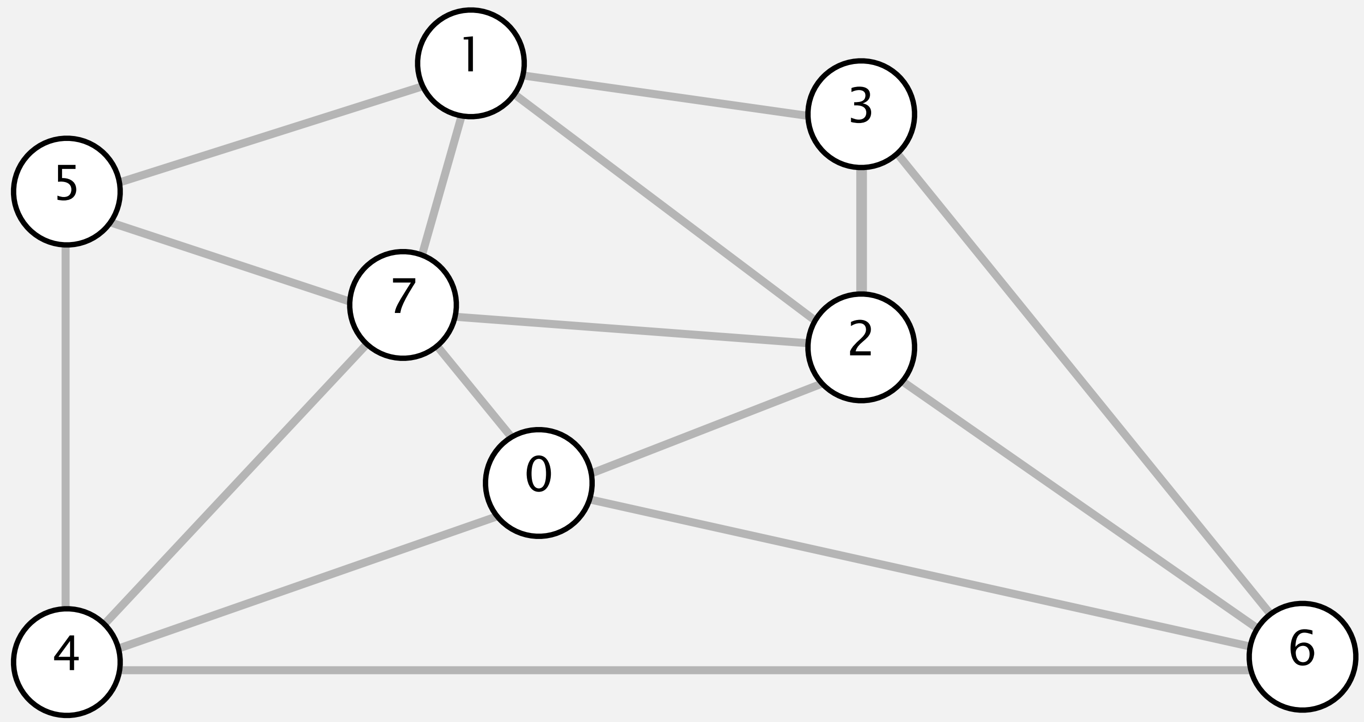
Kruskal's algorithm

- Sort edges in ascending order of weight.
- Starting from the one with the smallest weight, add it to the MST unless doing so would create a cycle.
- Running time of $|E|\log|V|$ in worst case.
- Uses union-find, a data structure we haven't covered.

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.



an edge-weighted graph

graph edges sorted by weight

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

in MST → 0-7 0.16

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
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6-0	0.58
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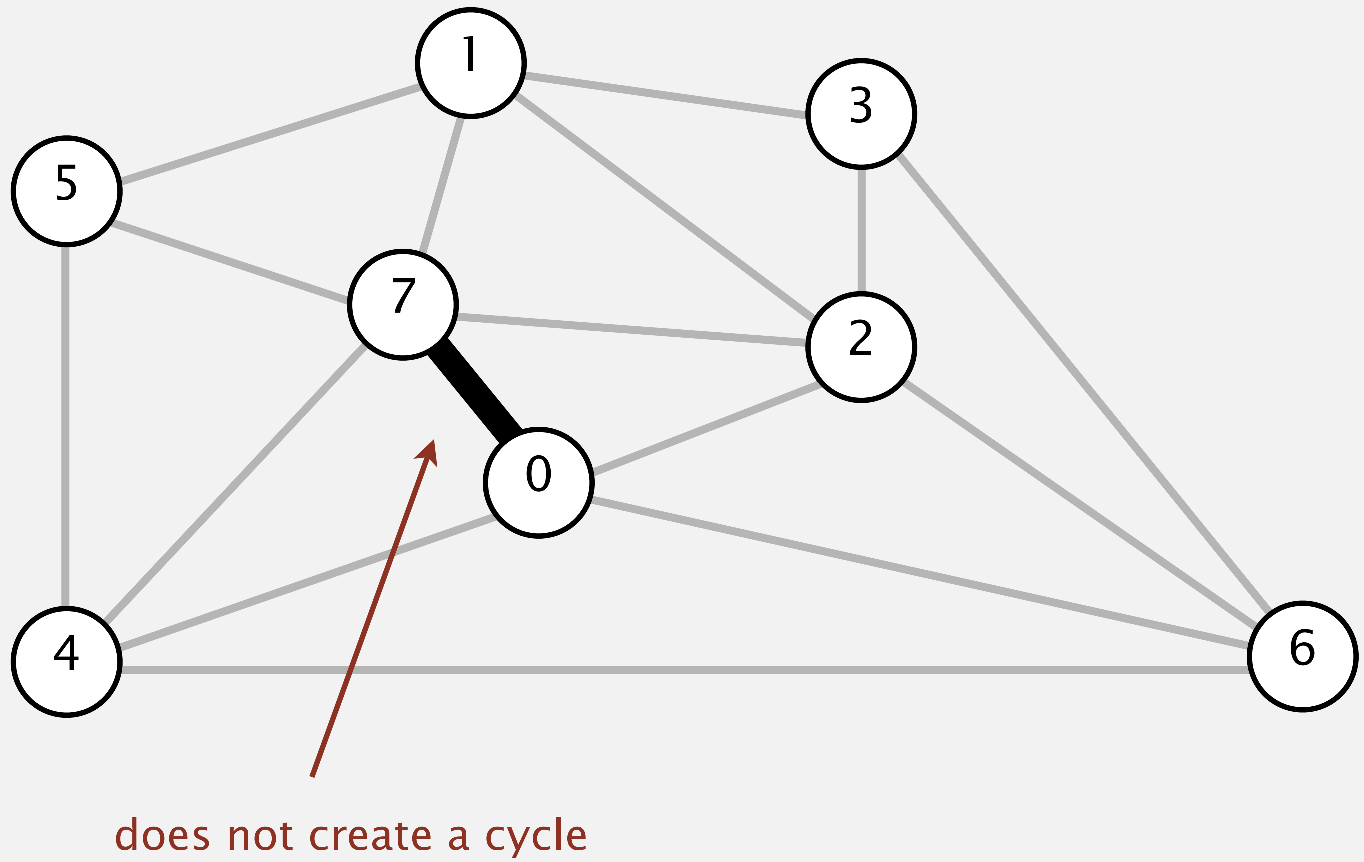

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
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0-4	0.38
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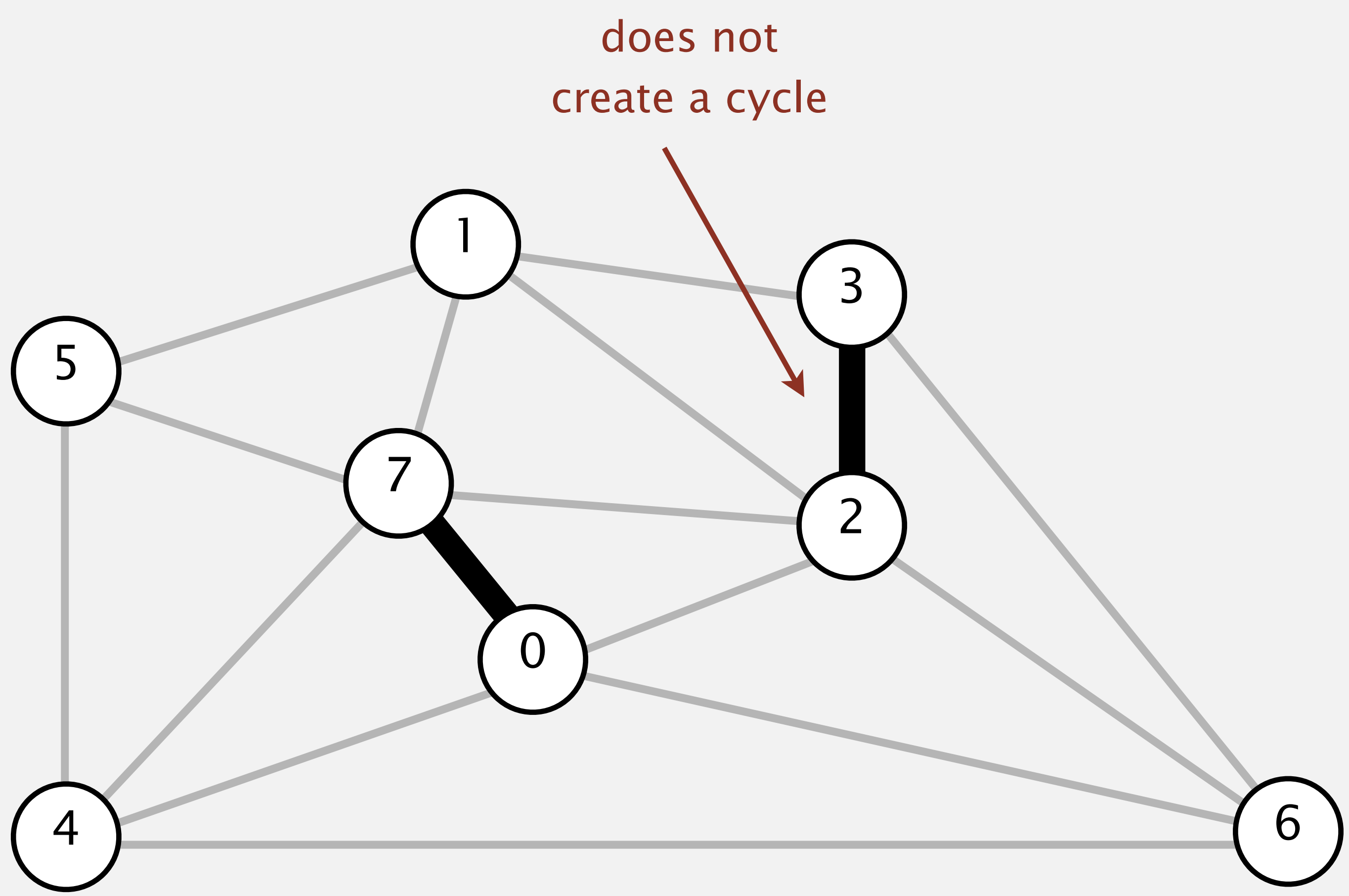
3-6	0.52
6-0	0.58
6-4	0.93



	0-7	0.16
in MST →	2-3	0.17

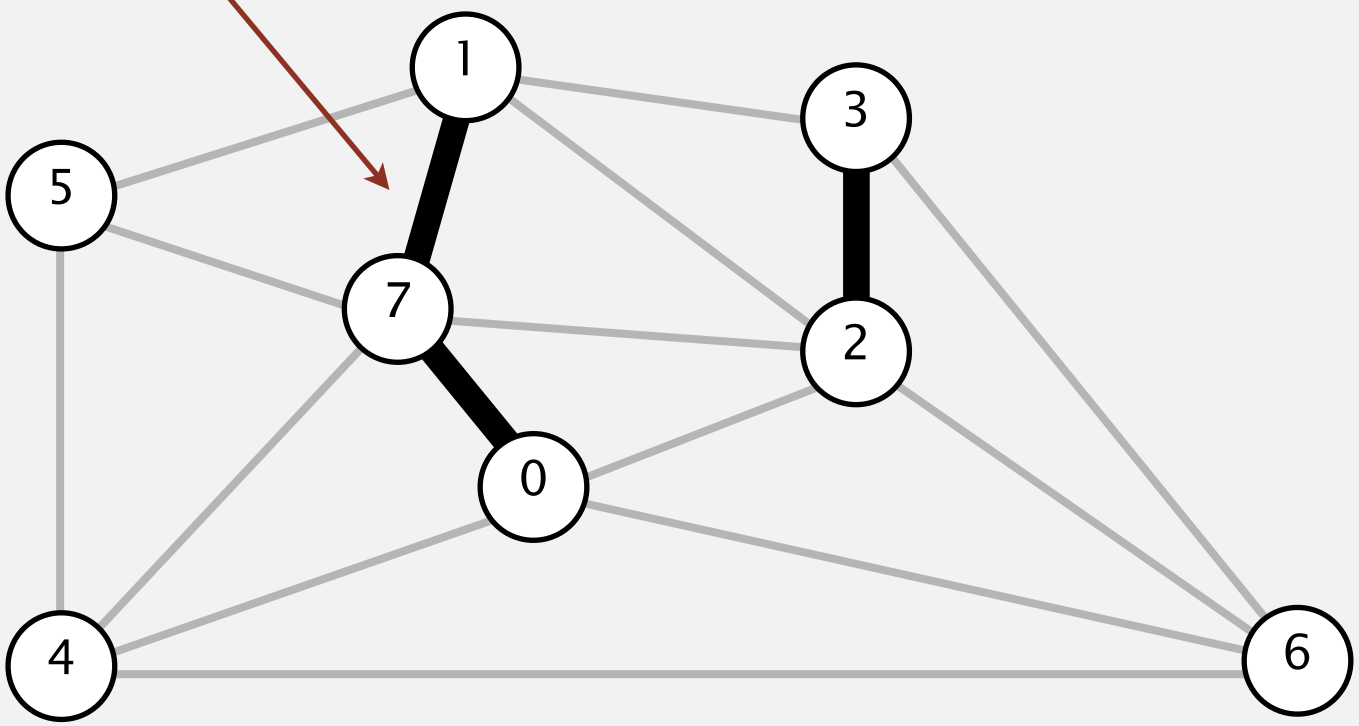
Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

does not create a cycle



in MST → 1-7 0.19

0-7	0.16
2-3	0.17
1-7	0.19

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
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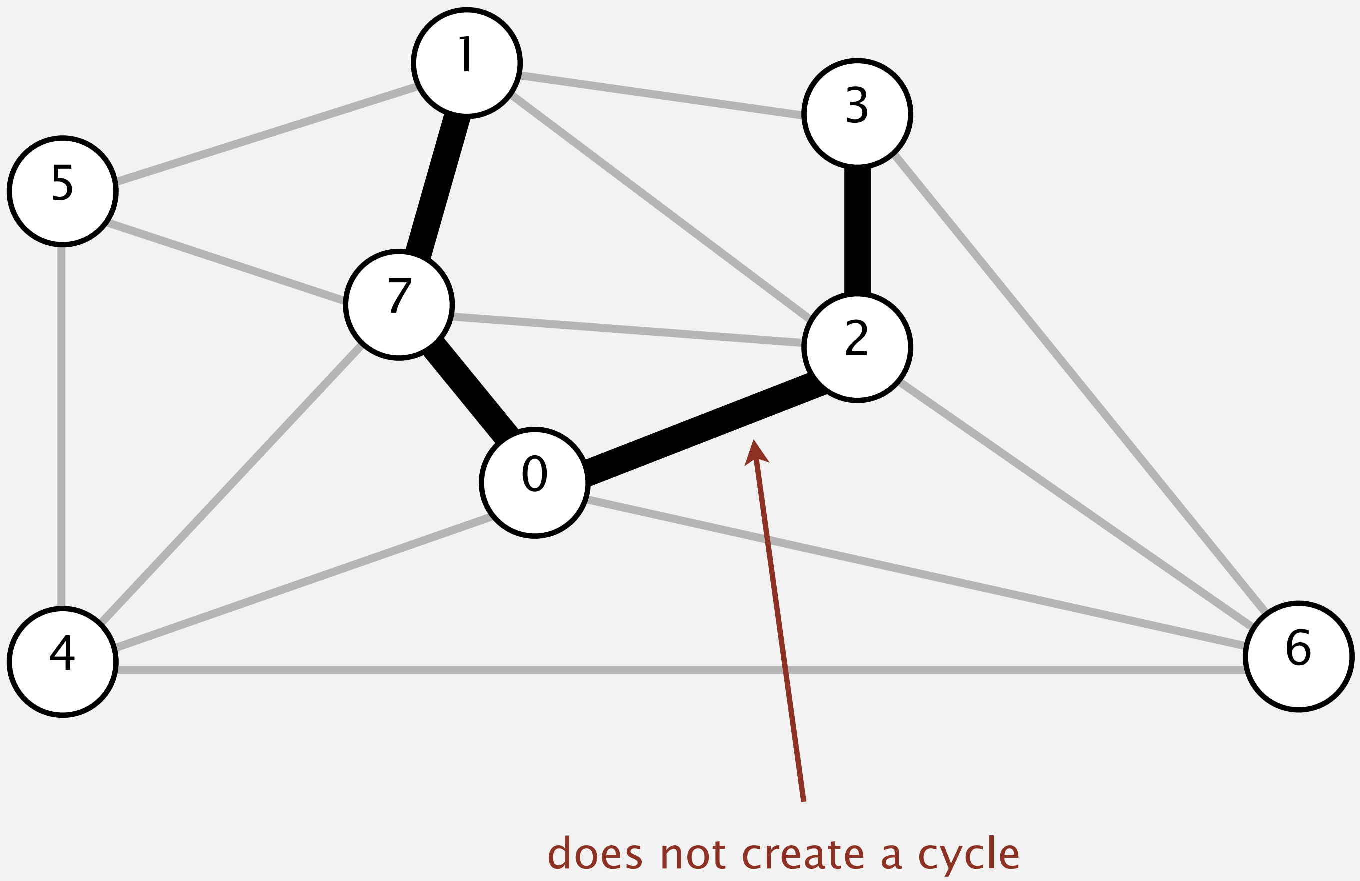

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26

in MST → 0-2 0.26

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
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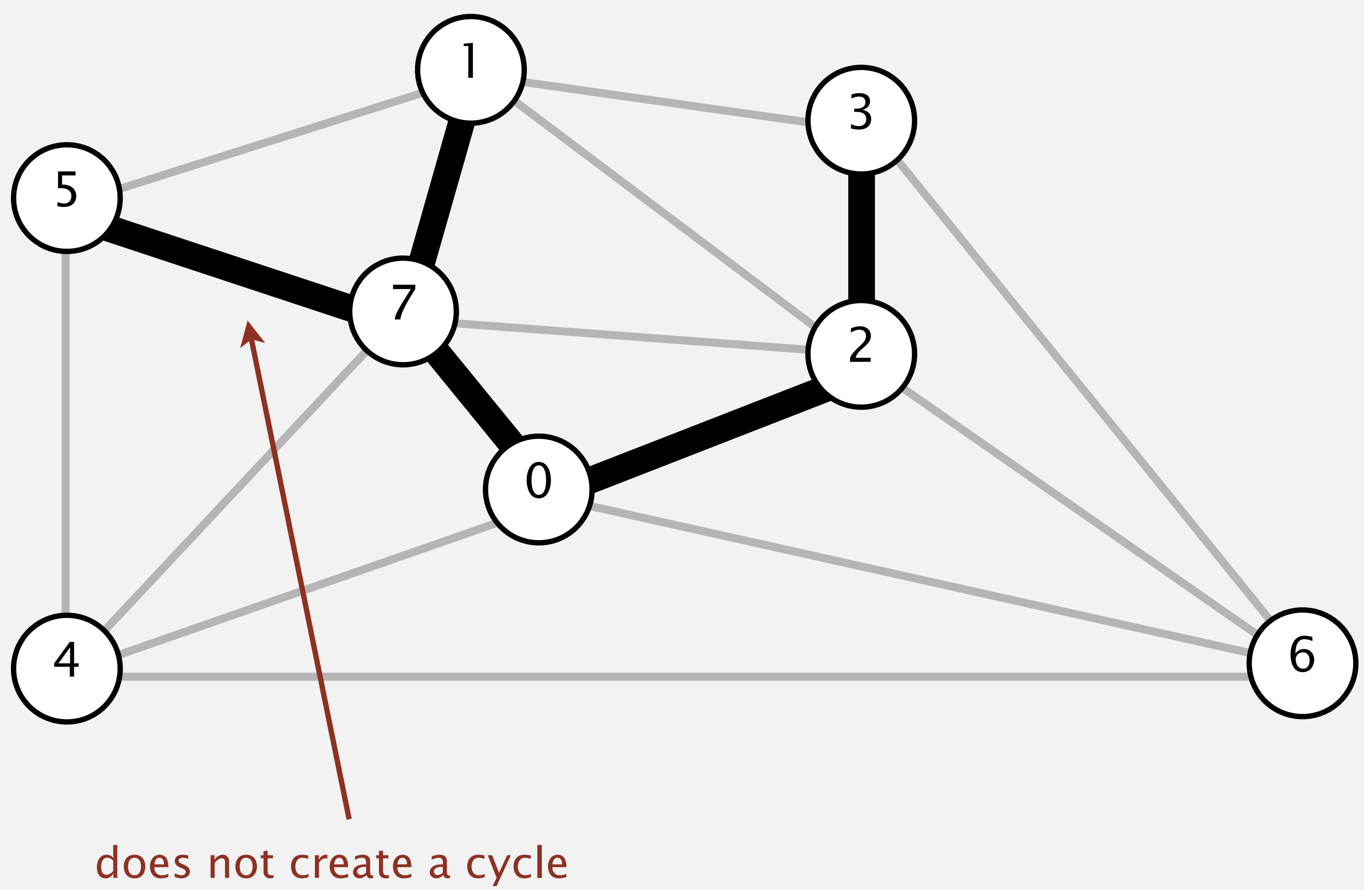

	0-7	0.16
	2-3	0.17
	1-7	0.19
	0-2	0.26
in MST →	5-7	0.28

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
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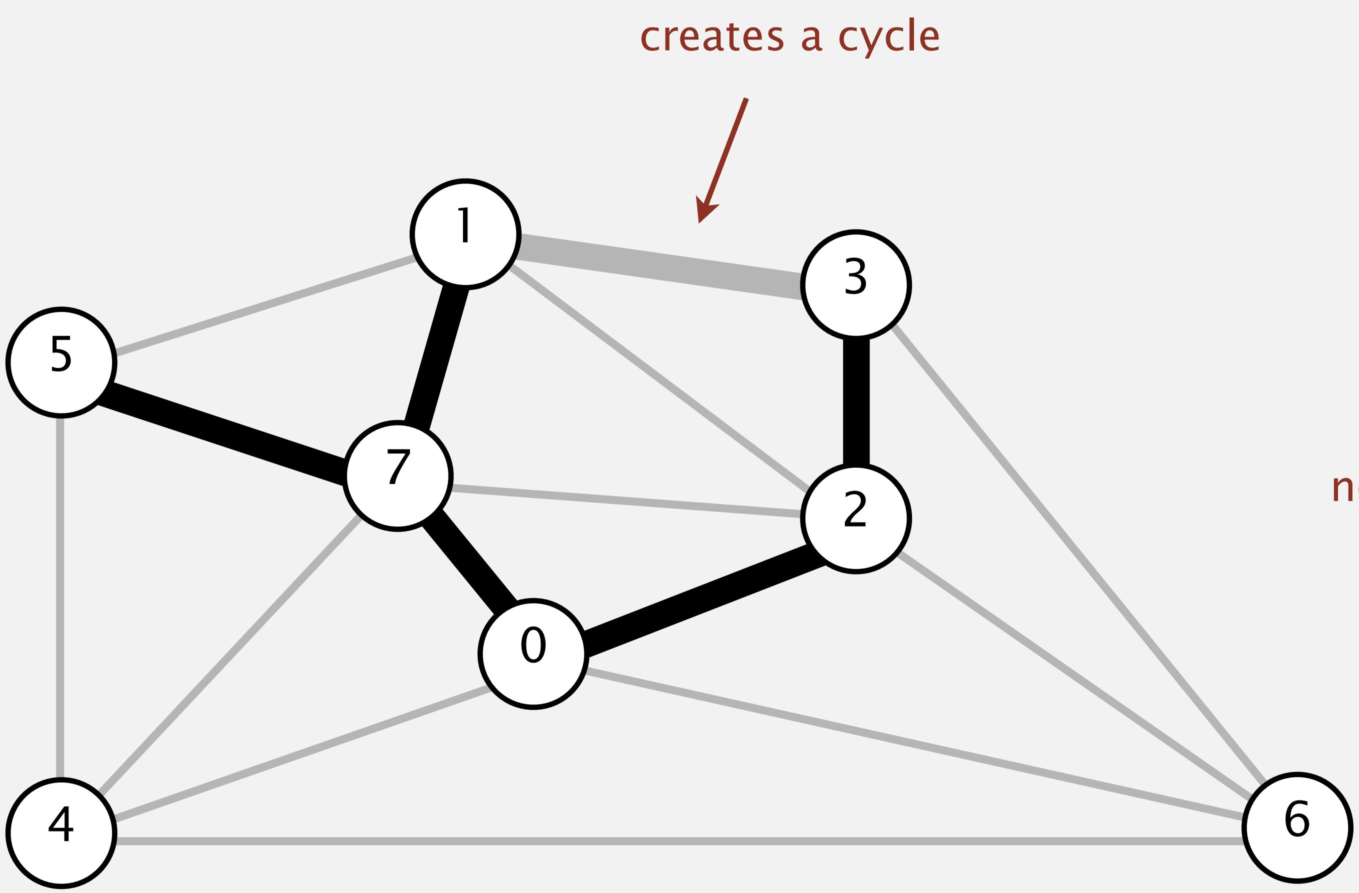

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29

not in MST → 1-3 0.29

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
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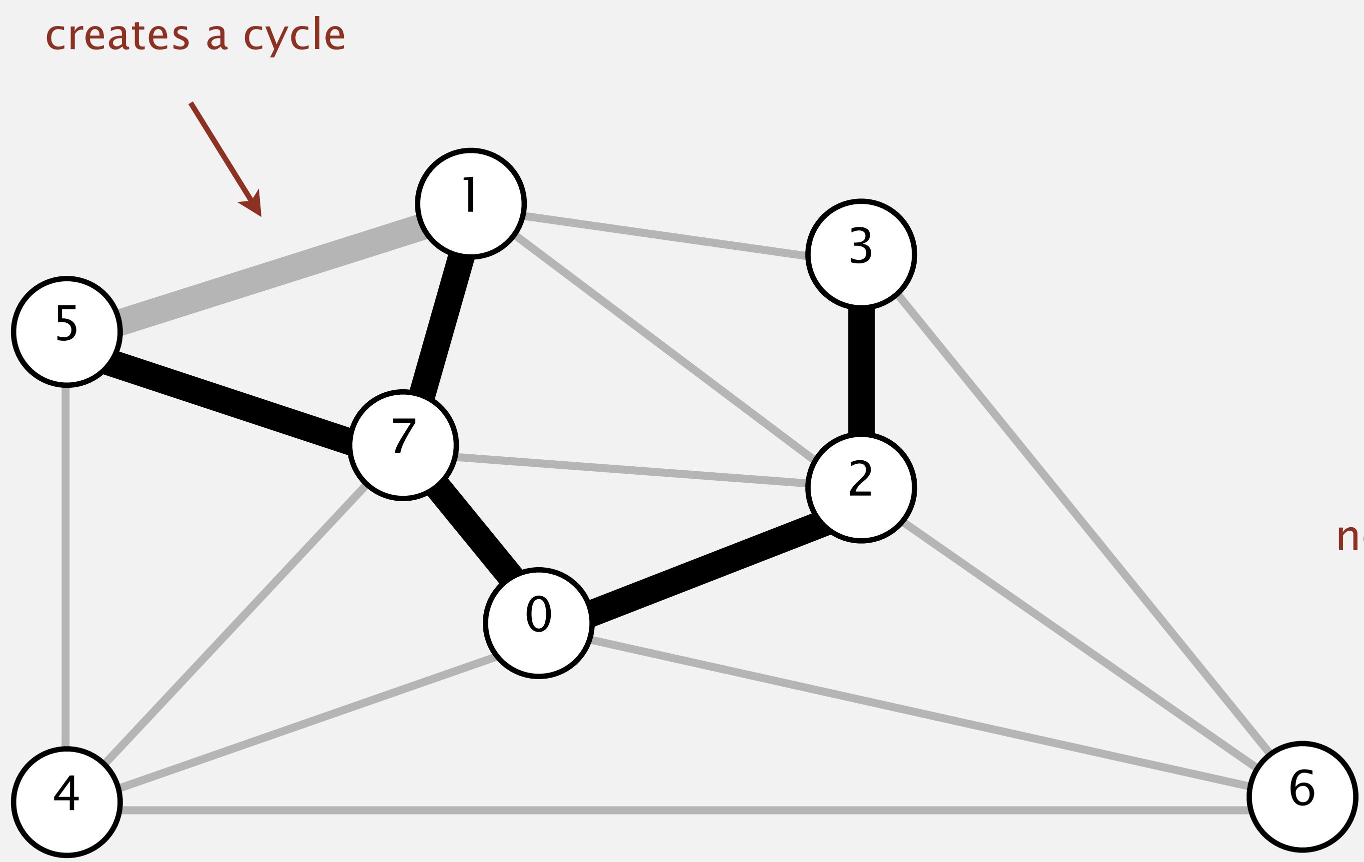

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32

not in MST →

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
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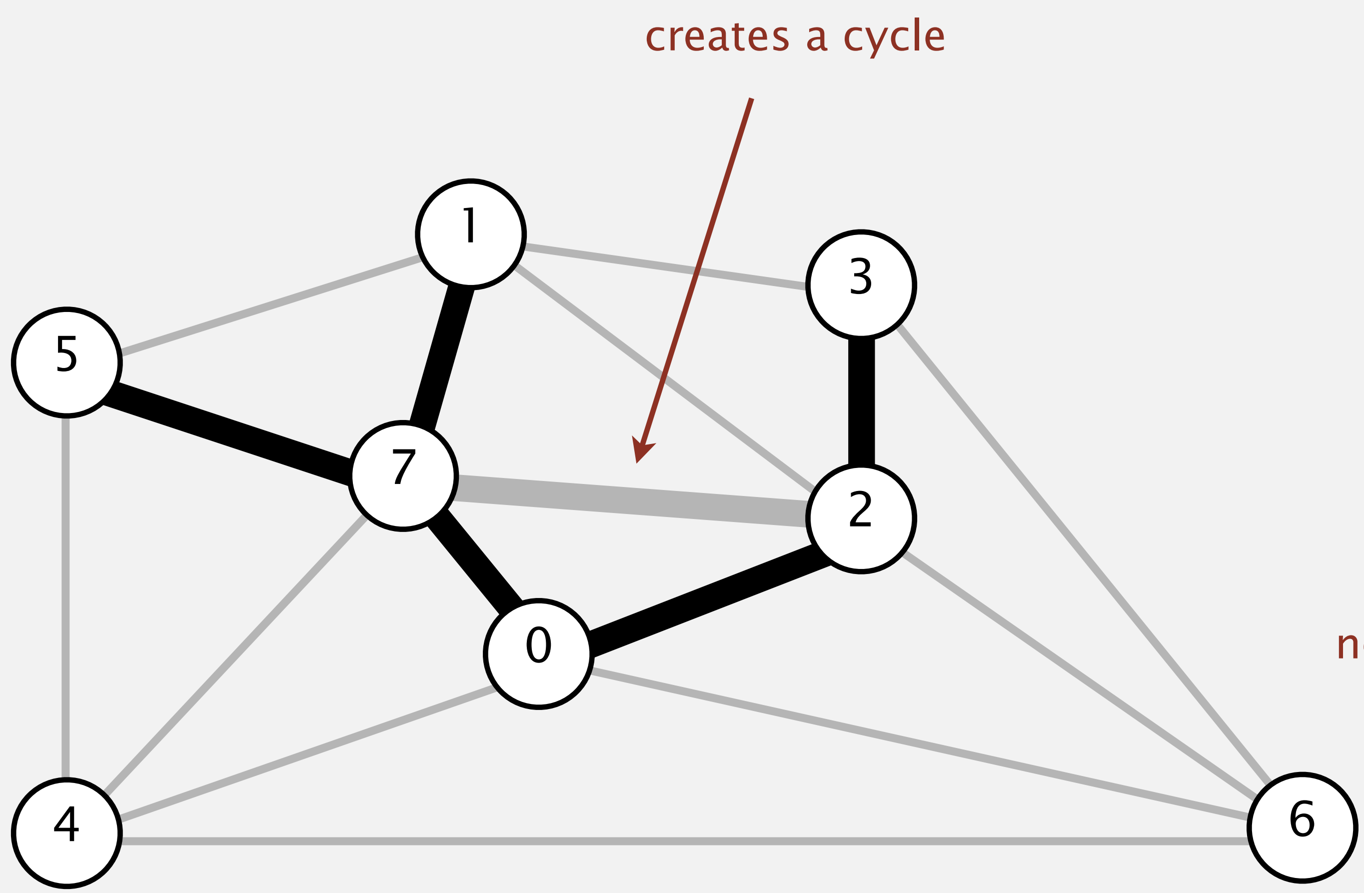

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34

not in MST → 2-7 0.34

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
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1-3	0.29
1-5	0.32
2-7	0.34
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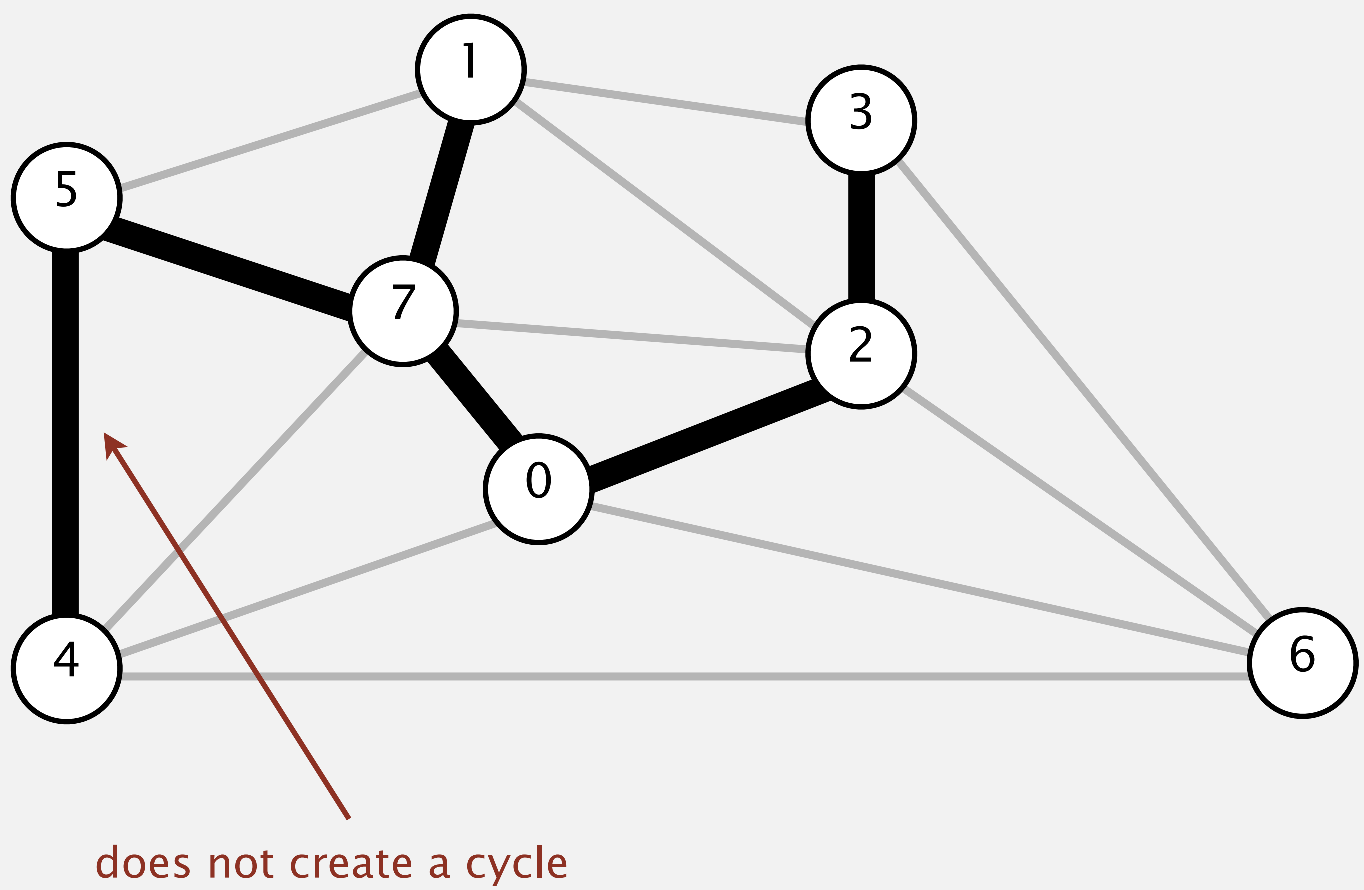

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35

in MST → 4-5 0.35

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
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6-0	0.58
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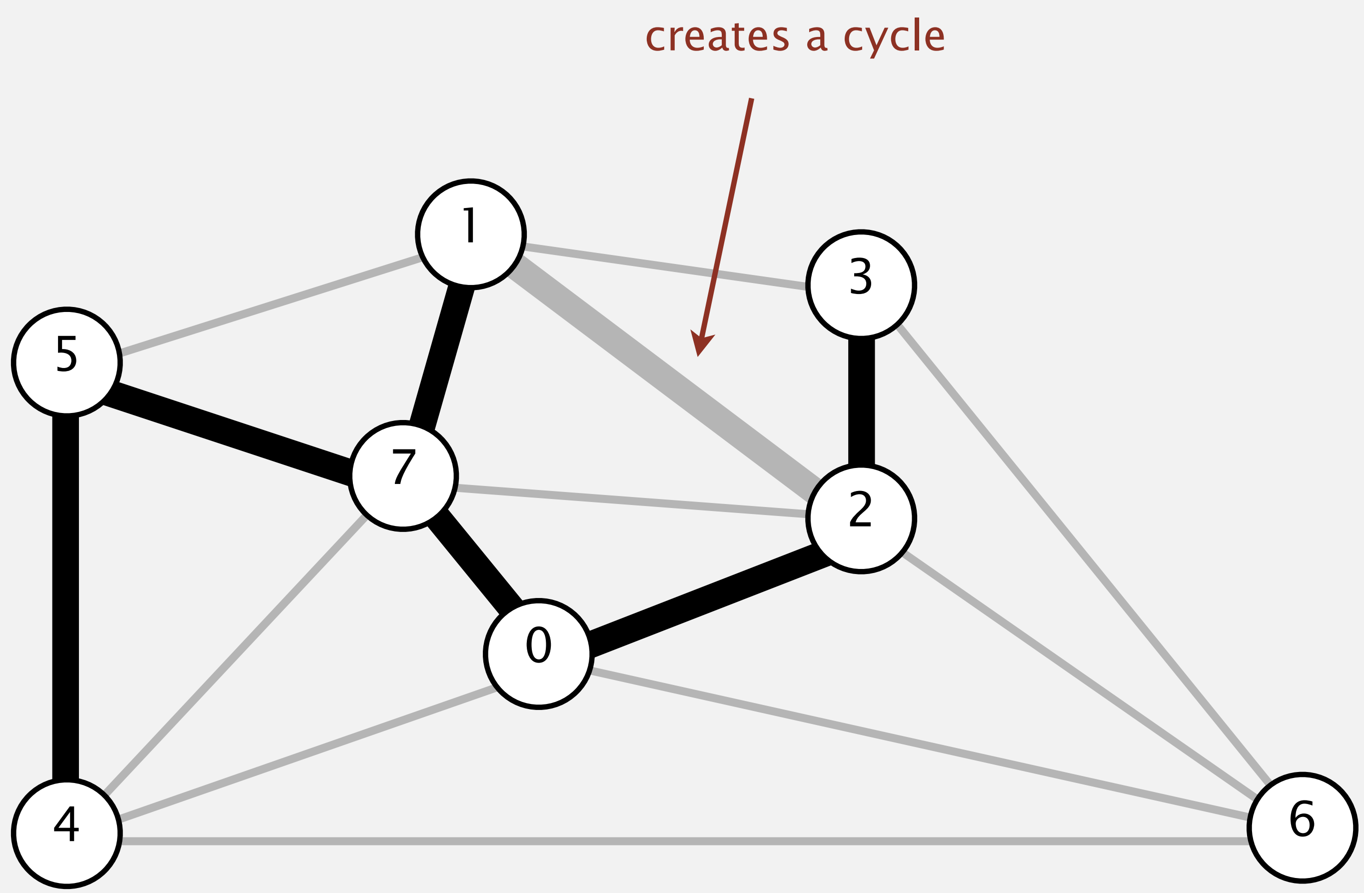

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36

not in MST →

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
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1-5	0.32
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4-5	0.35
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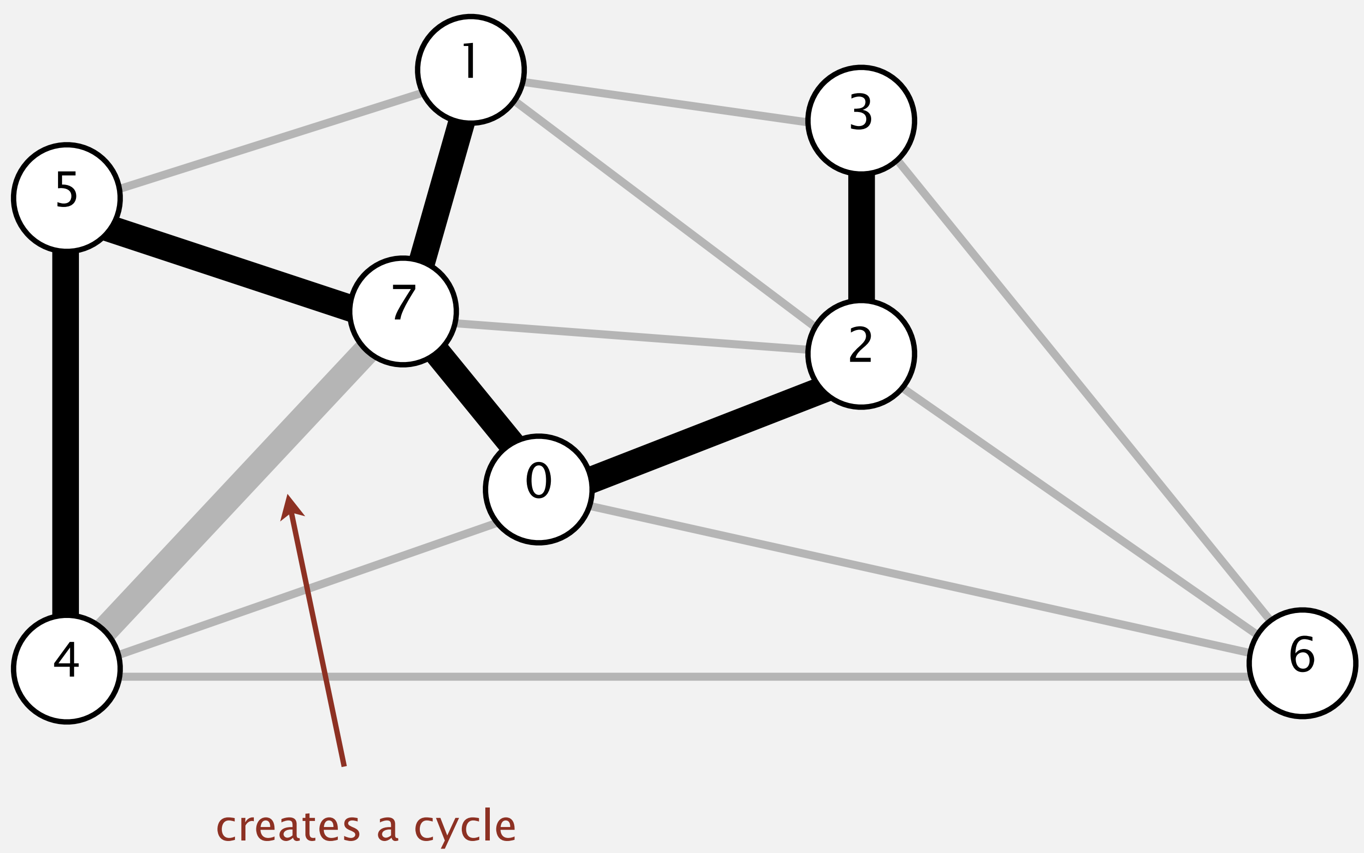

0-7	0.16
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1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37

not in MST → 4-7 0.37

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
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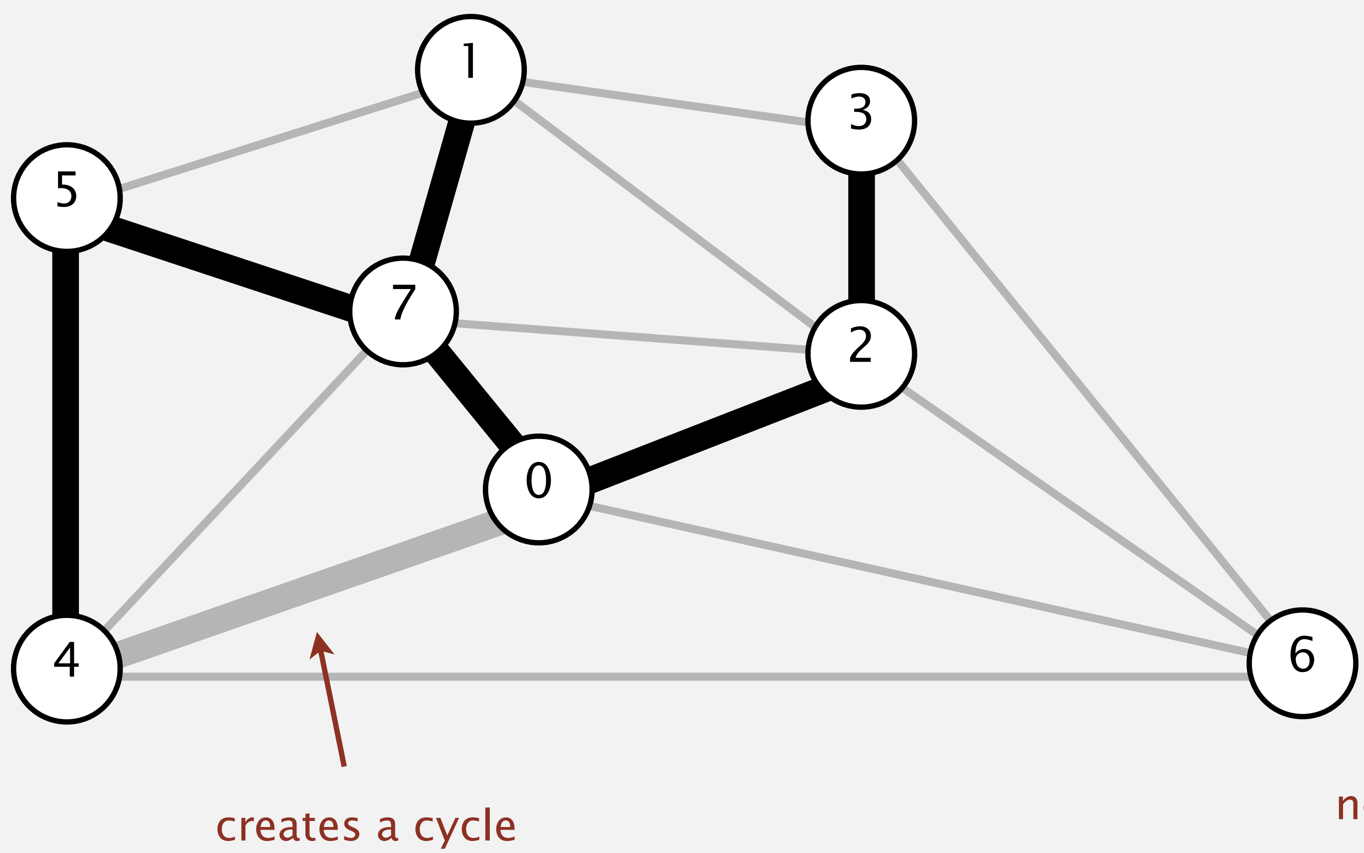
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38

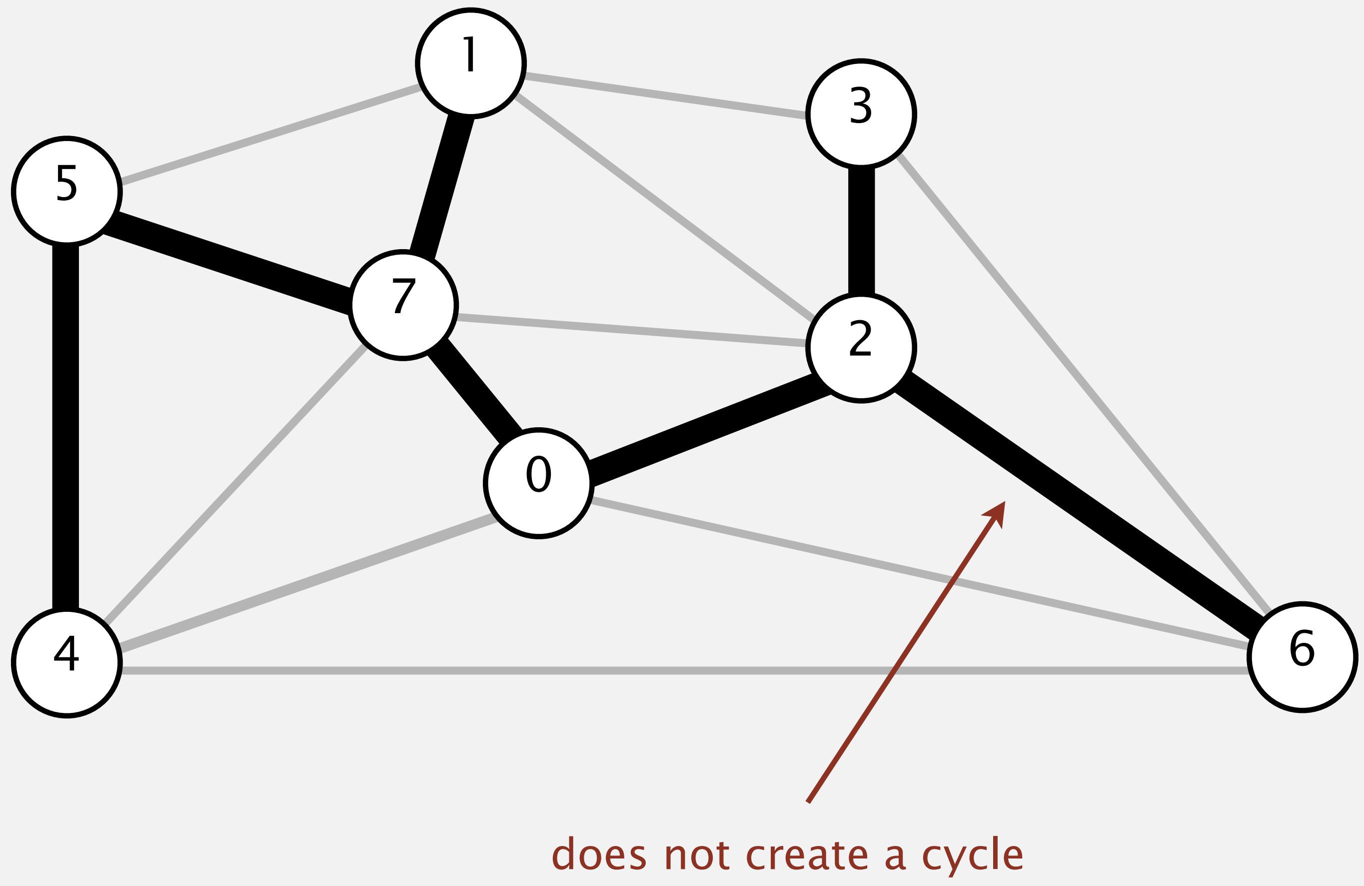
not in MST → 0-4 0.38

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



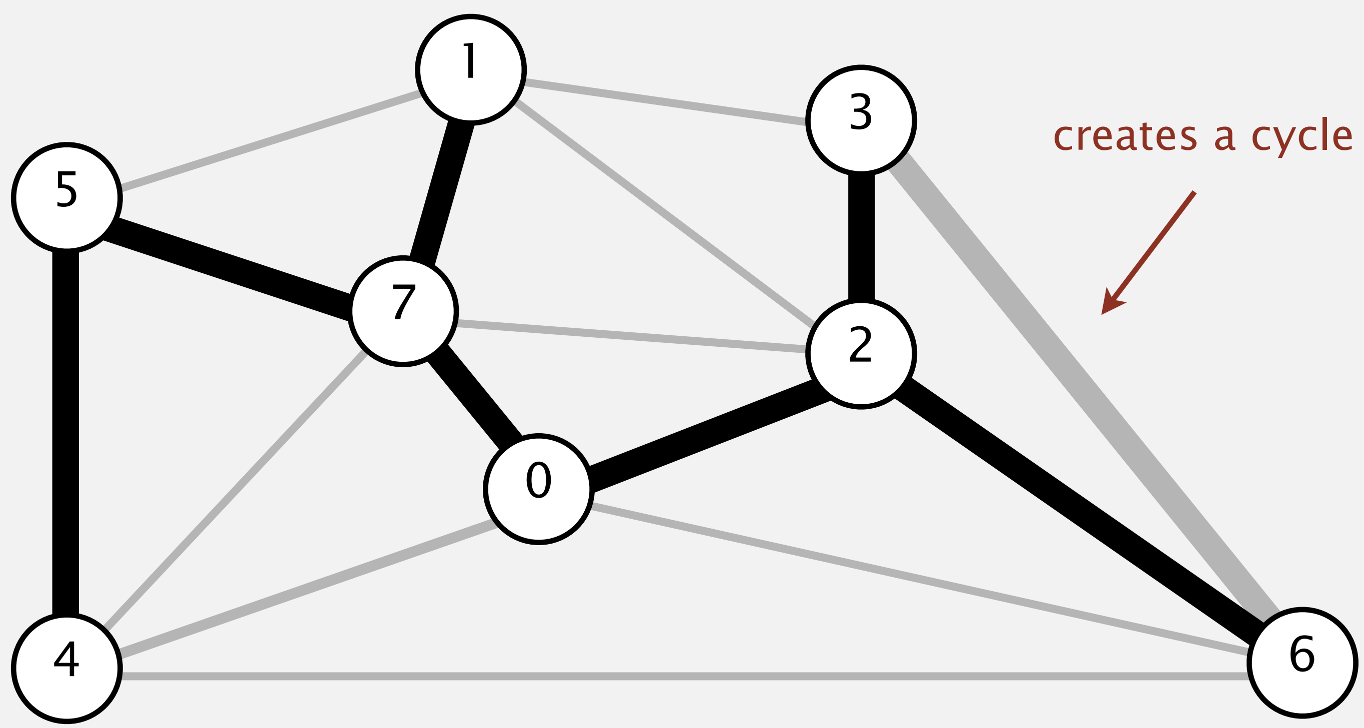
0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
in MST → 6-2	0.40

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



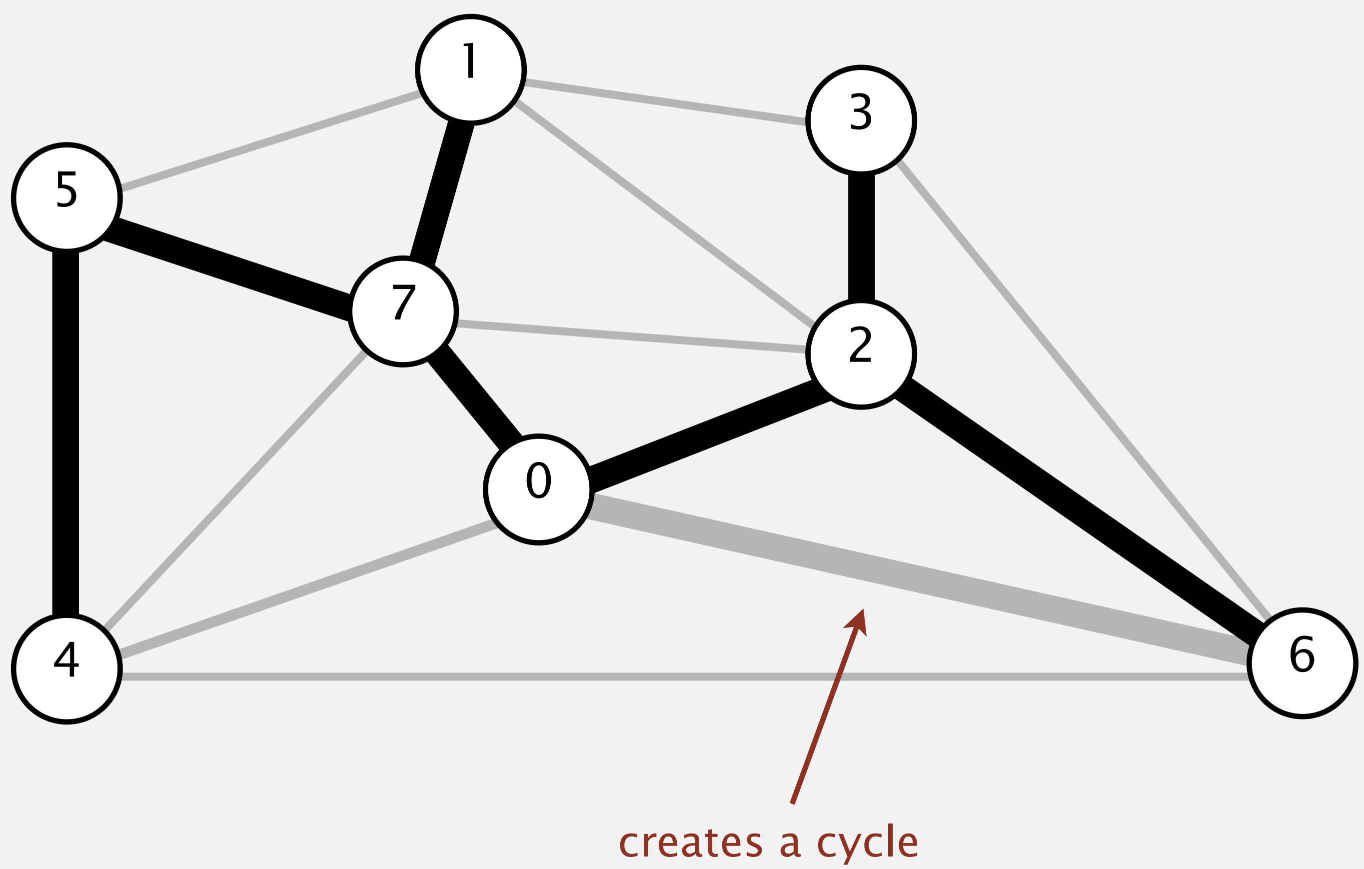
	0-7	0.16
	2-3	0.17
	1-7	0.19
	0-2	0.26
	5-7	0.28
	1-3	0.29
	1-5	0.32
	2-7	0.34
	4-5	0.35
	1-2	0.36
	4-7	0.37
	0-4	0.38
	6-2	0.40
not in MST →	3-6	0.52

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
not in MST → 6-0	0.58

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
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1-5	0.32
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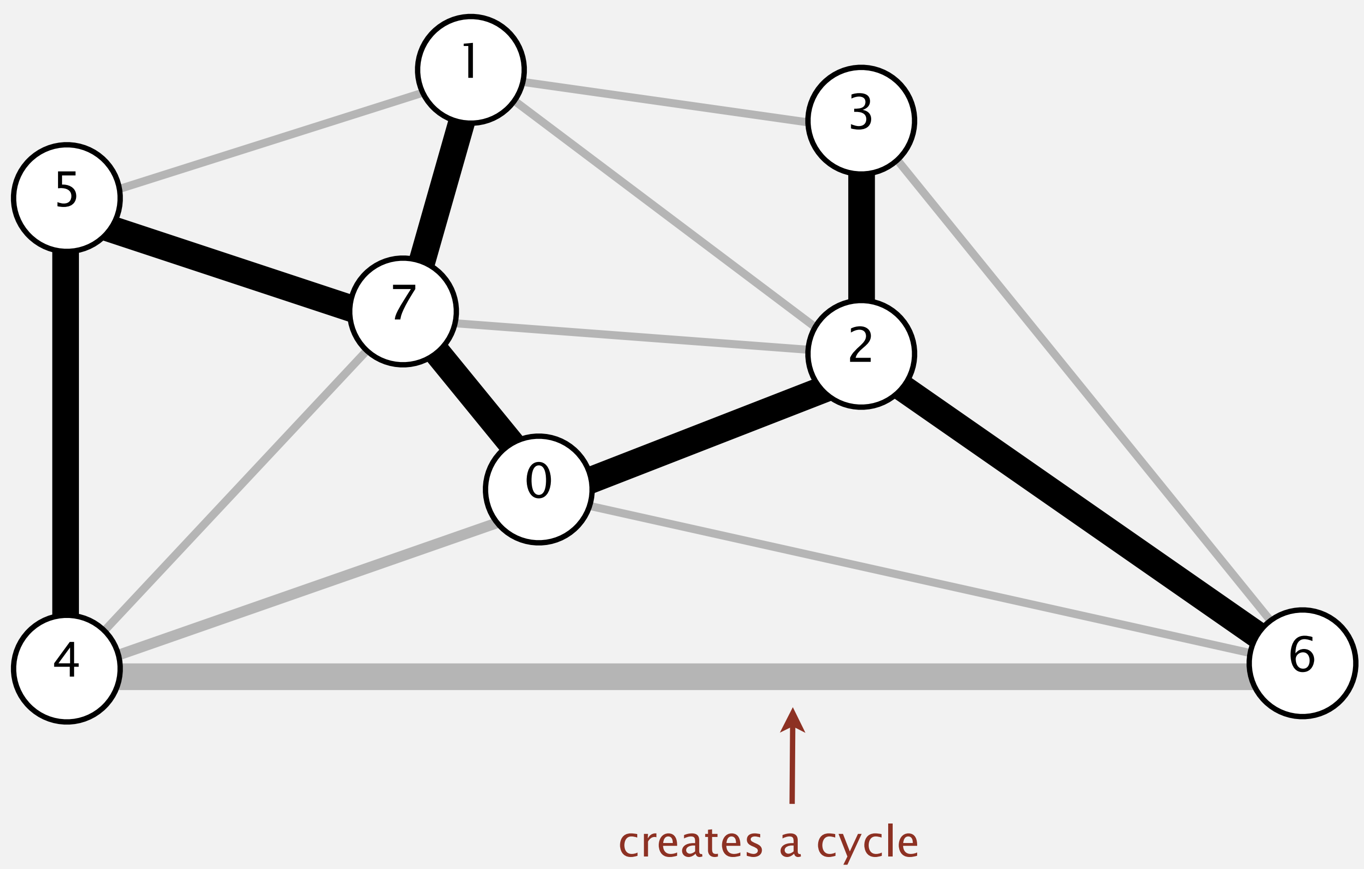
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



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2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

not in MST → 6-4 0.93

Kruskal's algorithm demo

Consider edges in ascending order of weight.

- Add next edge to tree T unless doing so would create a cycle.

0-7	0.16
2-3	0.17
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0-2	0.26
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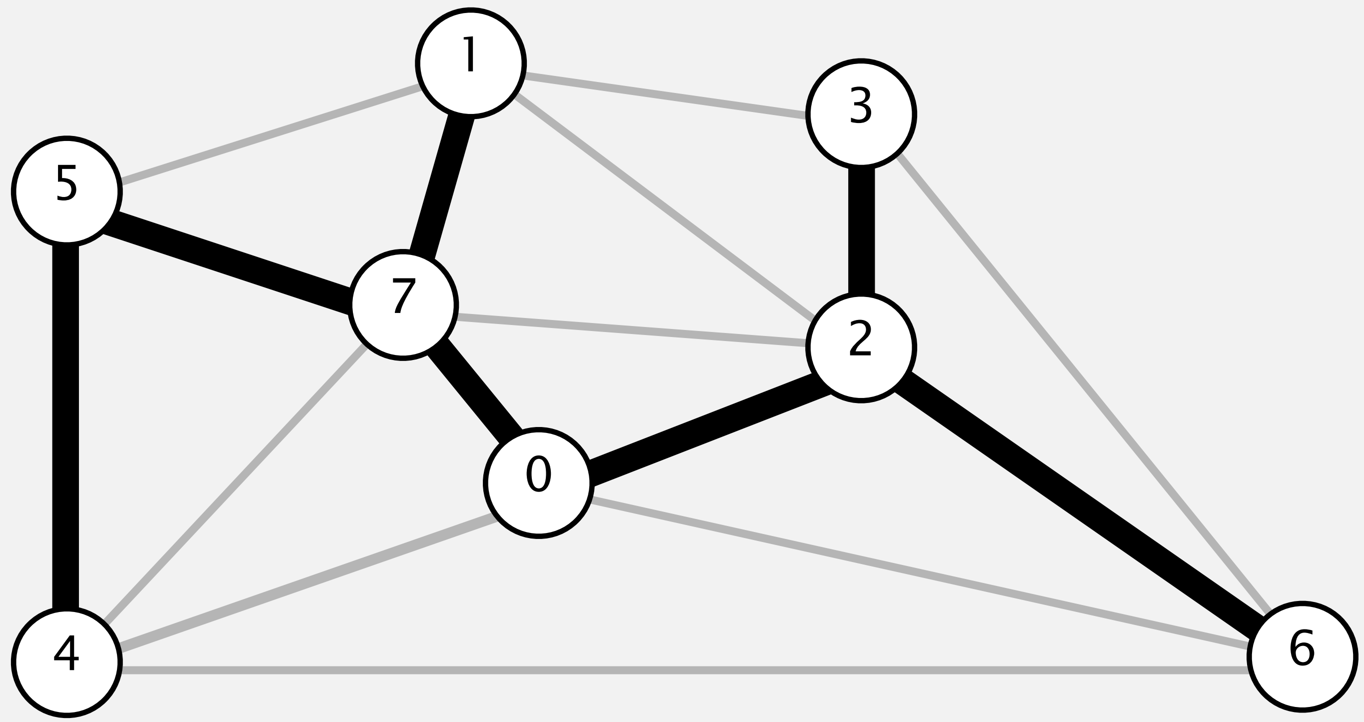

a minimum spanning tree

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
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1-2	0.36
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0-4	0.38
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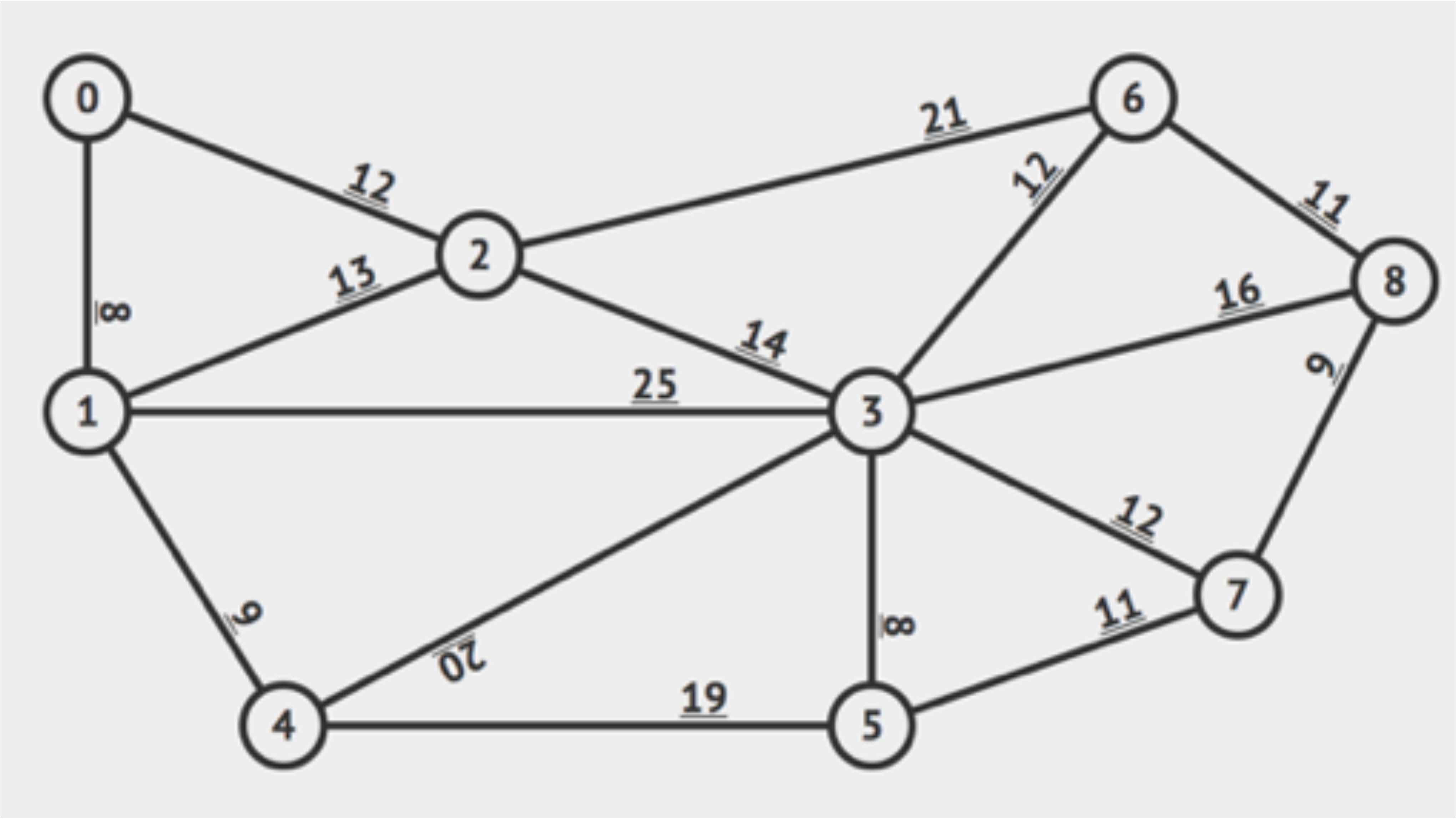
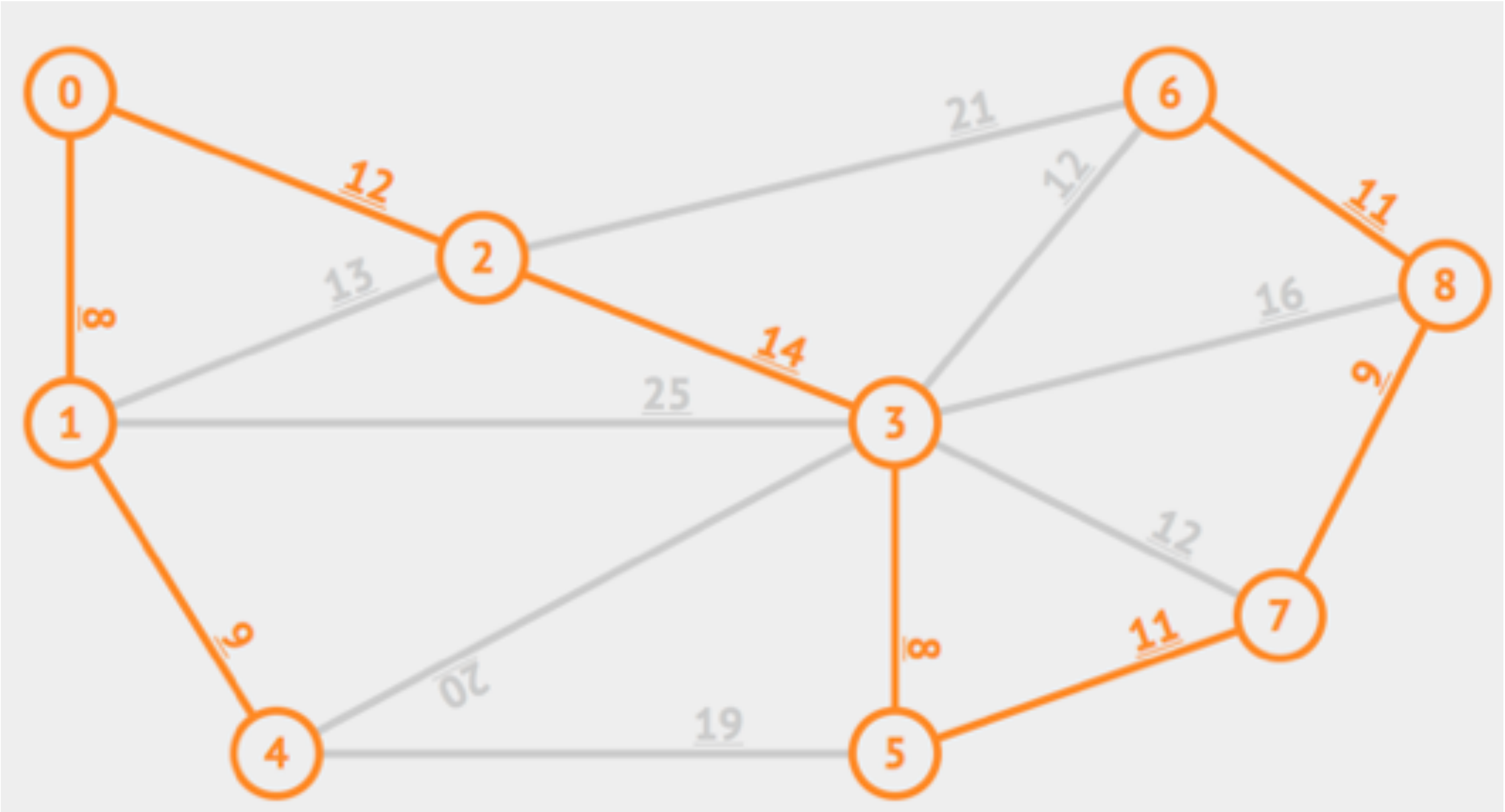
3-6	0.52
6-0	0.58
6-4	0.93

Practice Time



Answer



Lecture 24: Minimum Spanning Trees

- Introduction
- Kruskal's Algorithm
- Prim's Algorithm

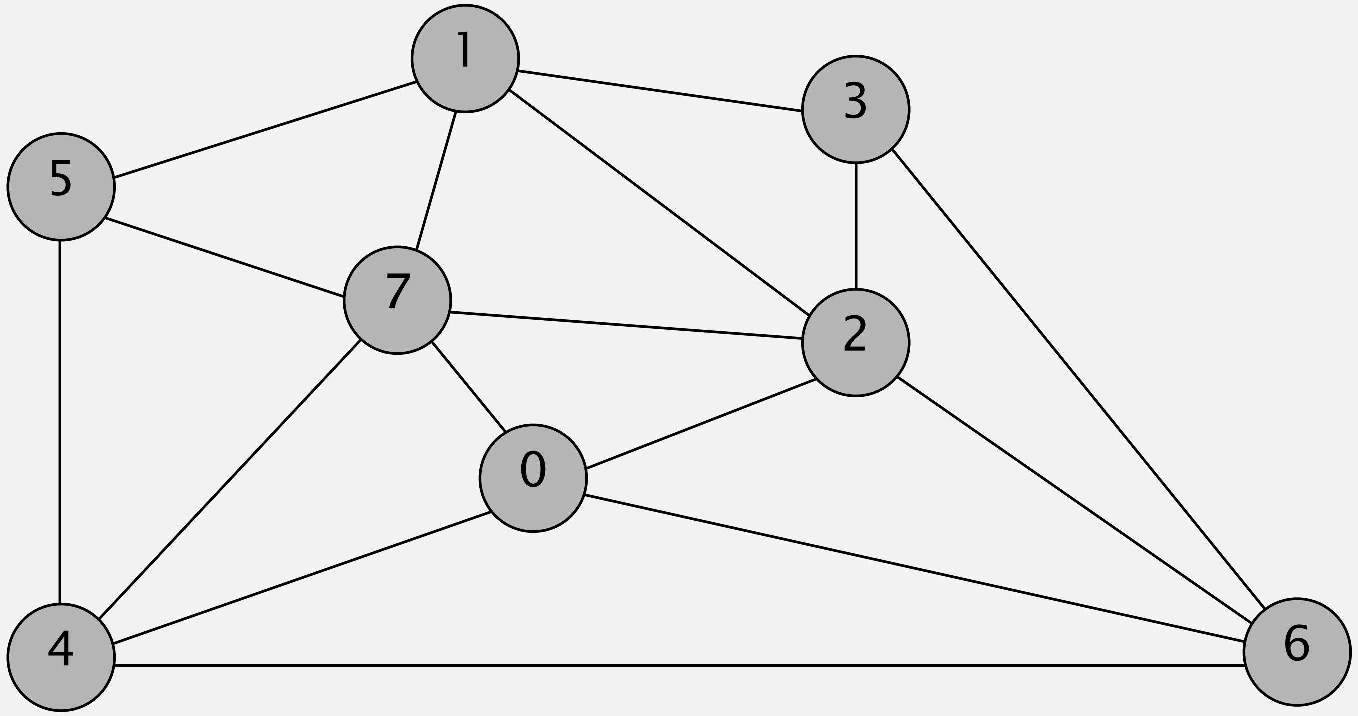
Prim's algorithm

- Start with a random vertex (here, 0) and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $|V| - 1$ edges.

- Two versions, lazy and eager. We will see lazy, here…
- Uses min-priority queue.
- Running time of $|E| \log |V|$ in worst case, as well.

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



an edge-weighted graph

0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
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1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

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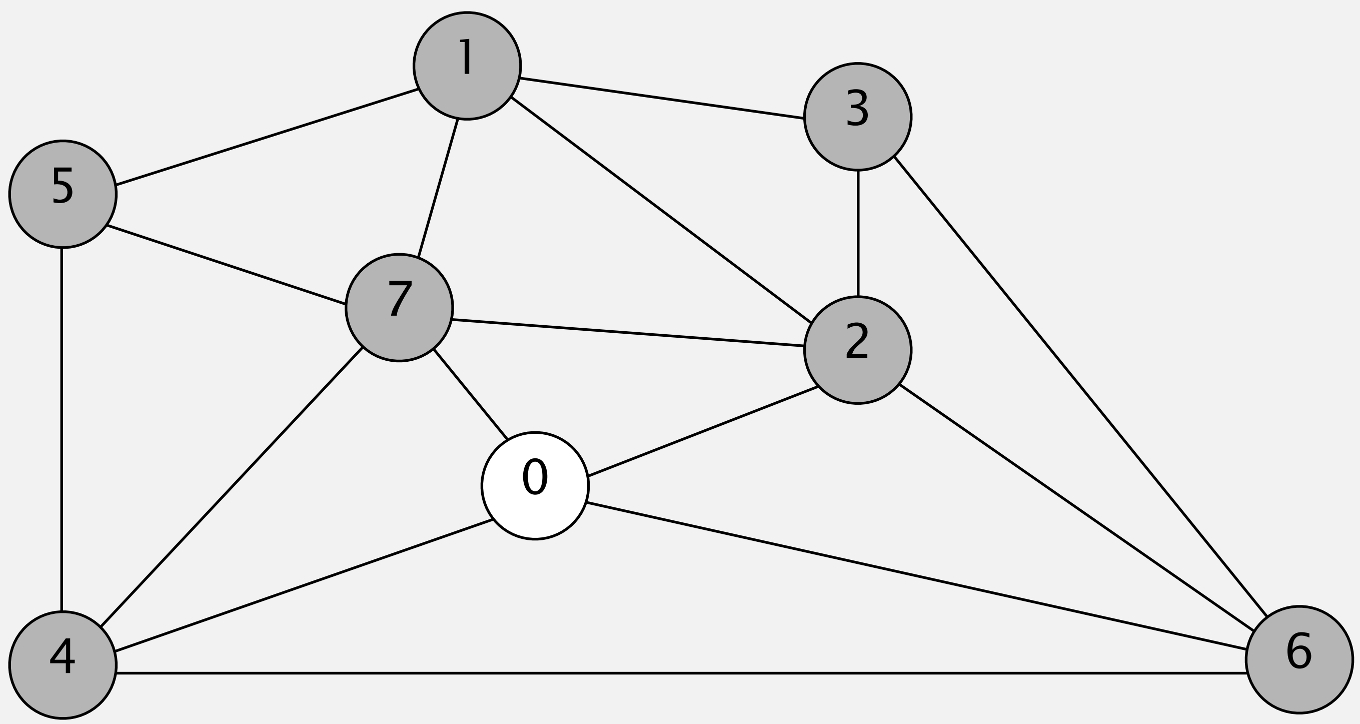
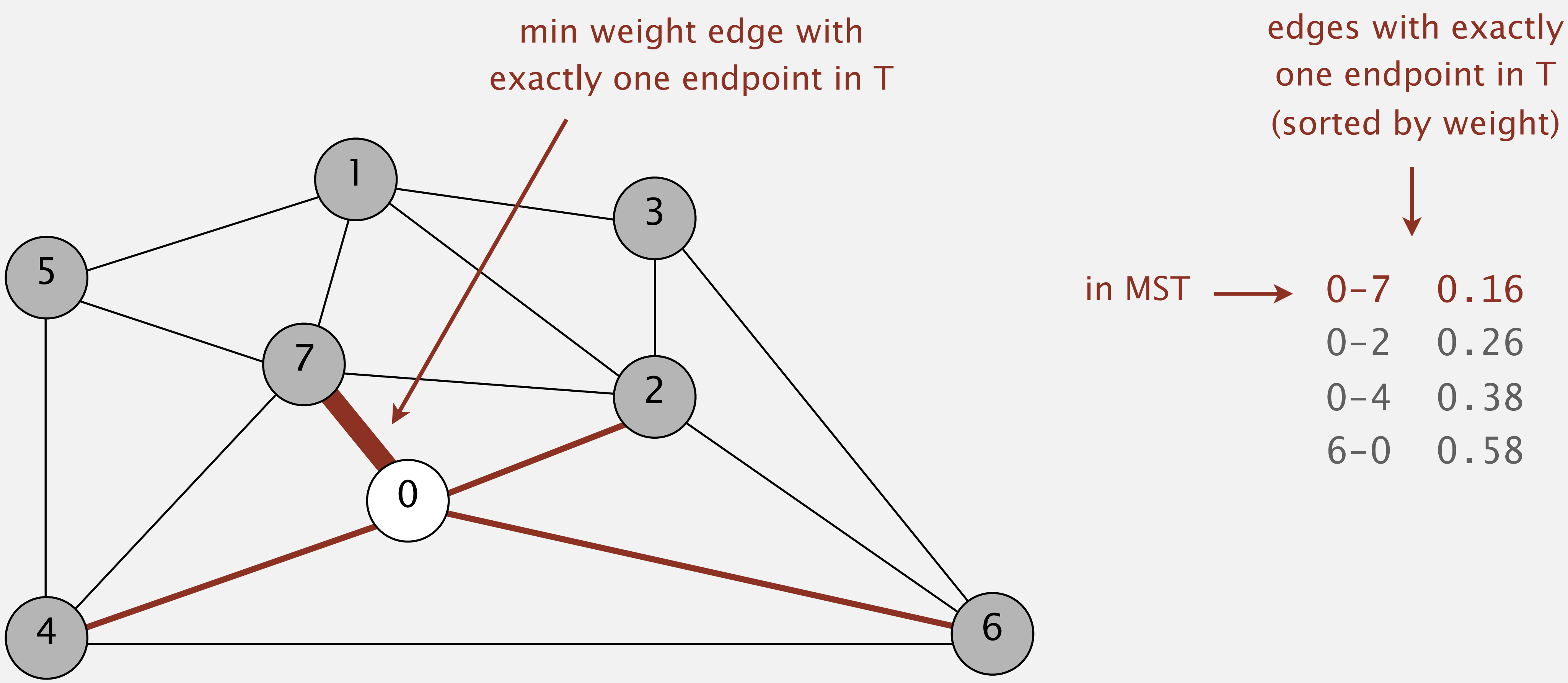
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



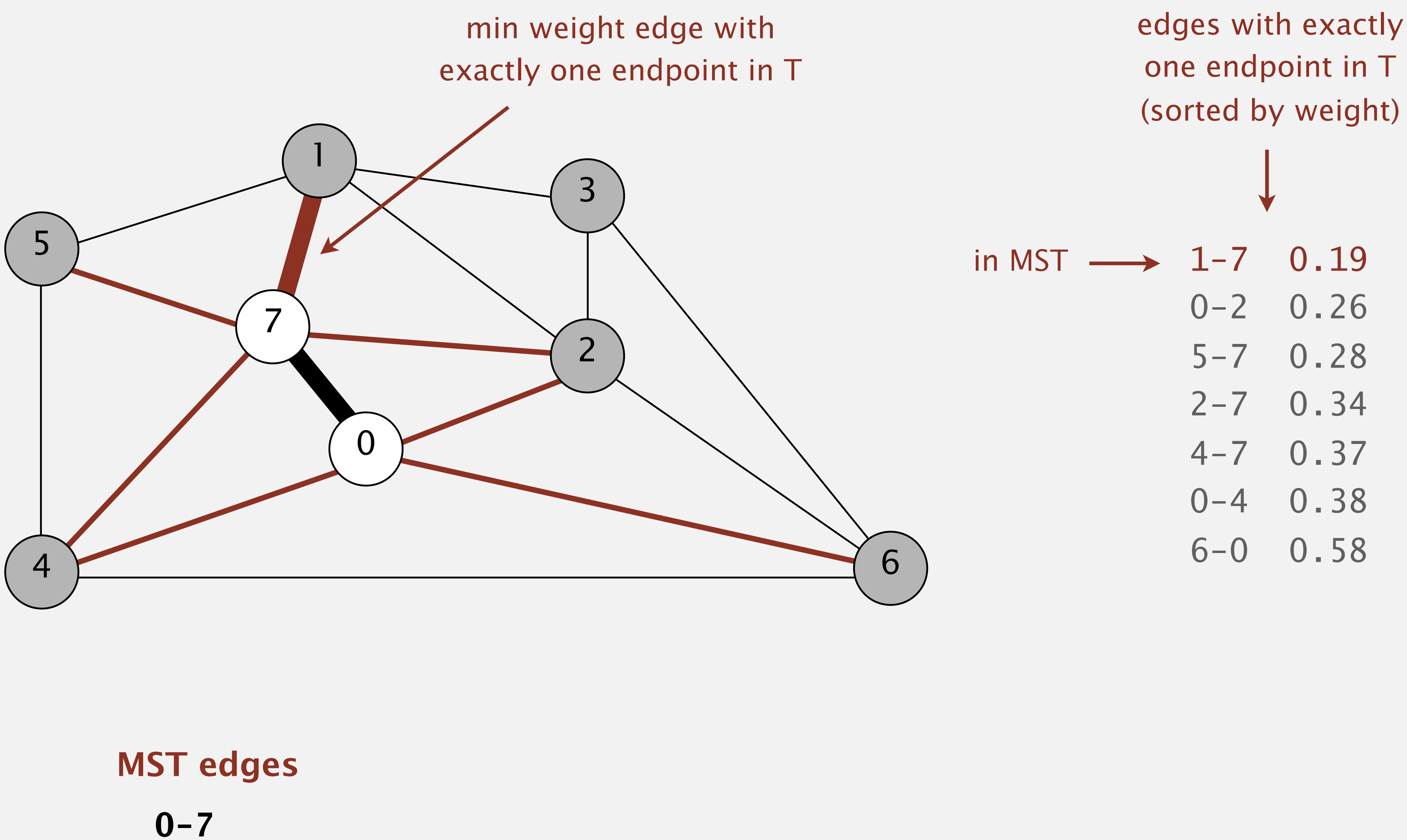
Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



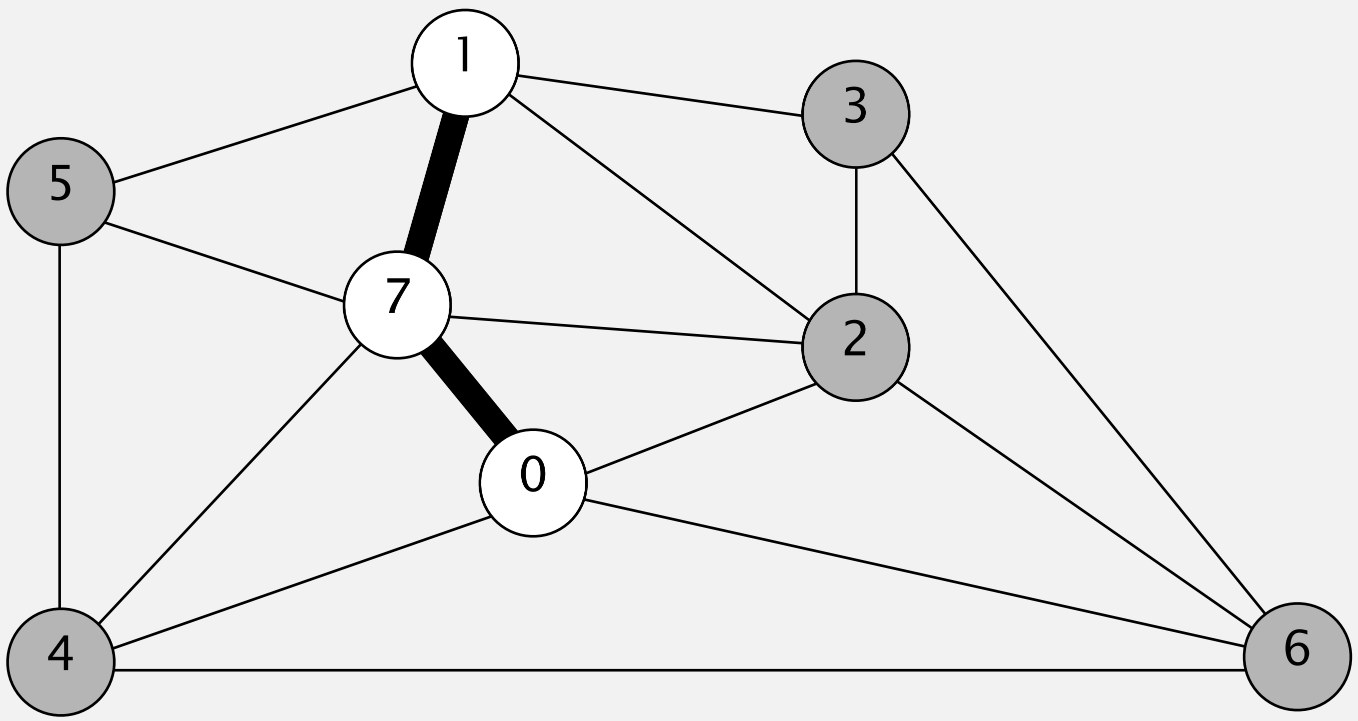
Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

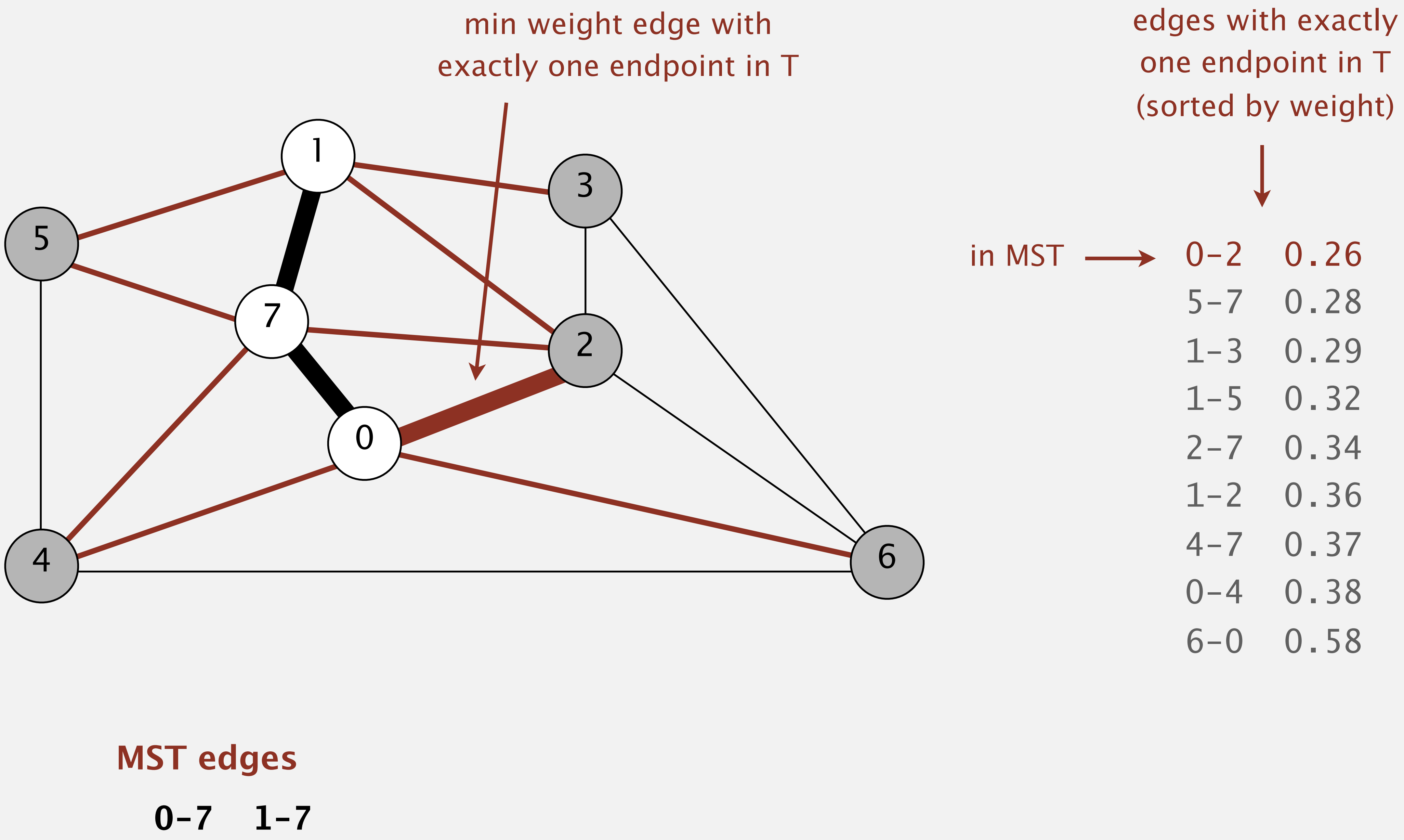


MST edges

0-7 1-7

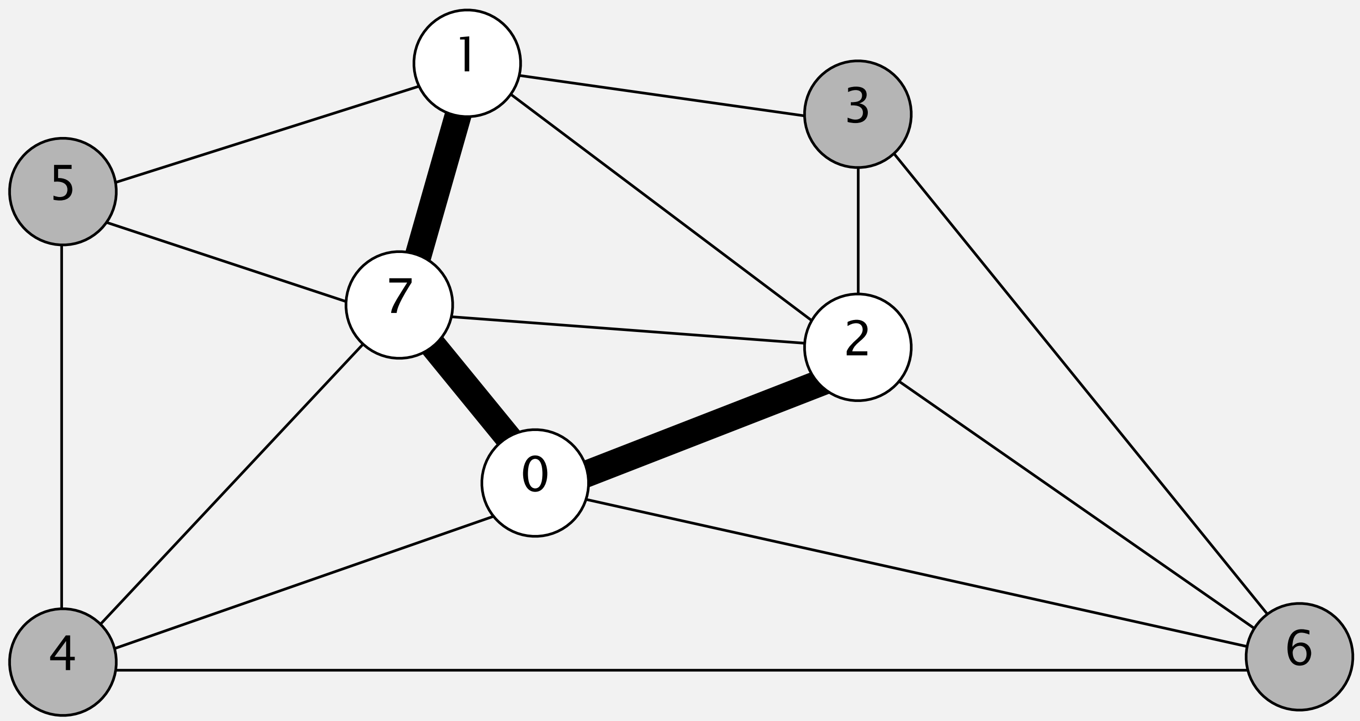
Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

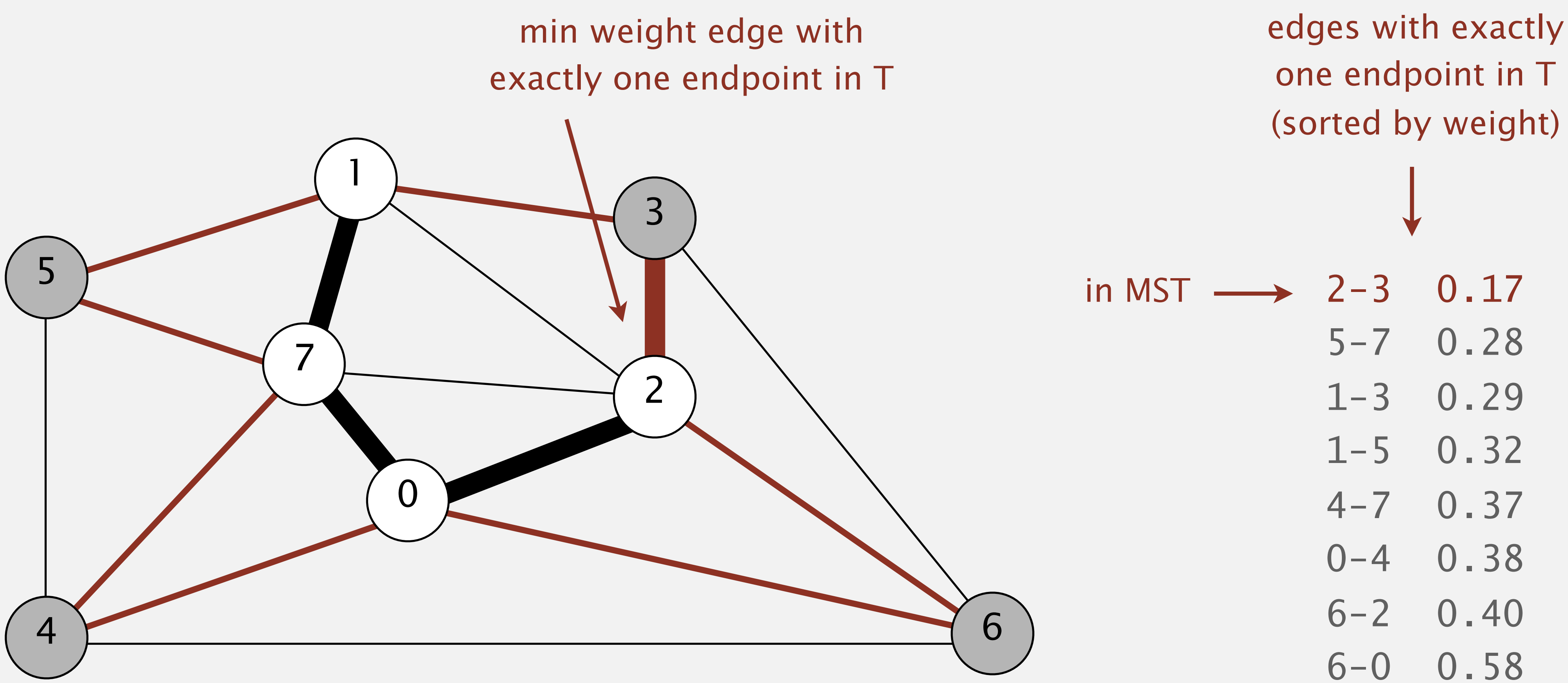


MST edges

0–7 1–7 0–2

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

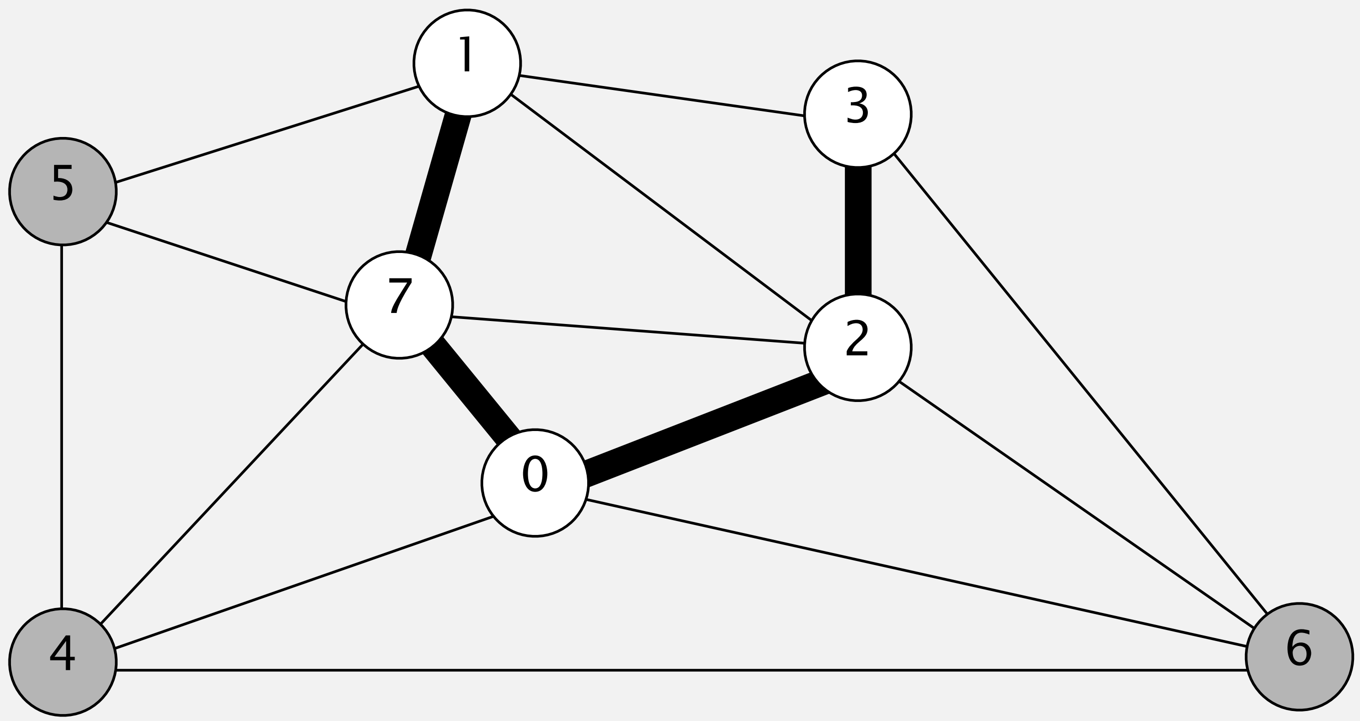


MST edges

0-7 1-7 0-2

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

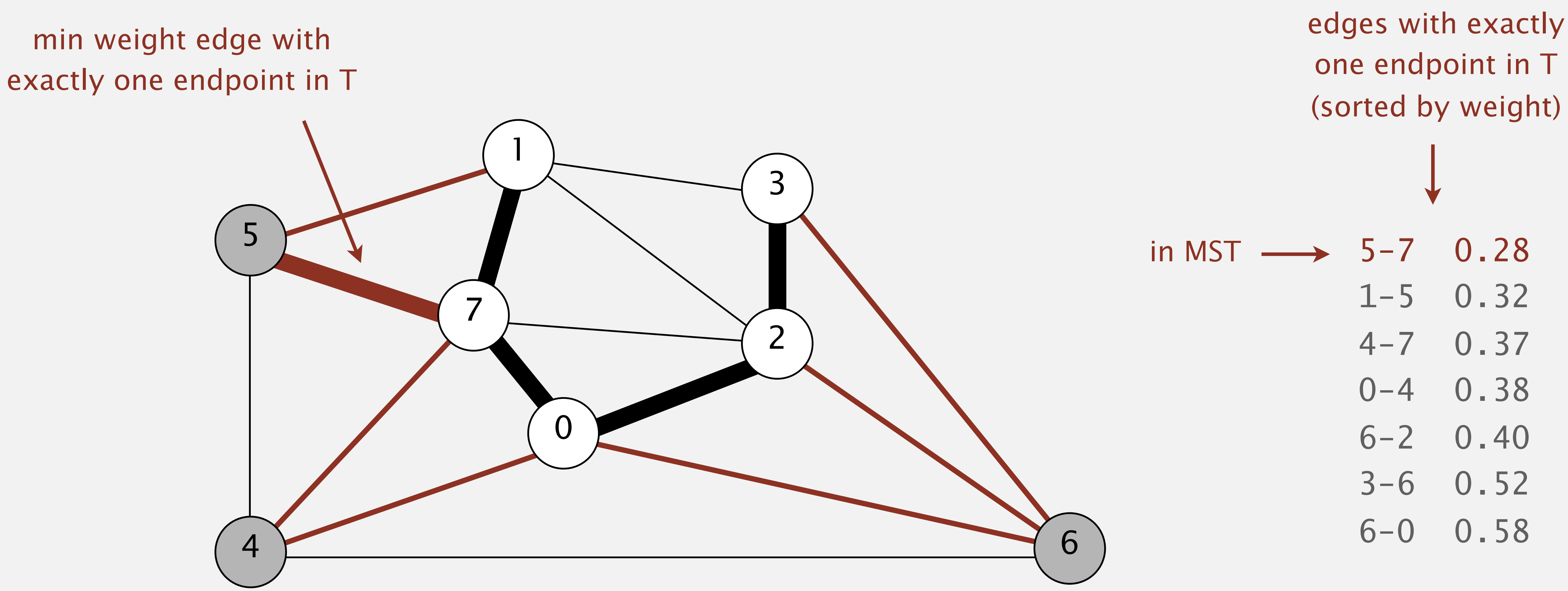


MST edges

0–7 1–7 0–2 2–3

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

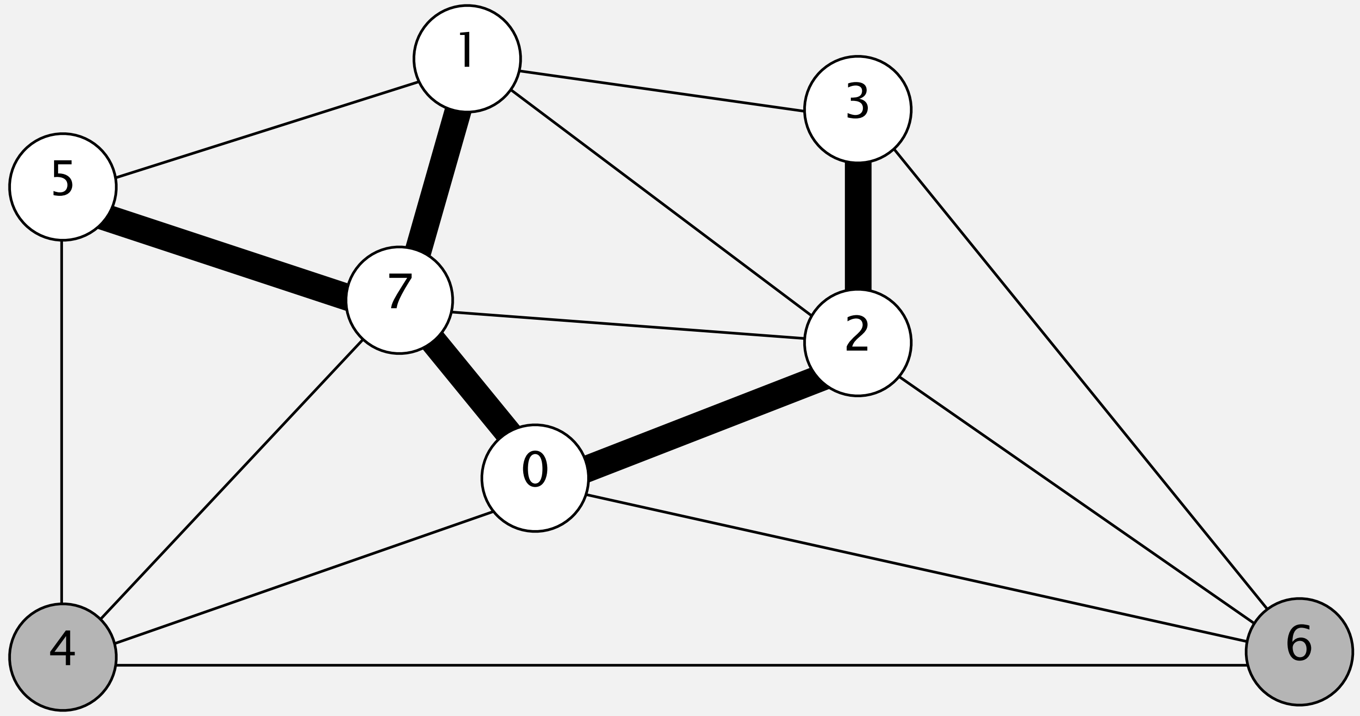


MST edges

0-7 1-7 0-2 2-3

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

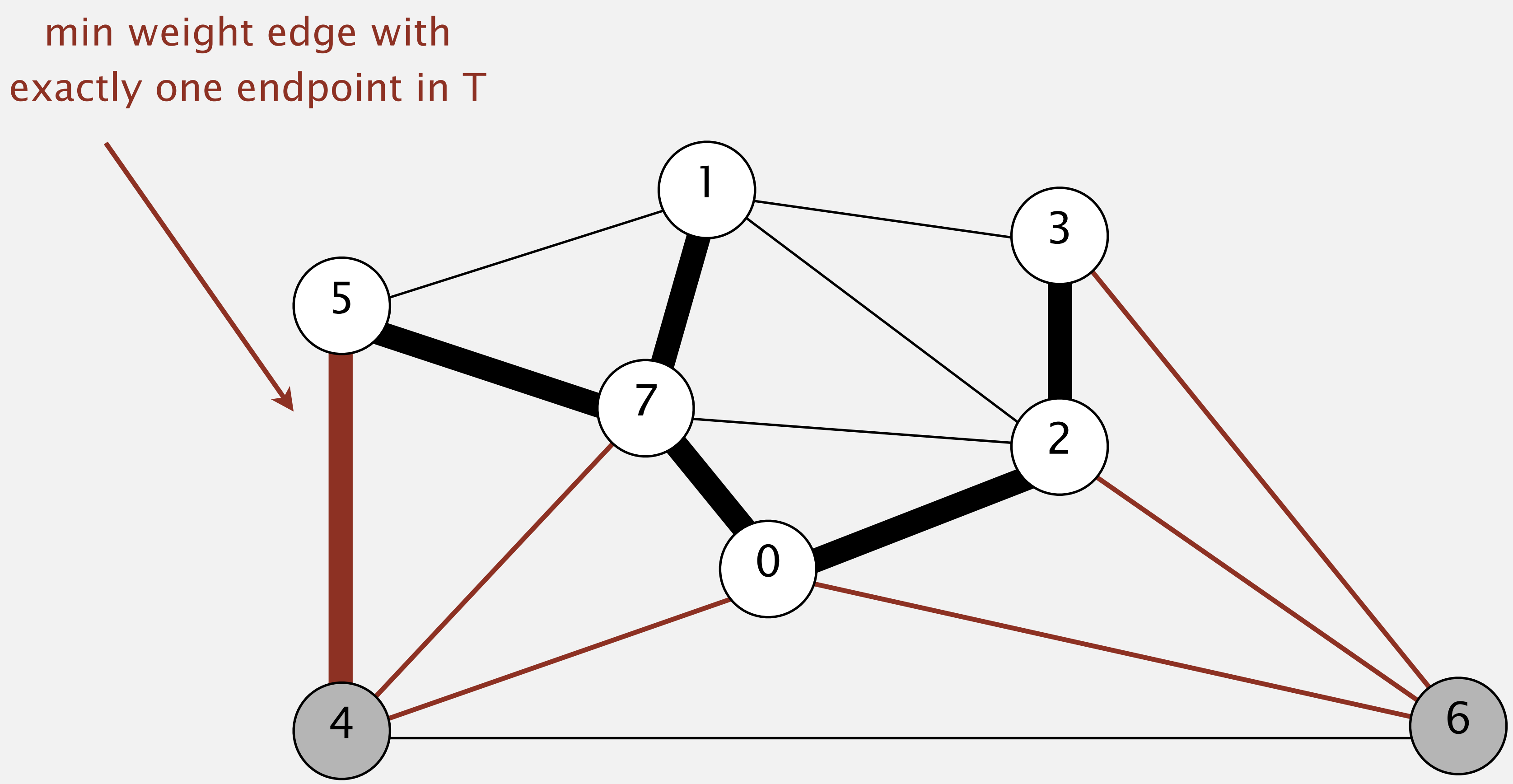


MST edges

0–7 1–7 0–2 2–3 5–7

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



edges with exactly one endpoint in T (sorted by weight)

in MST →

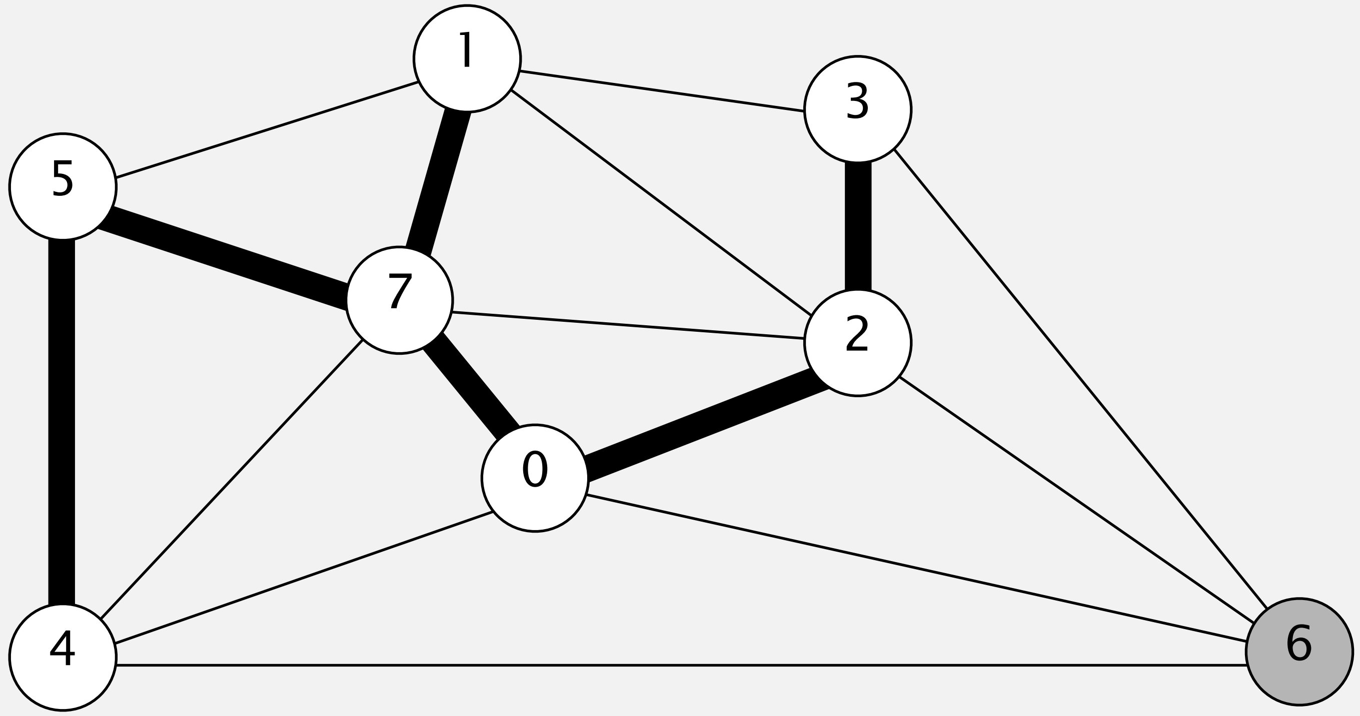
4-5	0.35
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58

MST edges

0-7 1-7 0-2 2-3 5-7

Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

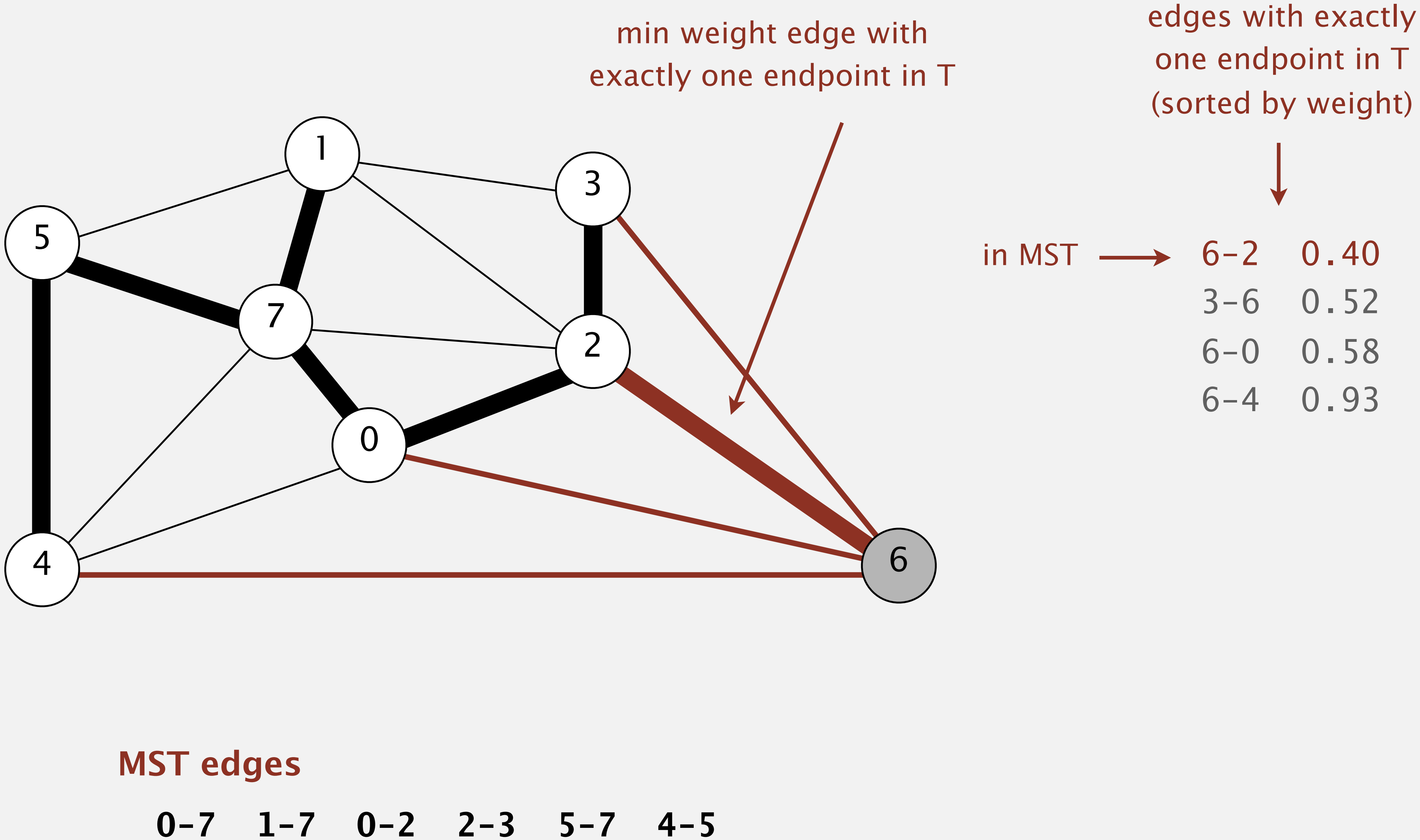


MST edges

0-7 1-7 0-2 2-3 5-7 4-5

Prim's algorithm demo

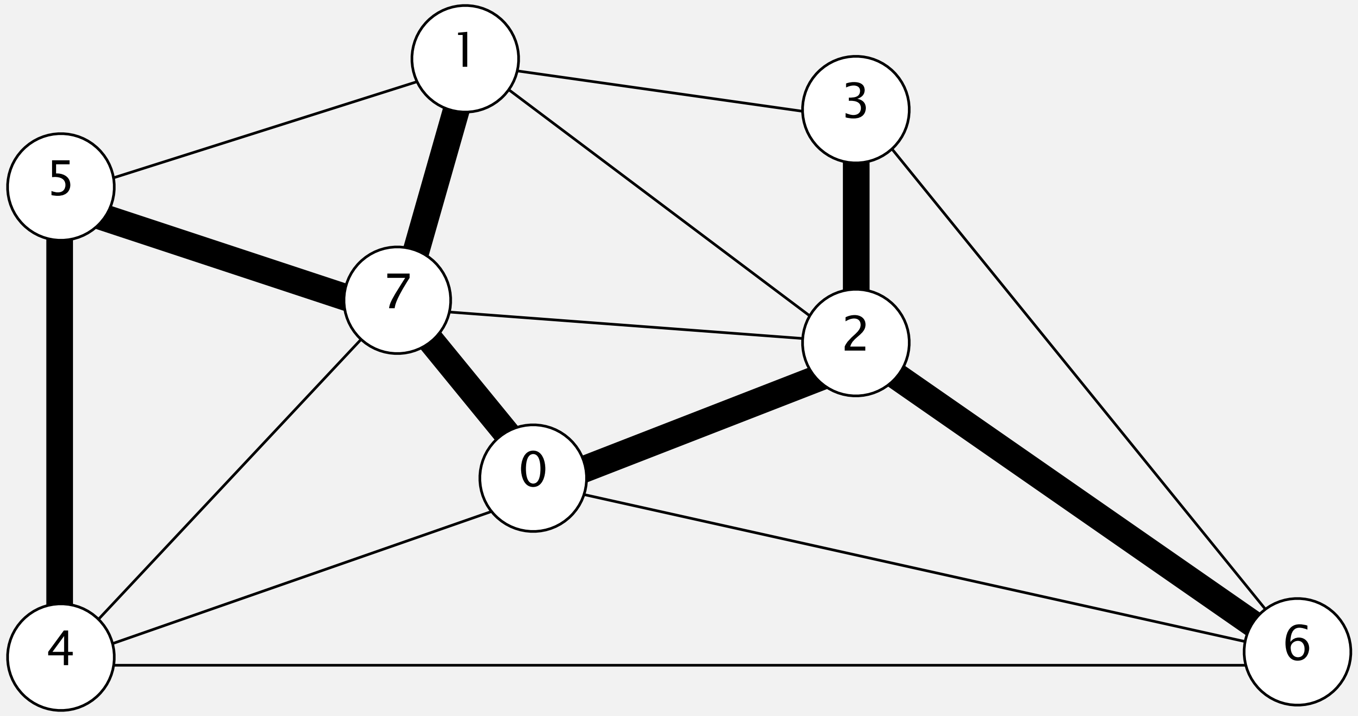
- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.



Prim's algorithm demo

- Start with vertex 0 and greedily grow tree T.
- Add to T the min weight edge with exactly one endpoint in T.
- Repeat until $V - 1$ edges.

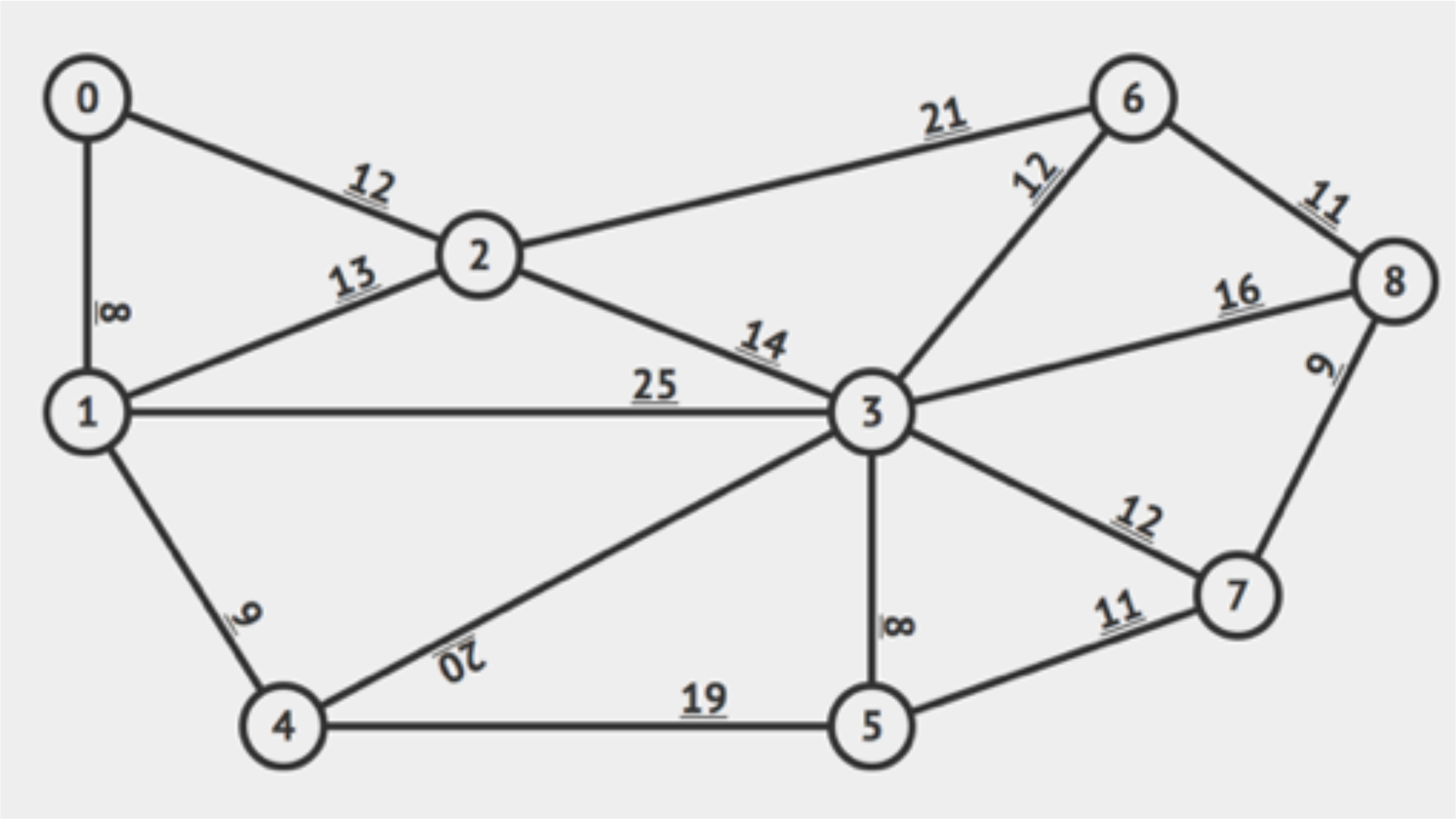
0-7	0.16
2-3	0.17
1-7	0.19
0-2	0.26
5-7	0.28
1-3	0.29
1-5	0.32
2-7	0.34
4-5	0.35
1-2	0.36
4-7	0.37
0-4	0.38
6-2	0.40
3-6	0.52
6-0	0.58
6-4	0.93



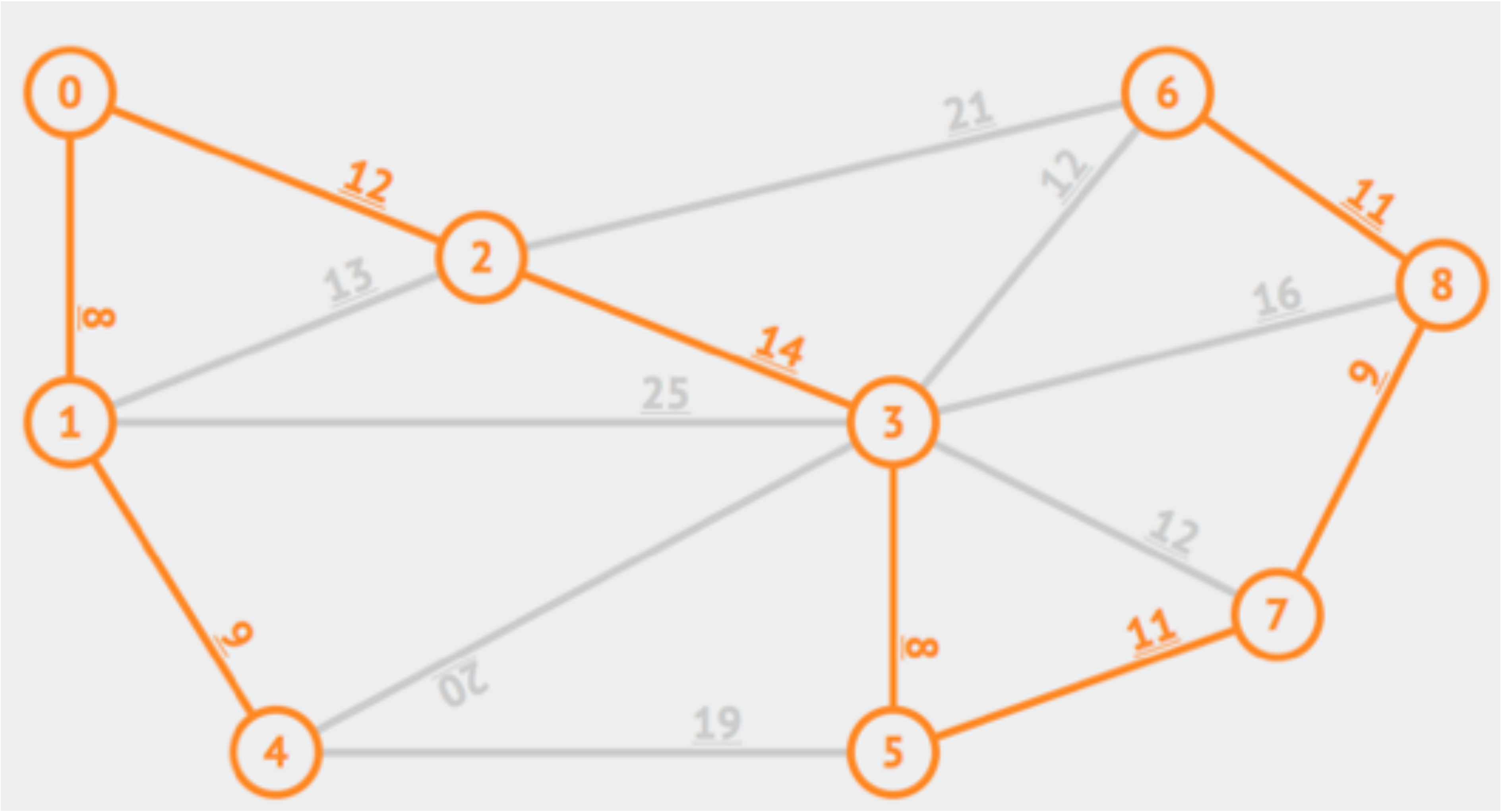
MST edges

0-7 1-7 0-2 2-3 5-7 4-5 6-2

Practice Time



Answer



Lecture 24: Minimum Spanning Trees

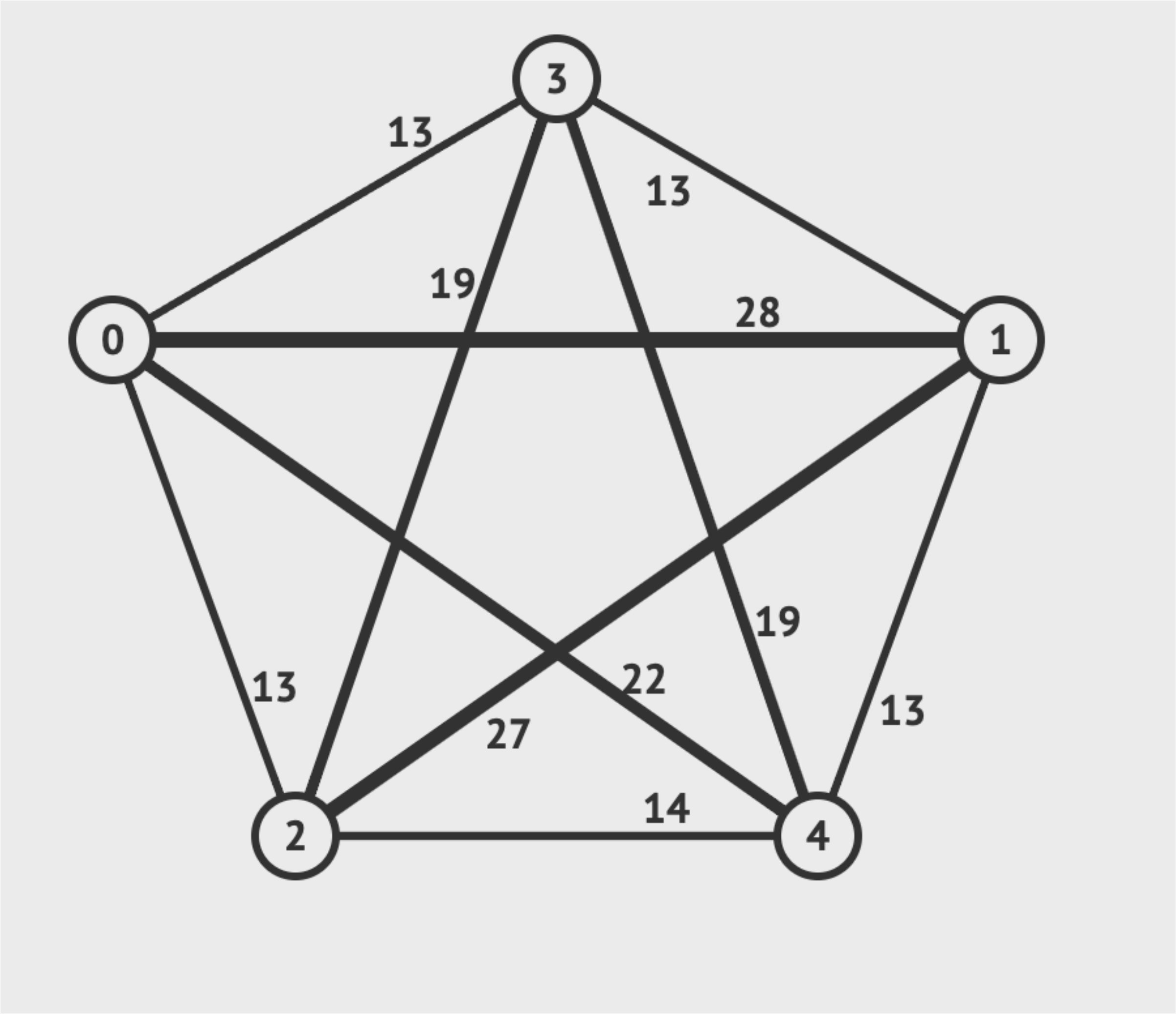
- Introduction
- Kruskal's Algorithm
- Prim's Algorithm

Readings:

- Recommended Textbook: Chapter 4.3 (Pages 604-629)
- Website:
  - https://algs4.cs.princeton.edu/43mst/
- Visualization:
  - https://visualgo.net/en/mst

Problem

- Run Kruskal's and Prim's algorithm (starting at index 0) on the following graph:



Problem

- Run Kruskal's and Prim's algorithm (starting at index 0) on the following graph.
- Both will provide the same MST: