CS062

DATA STRUCTURES AND ADVANCED PROGRAMMING

23: Shortest Paths

Alexandra Papoutsaki
she/her/hers

Lecture 23: Shortest Paths

- Introduction to Shortest Paths
- API
- Properties
- Dijkstra's Algorithm

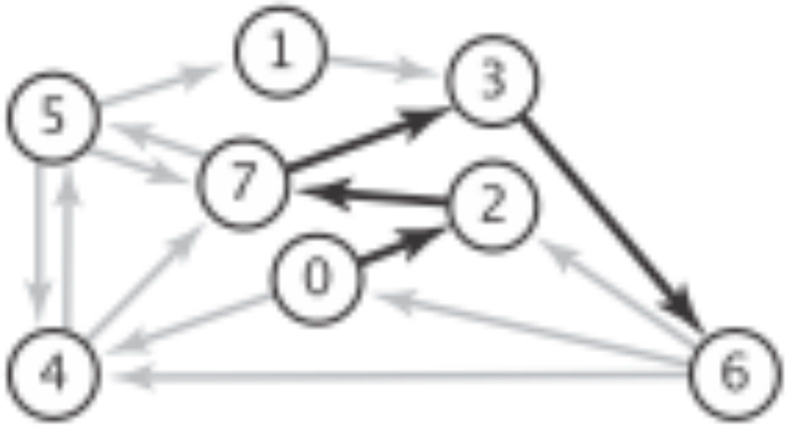
Some slides adopted from Algorithms 4th Edition or COS226

Edge-weighted digraph

- Edge-weighted digraph: a digraph where we associate weights or costs with each edge.

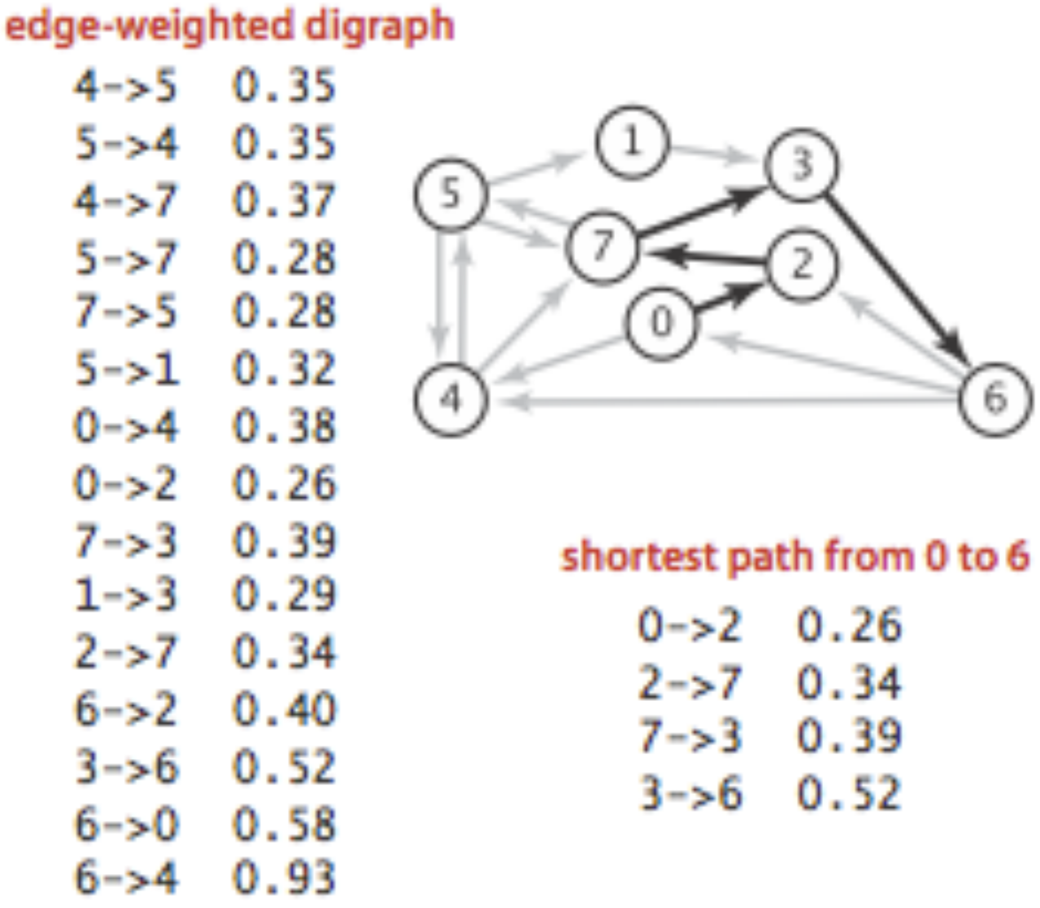
edge-weighted digraph

4->5	0.35
5->4	0.35
4->7	0.37
5->7	0.28
7->5	0.28
5->1	0.32
0->4	0.38
0->2	0.26
7->3	0.39
1->3	0.29
2->7	0.34
6->2	0.40
3->6	0.52
6->0	0.58
6->4	0.93



Shortest Paths

- Shortest path from vertex s to vertex t: a directed path from s to t with the property that no other such path has a lower weight (total weight sum of edges it consists).



edge-weighted digraph

4->5	0.35
5->4	0.35
4->7	0.37
5->7	0.28
7->5	0.28
5->1	0.32
0->4	0.38
0->2	0.26
7->3	0.39
1->3	0.29
2->7	0.34
6->2	0.40
3->6	0.52
6->0	0.58
6->4	0.93

shortest path from 0 to 6

0->2	0.26
2->7	0.34
7->3	0.39
3->6	0.52

An edge-weighted digraph and a shortest path

Shortest Path variants

- Single source: from one vertex s to every other vertex.
- Single sink: from every vertex to one vertex t.
- Source-sink: from one vertex s to another vertex t.
- All pairs: from every vertex to every other vertex.

- What version is there in your navigation app?

Shortest Paths Assumptions

- Not all vertices need to be reachable.
    - We will assume so in this lecture.
- Weights are non-negative.
    - There are algorithms that can handle negative weights.
- Shortest paths are not necessarily unique but they are simple.

Lecture 23: Shortest Paths

- Introduction to Shortest Paths
- API
- Properties
- Dijkstra's Algorithm

Weighted directed edge API

- `public class DirectedEdge`
    - `DirectedEdge(int v, int w, double weight)`
        - Constructs a weighted edge from `v` to `w` `(v->w)` with the provided `weight`.
    - `int from()`
        - Returns vertex source of this edge.
    - `int to()`
        - Returns vertex destination of this edge.
    - `double weight()`
        - Returns weight of this edge.
    - `String toString()`
        - Returns the string representation of this edge.

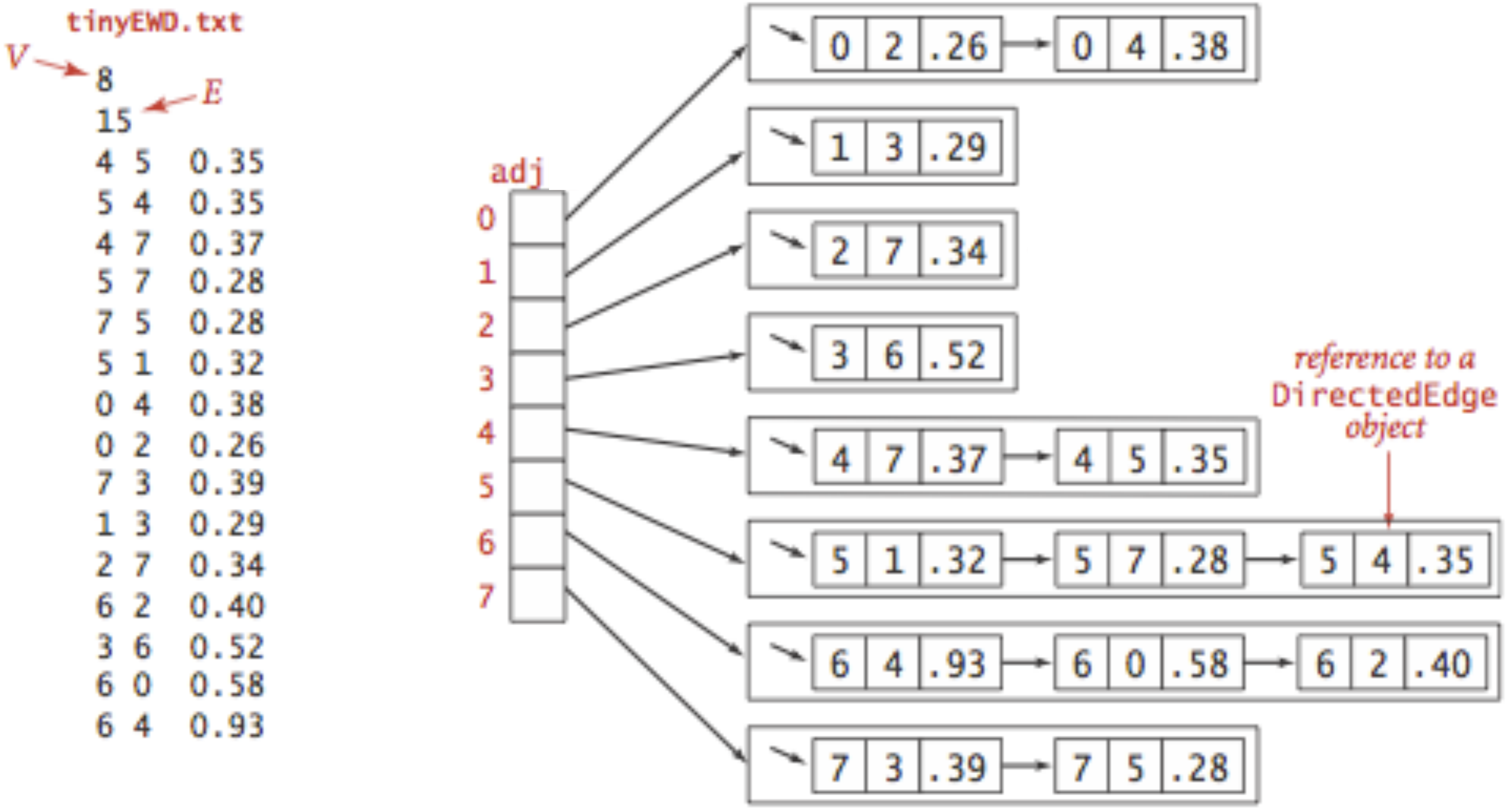
Weighted directed edge in Java

```java
public class DirectedEdge {
    private final int v;
    private final int w;
    private final double weight;

   public DirectedEdge(int v, int w, double weight) {
        this.v = v;
        this.w = w;
        this.weight = weight;
    }

   public int from() {
        return v;
    }

    public int to() {
        return w;
    }

    public double weight() {
        return weight;
    }
```

Edge-weighted digraph API

- `public class EdgeWeightedDigraph`
  - `EdgeWeightedDigraph(int v)`
    - Constructs an edge-weighted digraph with v vertices.
  - `void addEdge(DirectedEdge e)`
    - Add weighted directed edge e.
  - `Iterable<DirectedEdge> adj(int v)`
    - Returns edges adjacent from v.
  - `int V()`
    - Returns number of vertices.
  - `int E()`
    - Returns number of edges.
  - `Iterable<DirectedEdge> edges()`
    - Returns all edges.

Edge-weighted digraph adjacency list representation



tinyEWD.txt

```
8
15
4 5  0.35
5 4  0.35
4 7  0.37
5 7  0.28
7 5  0.28
5 1  0.32
0 4  0.38
0 2  0.26
7 3  0.39
1 3  0.29
2 7  0.34
6 2  0.40
3 6  0.52
6 0  0.58
6 4  0.93
```

Edge-weighted digraph representation

Edge-weighted digraph in Java

```
public class EdgeWeightedDigraph {
    private final int V;                // number of vertices in this digraph
    private int E;                      // number of edges in this digraph
    private ArrayList<ArrayList<DirectedEdge>> adj; // adj.get(v) = adjacency list for v

    public EdgeWeightedDigraph(int V) {
       this.V = V;
       this.E = 0;
       adj = new ArrayList<ArrayList<DirectedEdge>>(V);
        for (int v = 0; v < V; v++)
            adj.add(new ArrayList<DirectedEdge>());
    }
    public void addEdge(DirectedEdge e) {
        int v = e.from();
        int w = e.to();
        adj.get(v).add(e);
        E++;
    }

   public Iterable<DirectedEdge> adj(int v) {
       return adj.get(v);
    }
```

Single-source shortest path API

- Goal: find shortest path from `s` to every other vertex in the digraph.
- `public class SP`
  - `SP(EdgeWeightedDigraph G, int s)`
    - Shortest paths from `s` in digraph `G`.
  - `double distTo(int v)`
    - Length of shortest path from `s` to `v`.
  - `Iterable<DirectedEdge> pathTo(int v)`
    - Returns edges along the shortest path from `s` to `v`.
  - `boolean hasPathTo(int v)`
    - Returns whether there is a path from `s` to `v`.

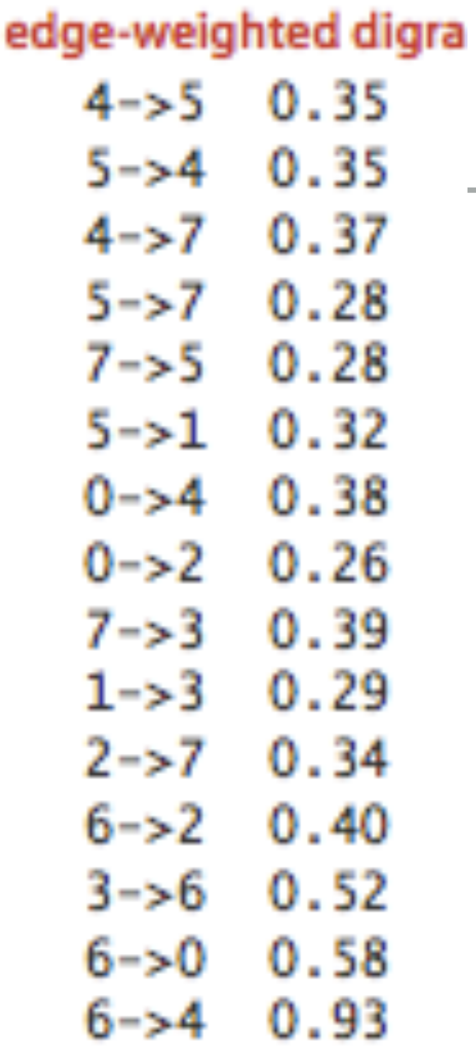
Lecture 23: Shortest Paths

- Introduction to Shortest Paths
- API
- Properties
- Dijkstra's Algorithm

Data structures for single-source shortest paths

- Goal: find shortest path from `s` to every other vertex in the digraph.
- Shortest-paths tree (SPT): a subgraph which will be a directed tree rooted at `s` which will contain all the vertices reachable from `s` and every tree path in the SPT is a shortest path in the digraph.
- Representation of shortest paths with two vertex-indexed arrays.
    - Edges on the shortest-paths tree: `edgeTo[v]` is the last edge on a shortest path from `s` to `v`.
    - Distance to the source: `distTo[v]` is the length of the shortest path from `s` to `v`.

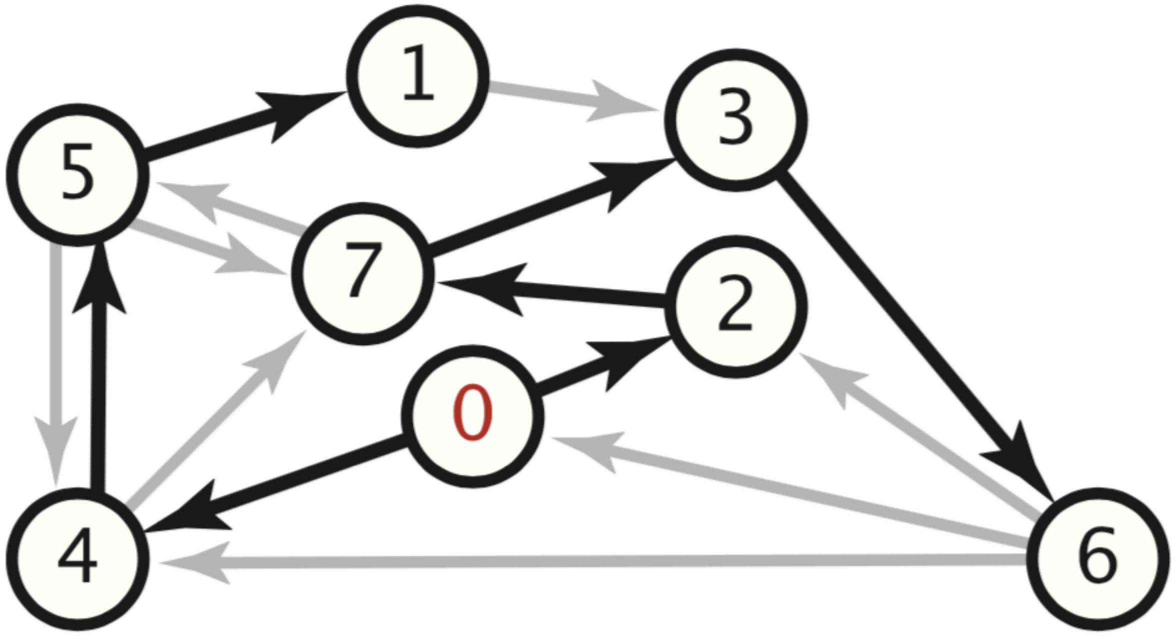
PROPERTIES

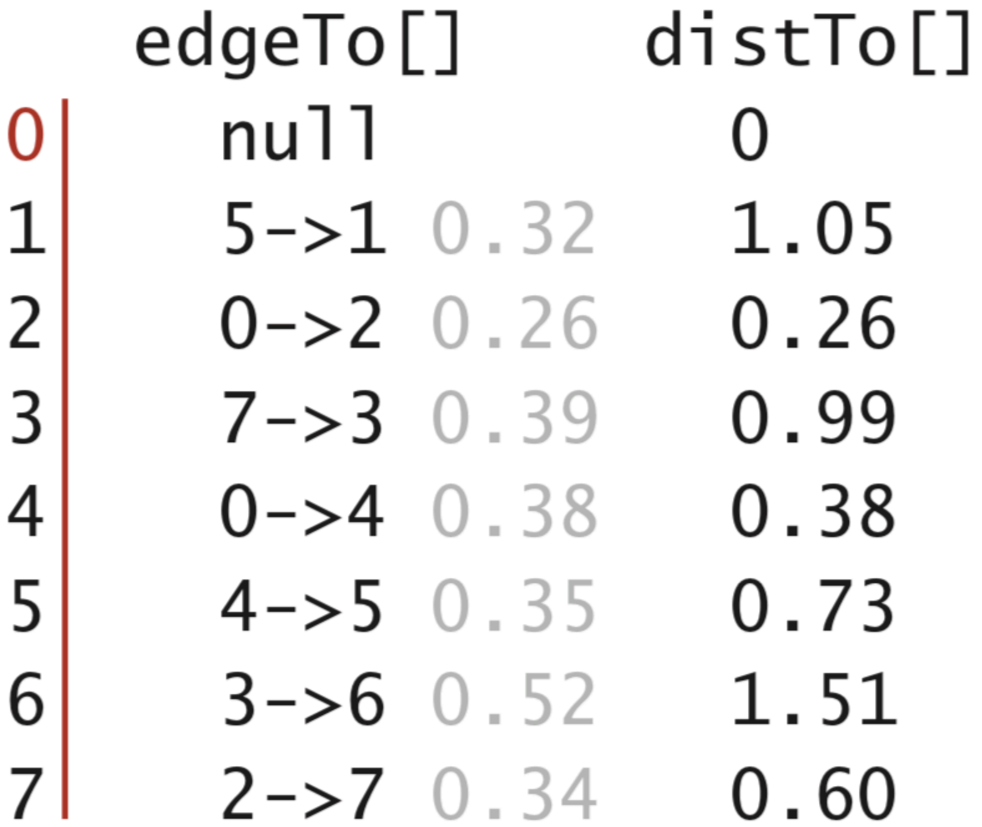


edge-weighted digra

4->5	0.35
5->4	0.35
4->7	0.37
5->7	0.28
7->5	0.28
5->1	0.32
0->4	0.38
0->2	0.26
7->3	0.39
1->3	0.29
2->7	0.34
6->2	0.40
3->6	0.52
6->0	0.58
6->4	0.93

```
public Iterable<DirectedEdge> pathTo(int v) {
       Stack<DirectedEdge> path = new Stack<DirectedEdge>();
       for (DirectedEdge e = edgeTo[v]; e != null; e = edgeTo[e.from()]) {
           path.push(e);
       }
       return path;
   }
```



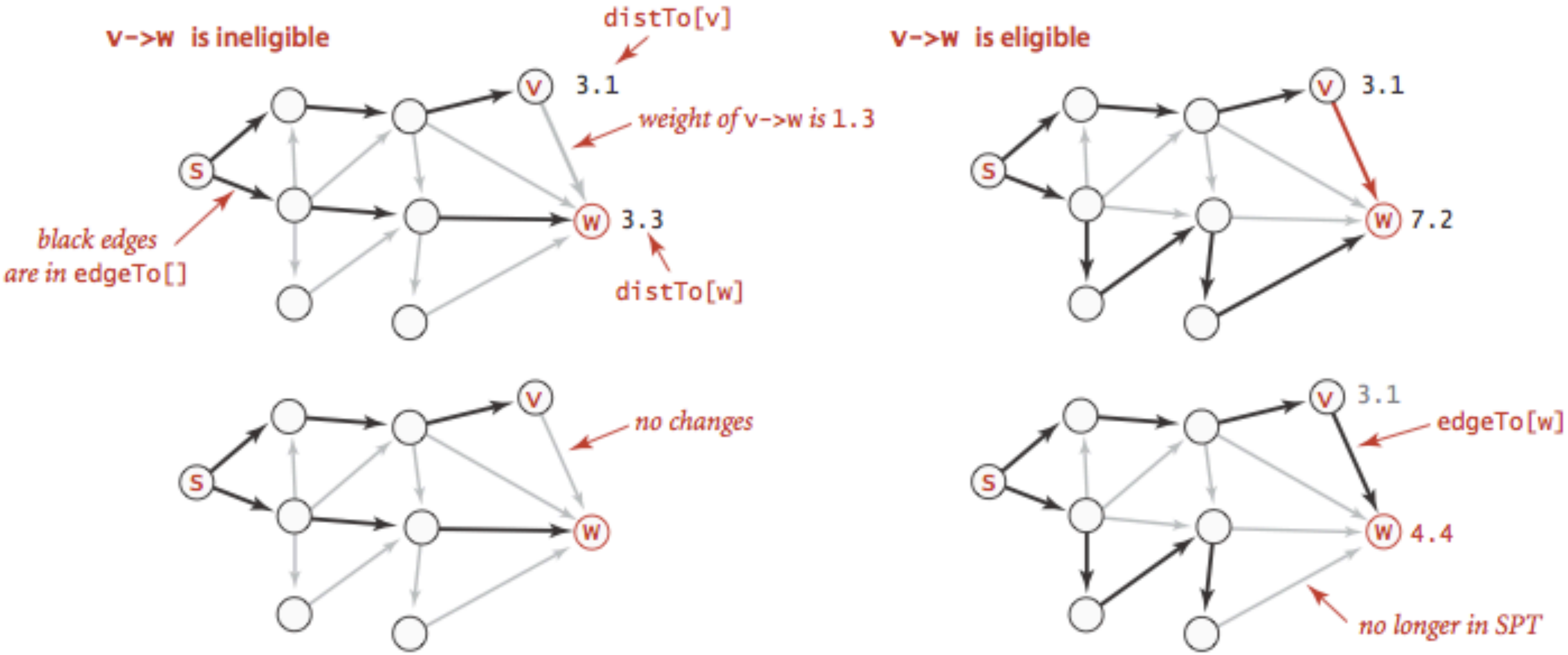


	edgeTo[]		distTo[]
0	null		0
1	5->1	0.32	1.05
2	0->2	0.26	0.26
3	7->3	0.39	0.99
4	0->4	0.38	0.38
5	4->5	0.35	0.73
6	3->6	0.52	1.51
7	2->7	0.34	0.60

Edge relaxation

- Relax edge `e = v->w`
  - `distTo[v]` is the length of the shortest **known** path from `s` to `v`.
  - `distTo[w]` is the length of the shortest **known** path from `s` to `w`.
  - `edgeTo[w]` is the last edge on shortest **known** path from `s` to `w`.
  - If `e = v->w` yields shorter path to `w`, update `distTo[w]` and `edgeTo[w]`.

Edge relaxation



Edge relaxation implementation

```
private void relax(DirectedEdge e) {
    int v = e.from(), w = e.to();
    if (distTo[w] > distTo[v] + e.weight()) {
       distTo[w] = distTo[v] + e.weight();
       edgeTo[w] = e;
    }
}
```

Framework for shortest-paths algorithm

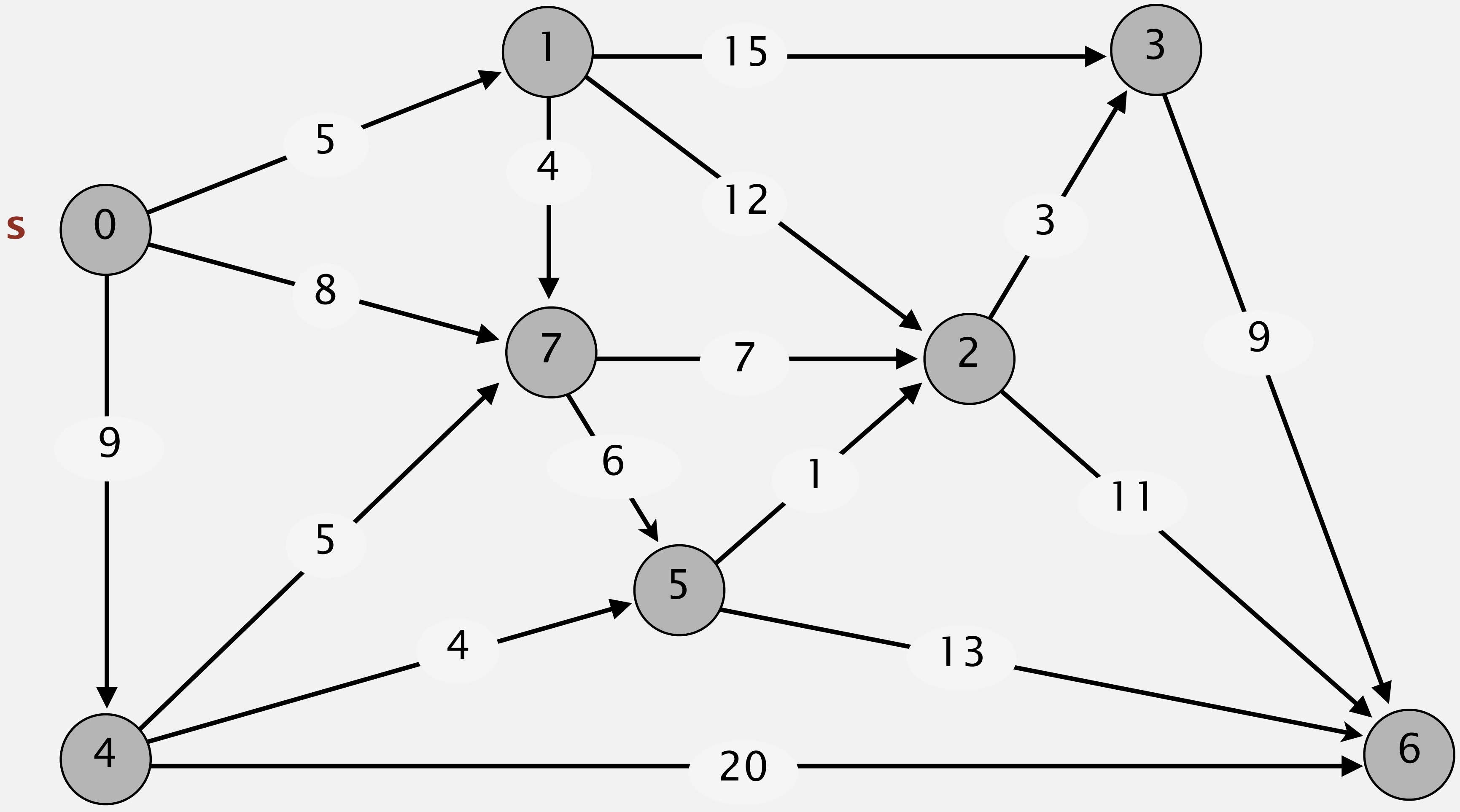
- Generic algorithm to compute a SPT from s
  - distTo[v]=∞ for each vertex v.
  - edgeTo[v]=null for each vertex v.
  - distTo[s]=0.
  - Repeat until done:
    - Relax any edge.
- distTo[v] is the length of a simple path from s to v.
- distTo[v] does not increase.

Lecture 23: Shortest Paths

- Introduction to Shortest Paths
- API
- Properties
- Dijkstra's Algorithm

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges adjacent from that vertex.



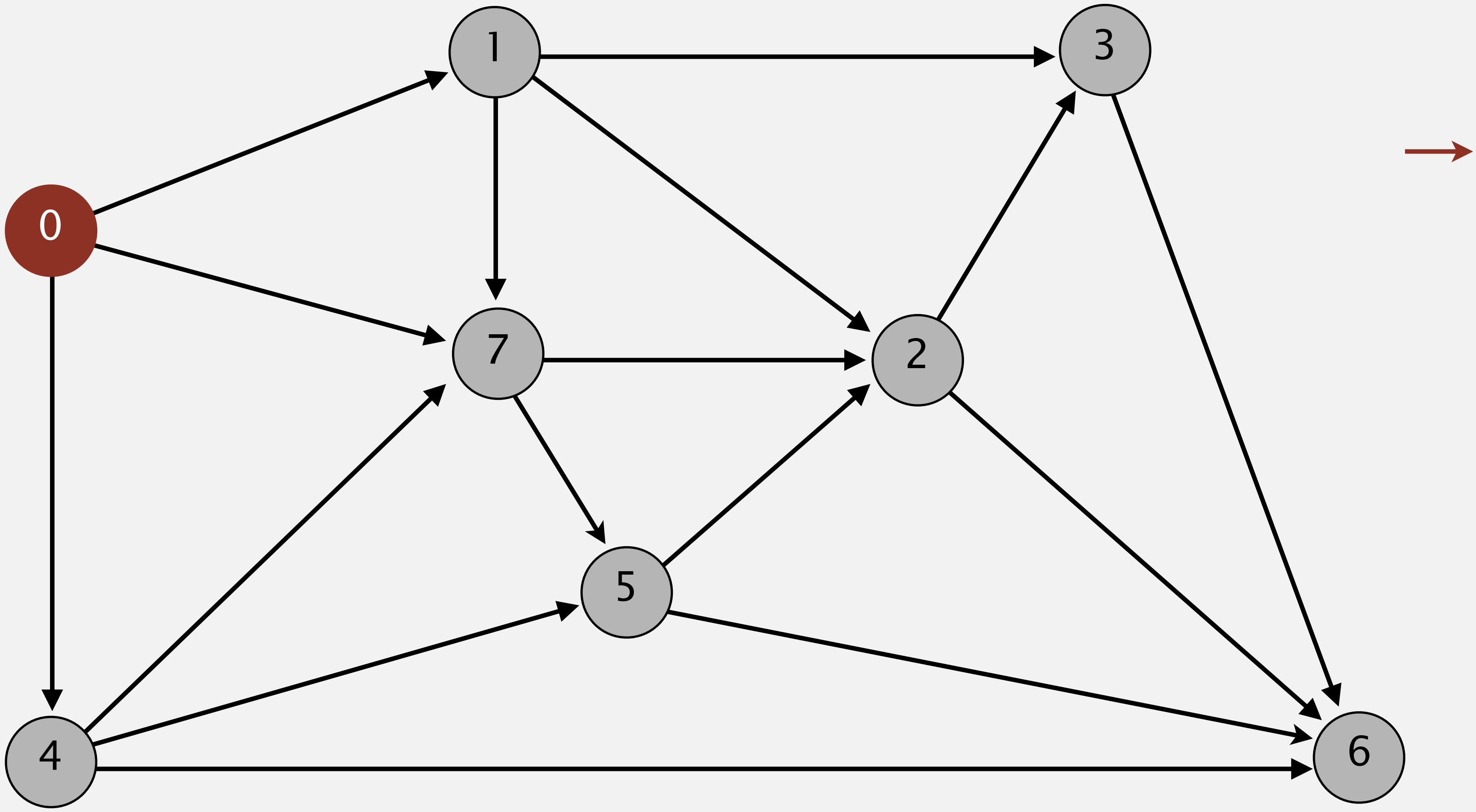
an edge-weighted digraph

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges adjacent from that vertex.



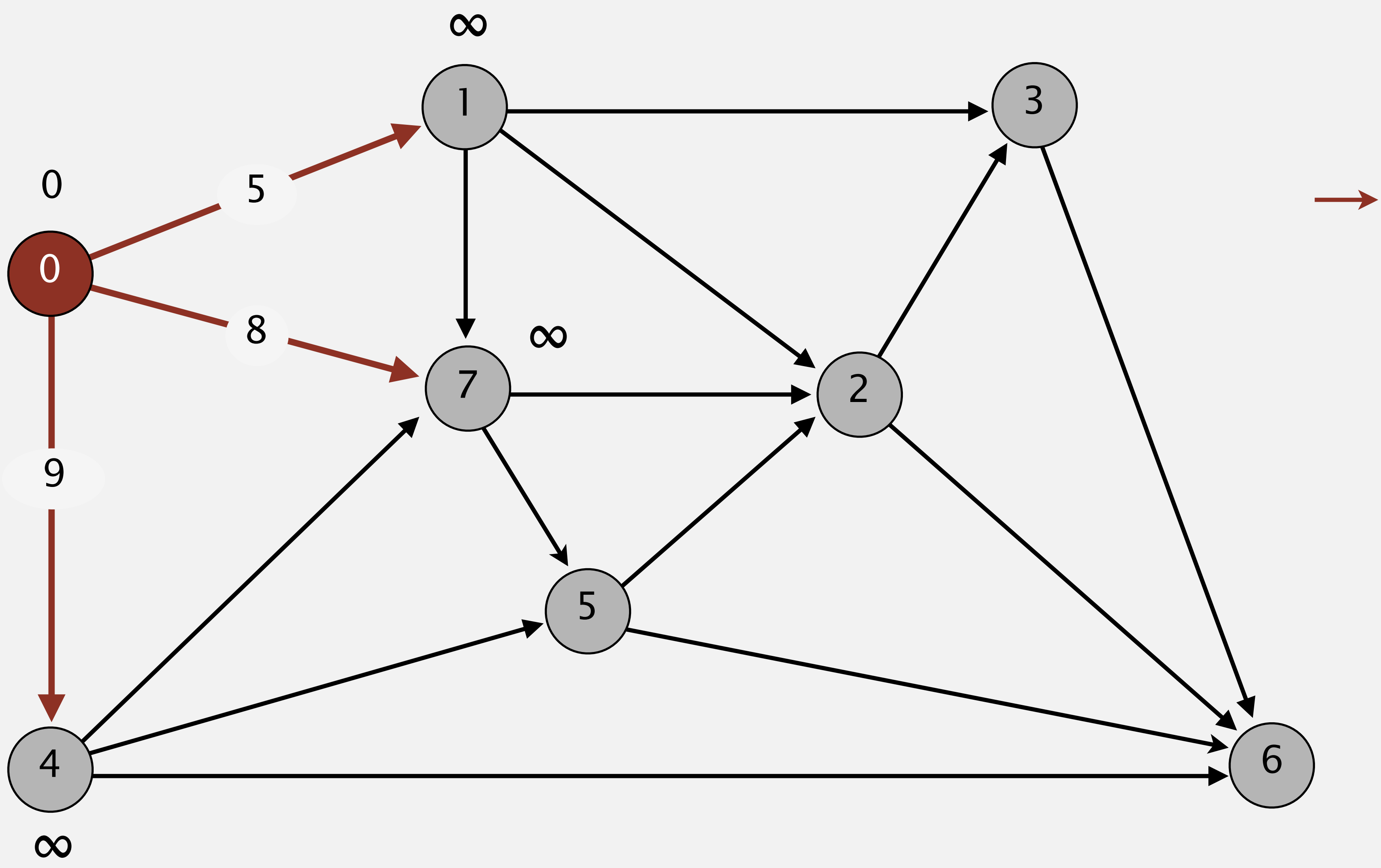
v	distTo[]	edgeTo[]
→ 0	0.0	-
1		
2		
3		
4		
5		
6		
7		

choose source vertex 0

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges adjacent from that vertex.



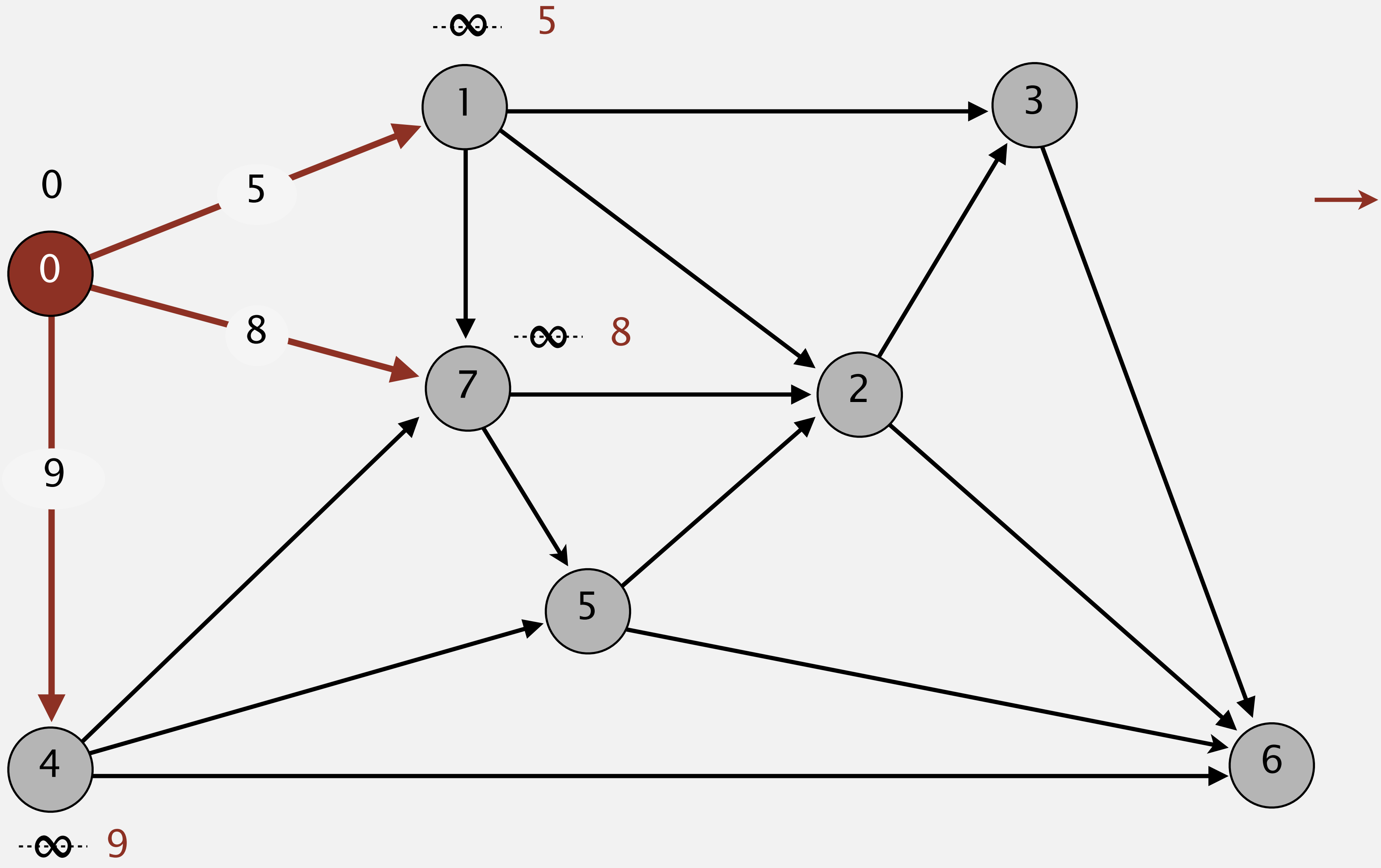
v	distTo[]	edgeTo[]
0	0.0	-
1		
2		
3		
4		
5		
6		
7		

relax all edges adjacent from 0

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges adjacent from that vertex.



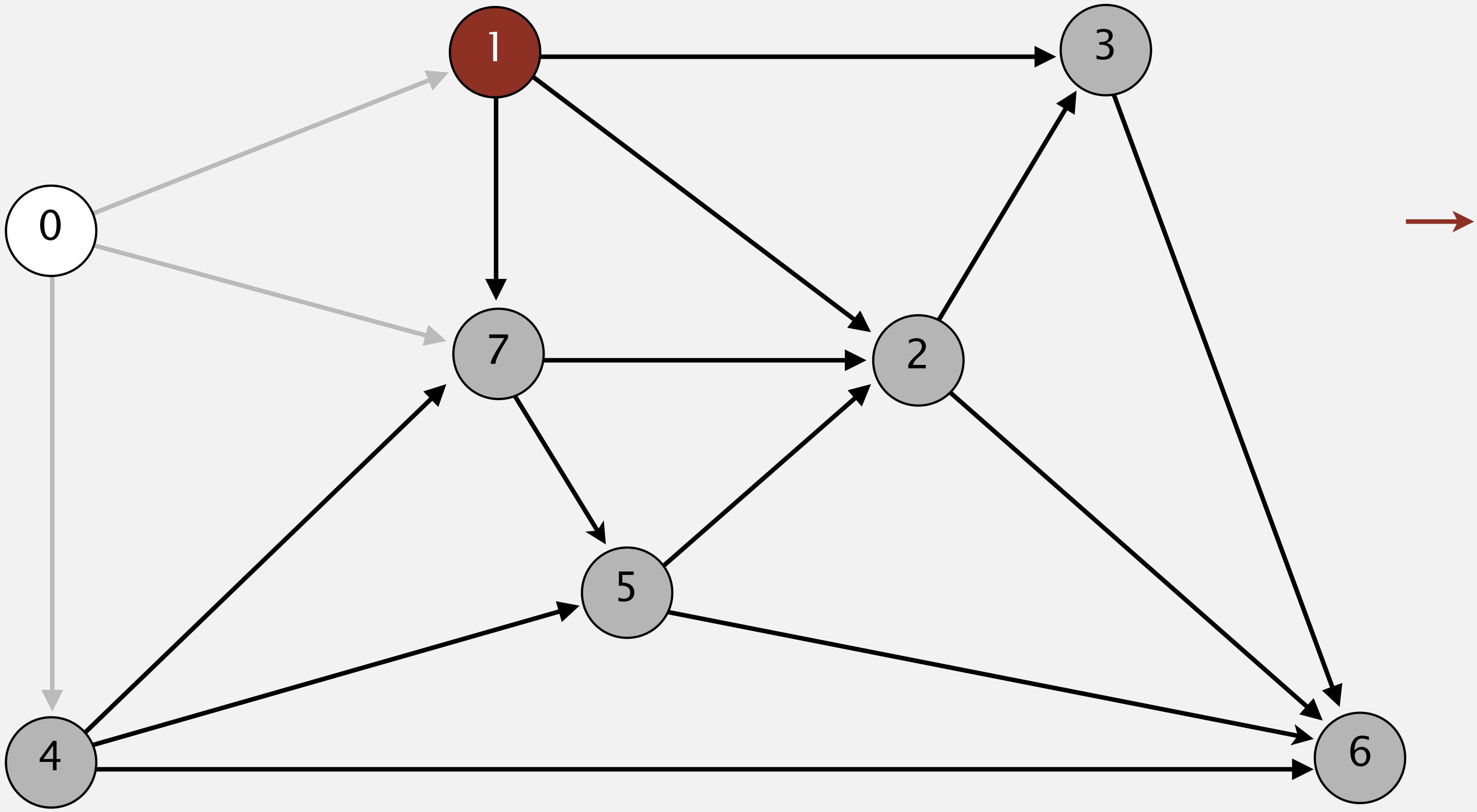
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2		
3		
4	9.0	0→4
5		
6		
7	8.0	0→7

relax all edges adjacent from 0

Dijkstra's algorithm demo

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges adjacent from that vertex.



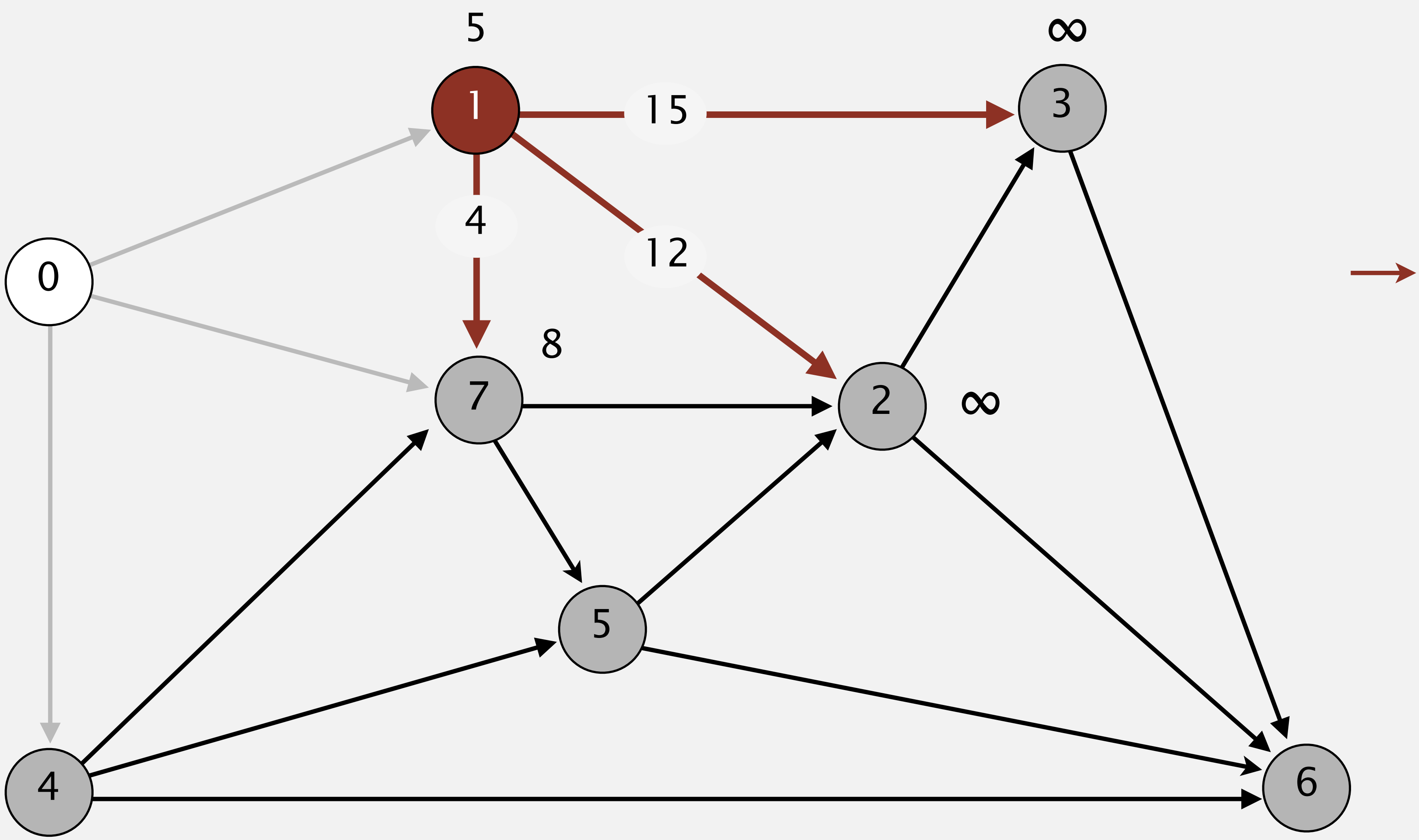
	v	distTo[]	edgeTo[]
	0	0.0	–
⟶	1	5.0	0→1
	2		
	3		
	4	9.0	0→4
	5		
	6		
	7	8.0	0→7

choose vertex 1

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges adjacent from that vertex.



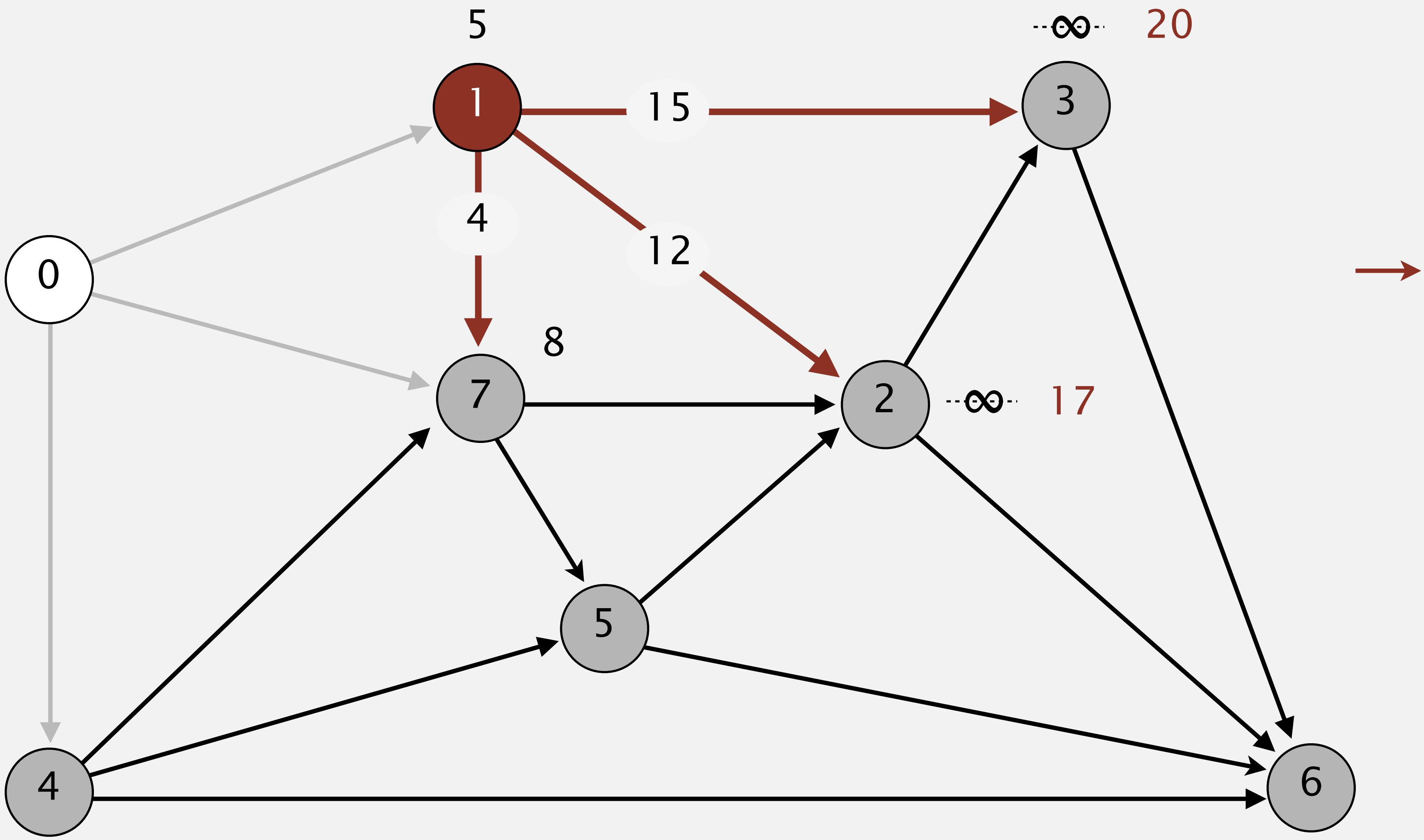
	v	distTo[]	edgeTo[]
	0	0.0	–
→	1	5.0	0→1
	2		
	3		
	4	9.0	0→4
	5		
	6		
	7	8.0	0→7

relax all edges adjacent from 1

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges adjacent from that vertex.



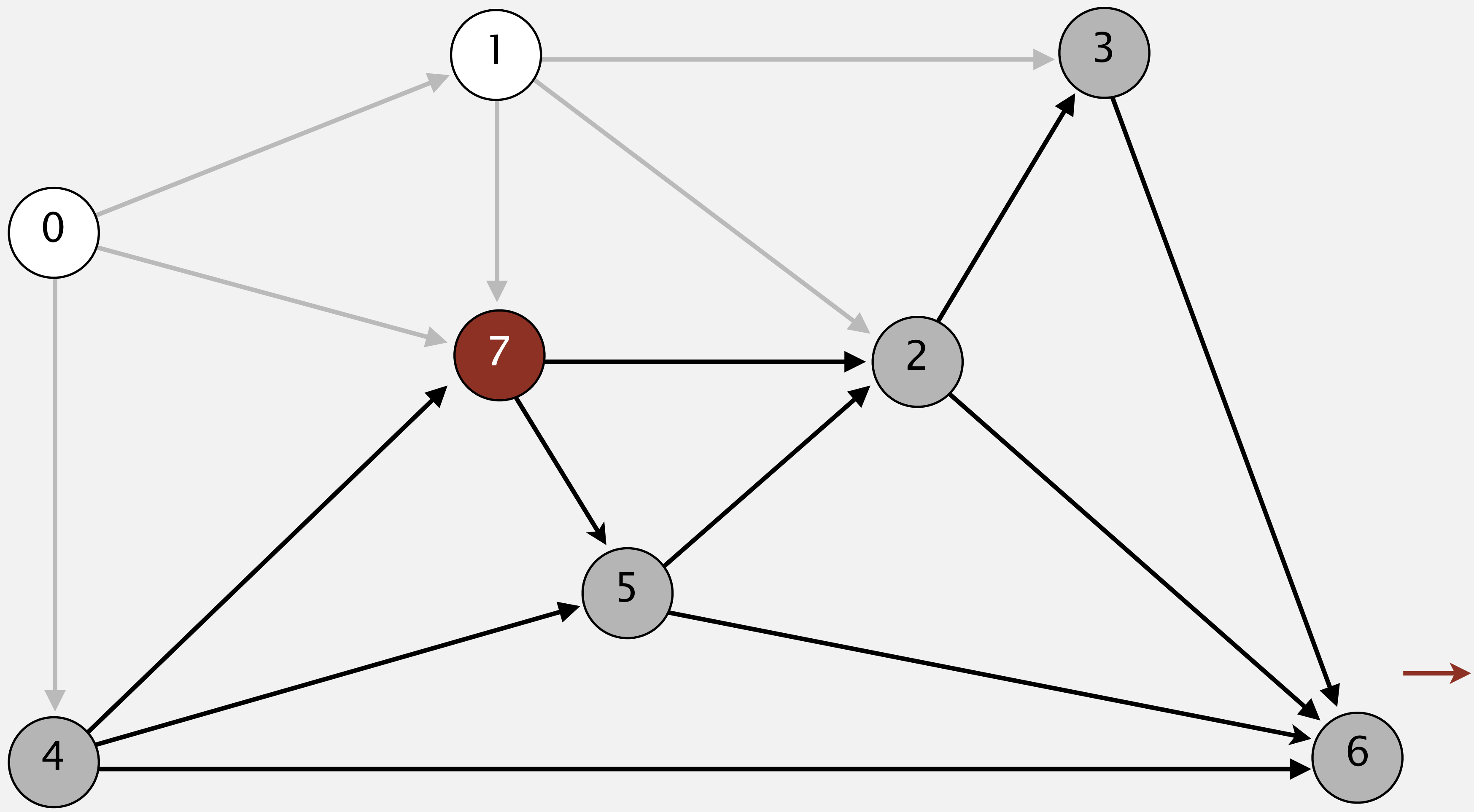
	v	distTo[]	edgeTo[]
	0	0.0	–
→	1	5.0	0→1
	2	17.0	1→2
	3	20.0	1→3
	4	9.0	0→4
	5		
	6		
	7	8.0 ✓	0→7

relax all edges adjacent from 1

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges adjacent from that vertex.



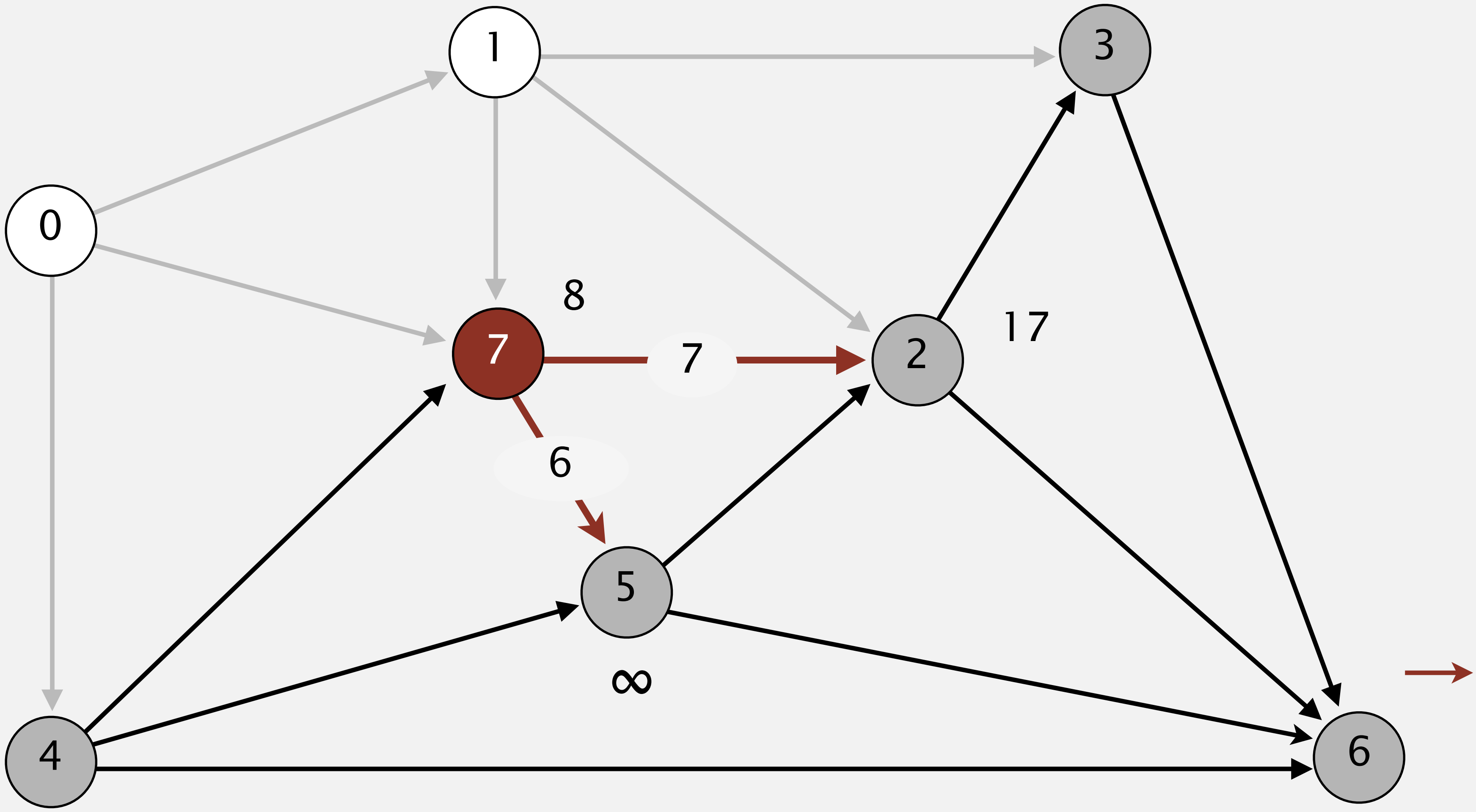
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
7	8.0	0→7

choose vertex 7

Dijkstra's algorithm demo

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges adjacent from that vertex.



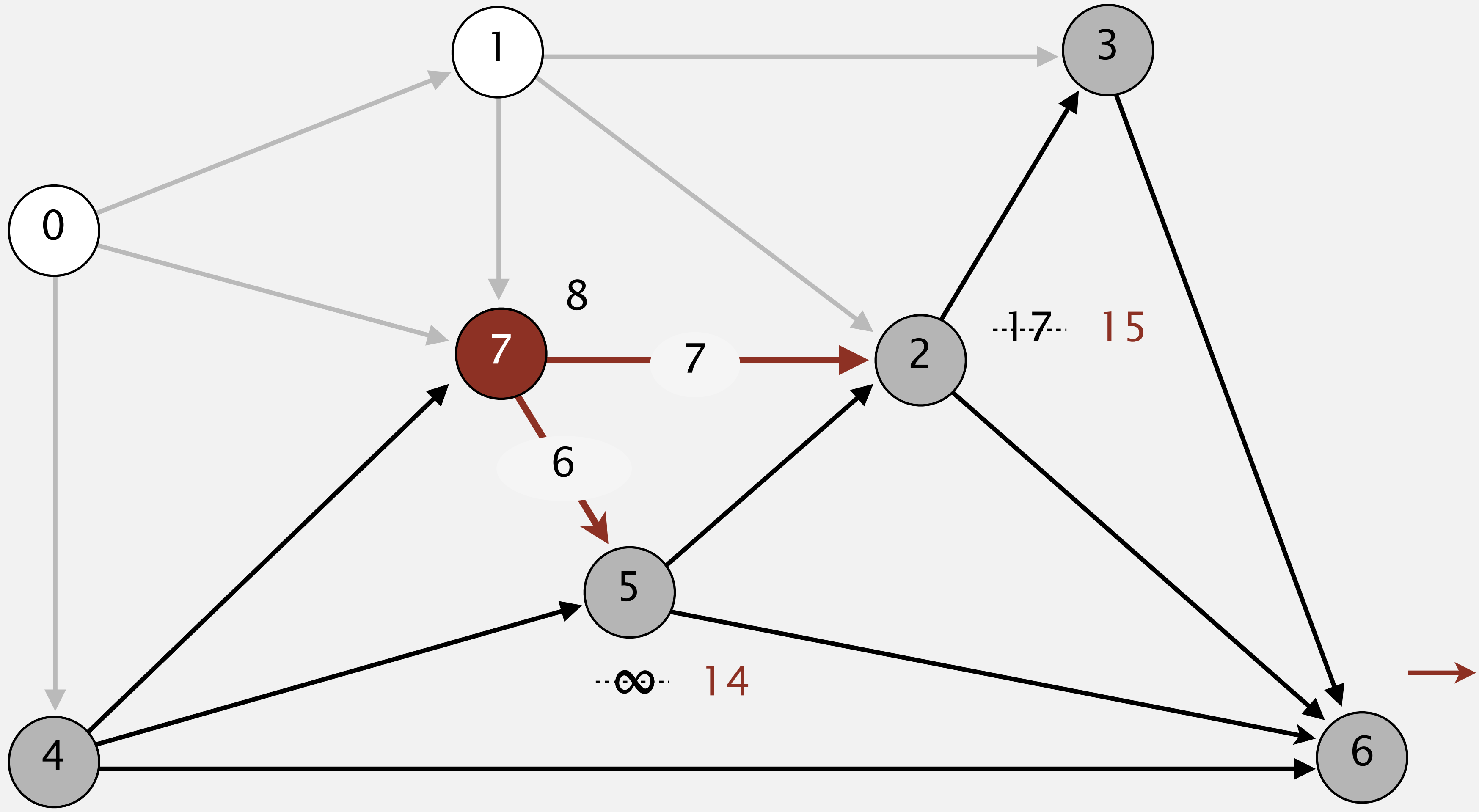
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	17.0	1→2
3	20.0	1→3
4	9.0	0→4
5		
6		
→ 7	8.0	0→7

relax all edges adjacent from 7

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges adjacent from that vertex.



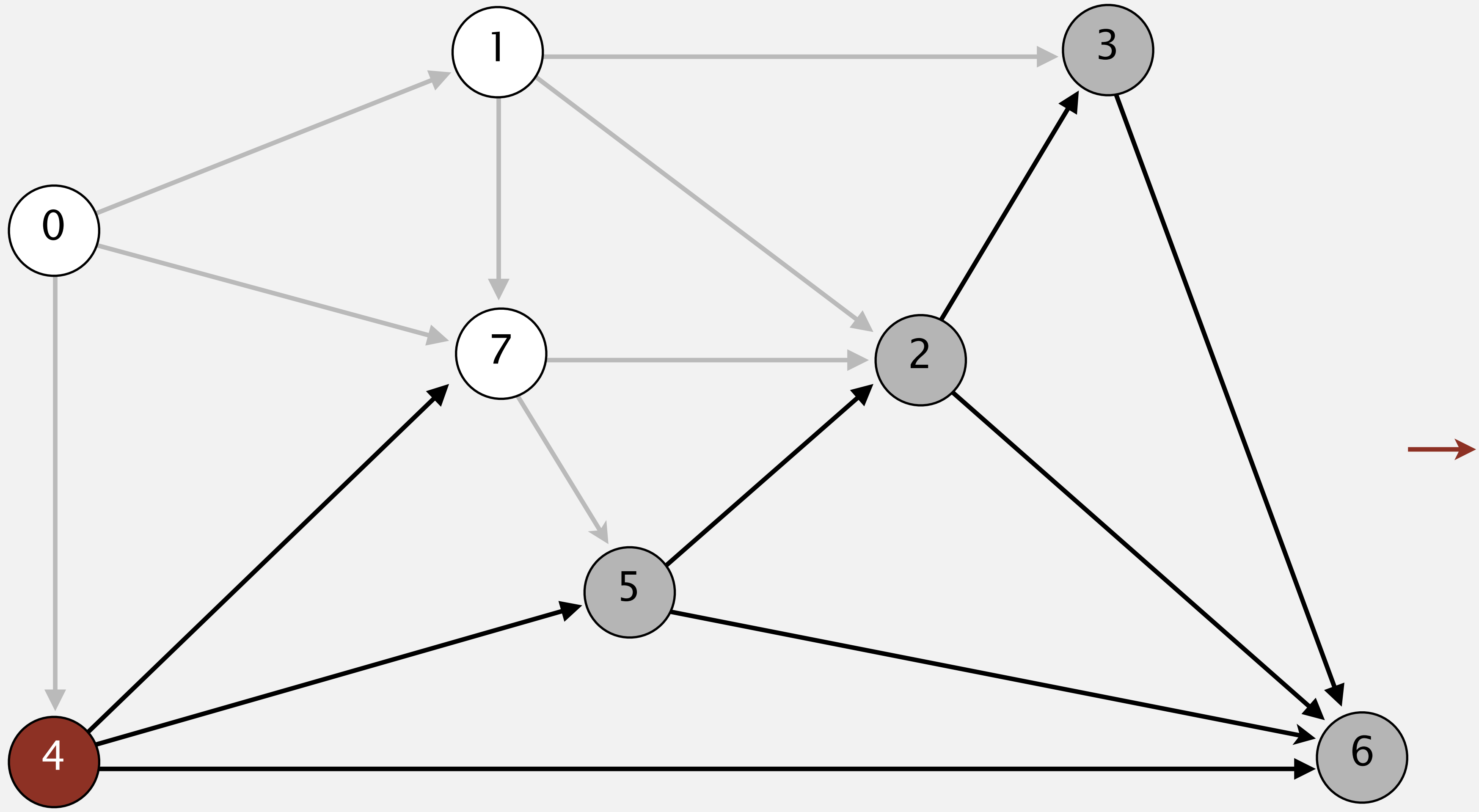
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
5	14.0	7→5
6		
7	8.0	0→7

relax all edges adjacent from 7

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges adjacent from that vertex.

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0



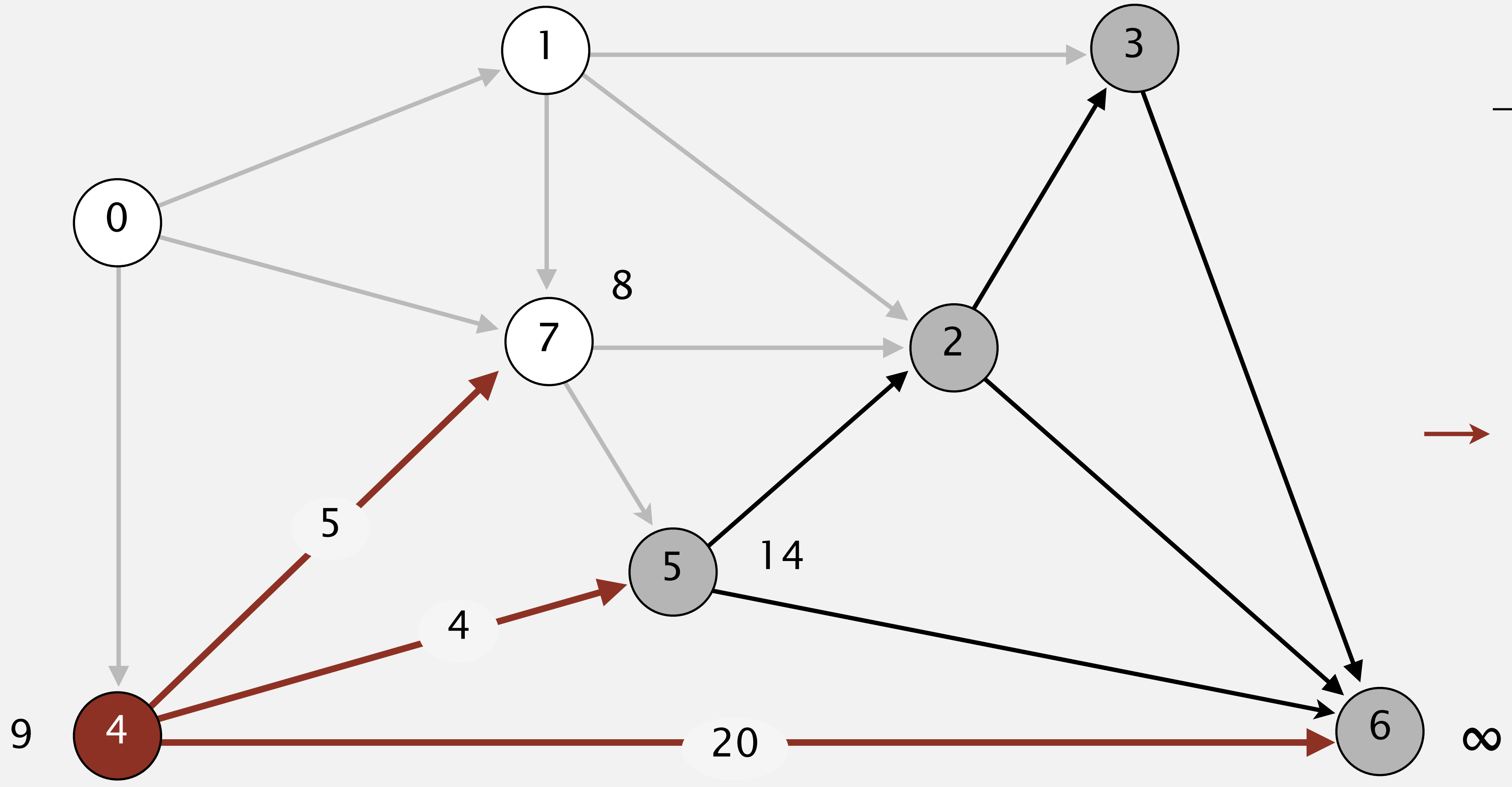
	v	distTo[]	edgeTo[]
	0	0.0	-
	1	5.0	0→1
	2	15.0	7→2
	3	20.0	1→3
→	4	9.0	0→4
	5	14.0	7→5
	6		
	7	8.0	0→7

select vertex 4

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges adjacent from that vertex.



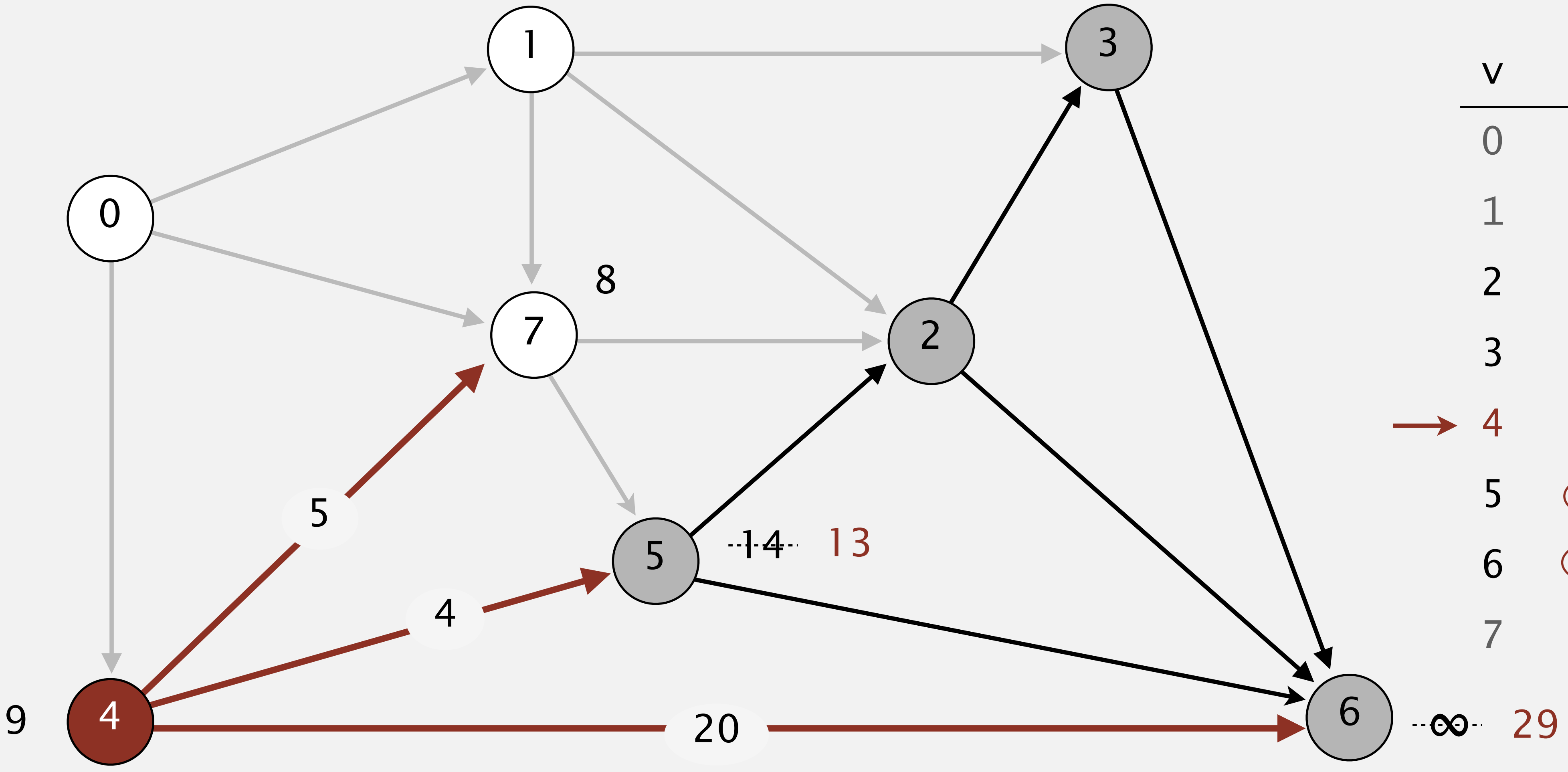
	v	distTo[]	edgeTo[]
	0	0.0	-
	1	5.0	0→1
	2	15.0	7→2
	3	20.0	1→3
→	4	9.0	0→4
	5	14.0	7→5
	6		
	7	8.0	0→7

relax all edges adjacent from 4

Dijkstra's algorithm demo

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges adjacent from that vertex.



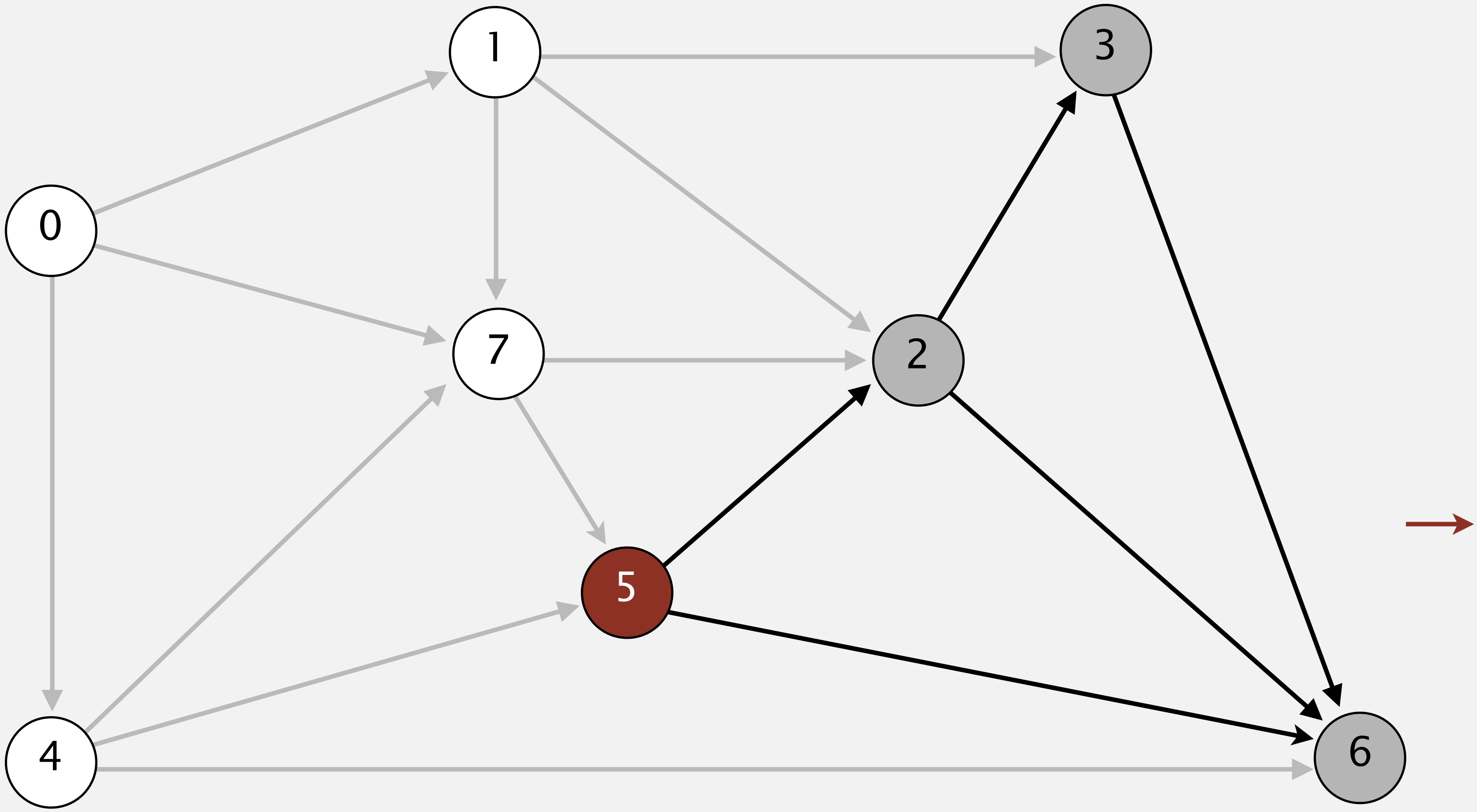
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
→ 4	9.0	0→4
5	13.0	4→5
6	29.0	4→6
7	8.0 ✔	0→7

relax all edges adjacent from 4

Dijkstra's algorithm demo

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges adjacent from that vertex.



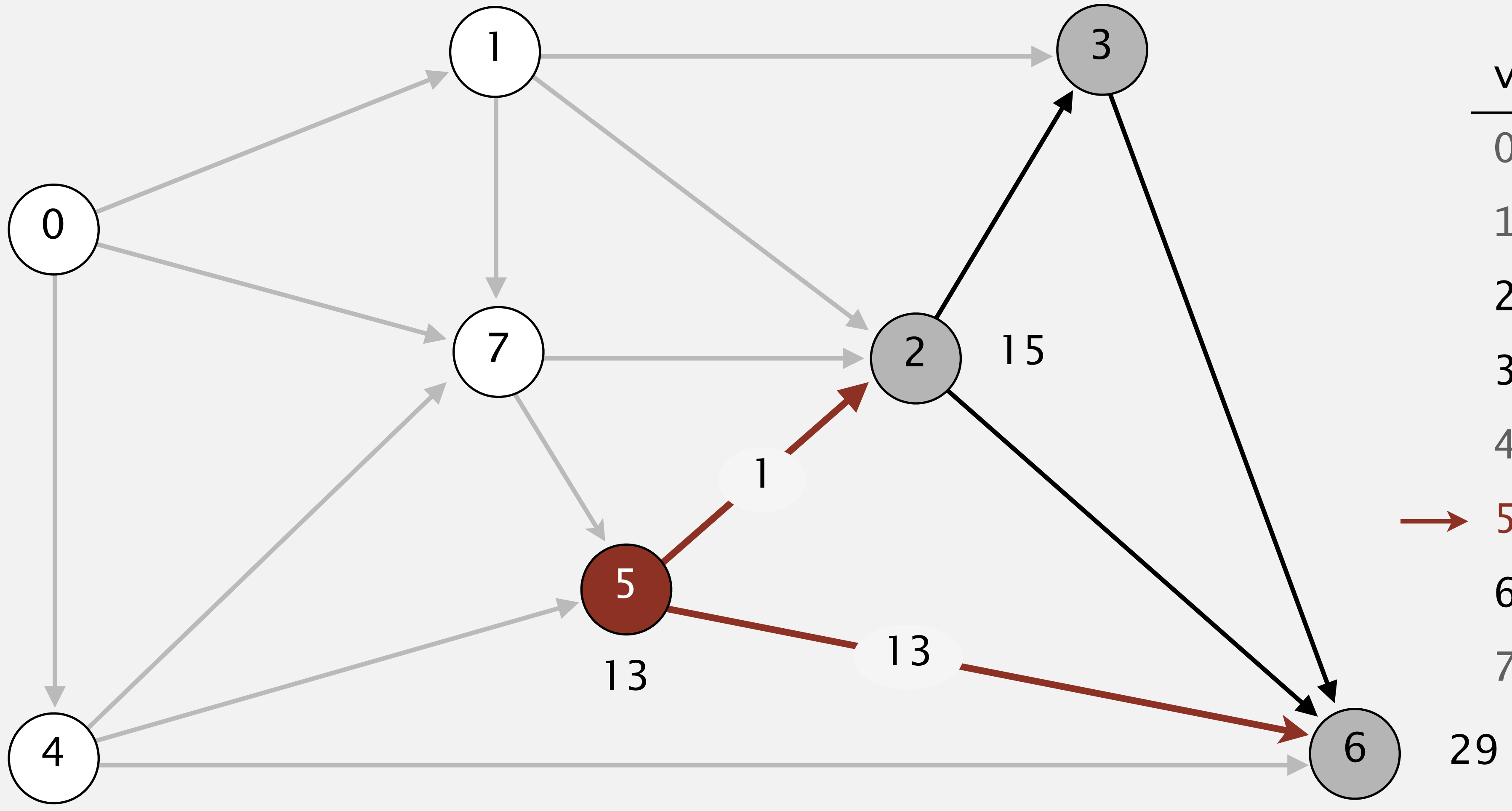
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	15.0	7→2
3	20.0	1→3
4	9.0	0→4
→ 5	13.0	4→5
6	29.0	4→6
7	8.0	0→7

select vertex 5

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges adjacent from that vertex.



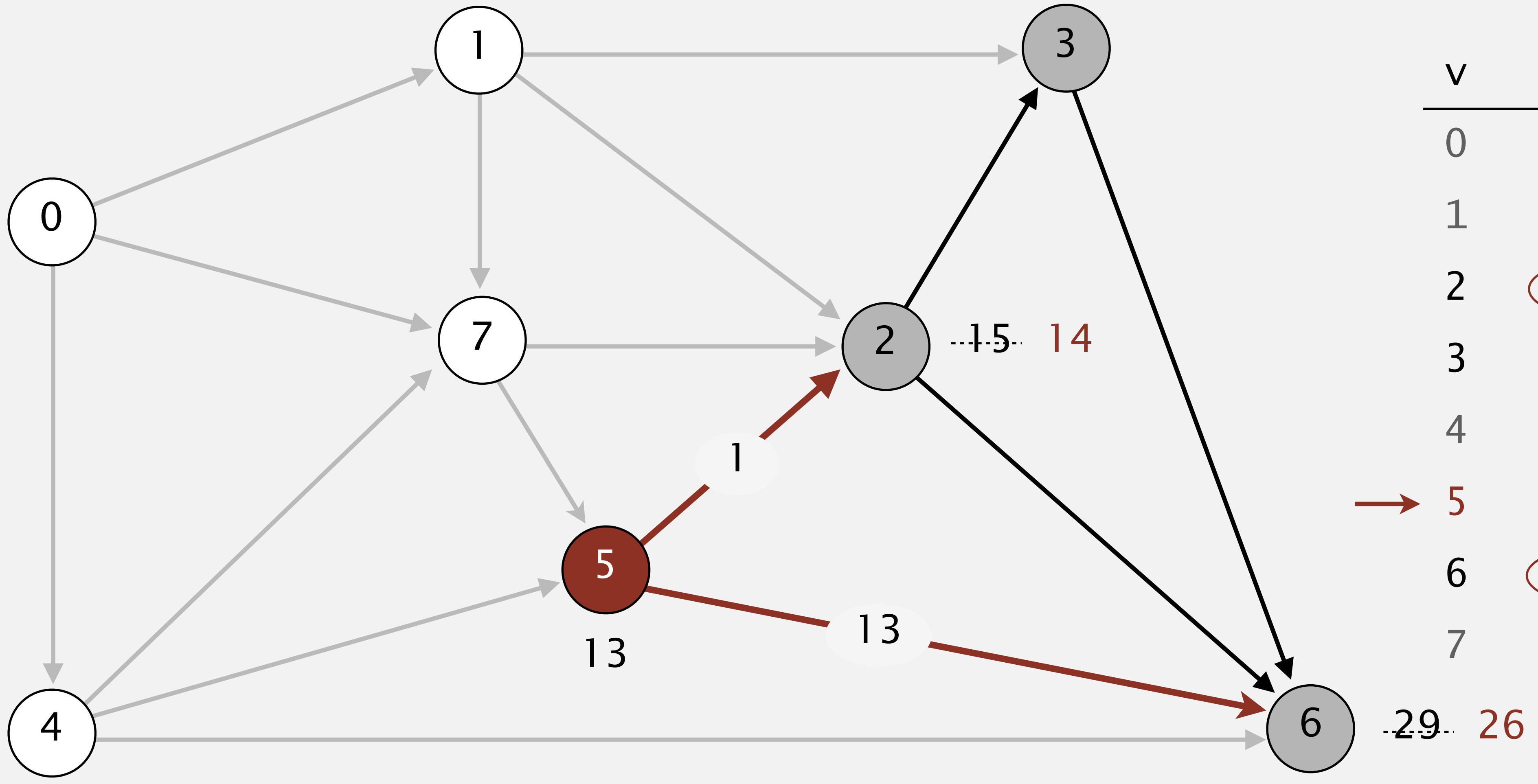
	v	distTo[]	edgeTo[]
	0	0.0	-
	1	5.0	0→1
	2	15.0	7→2
	3	20.0	1→3
	4	9.0	0→4
→	5	13.0	4→5
	6	29.0	4→6
	7	8.0	0→7

relax all edges adjacent from 5

Dijkstra's algorithm demo

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges adjacent from that vertex.



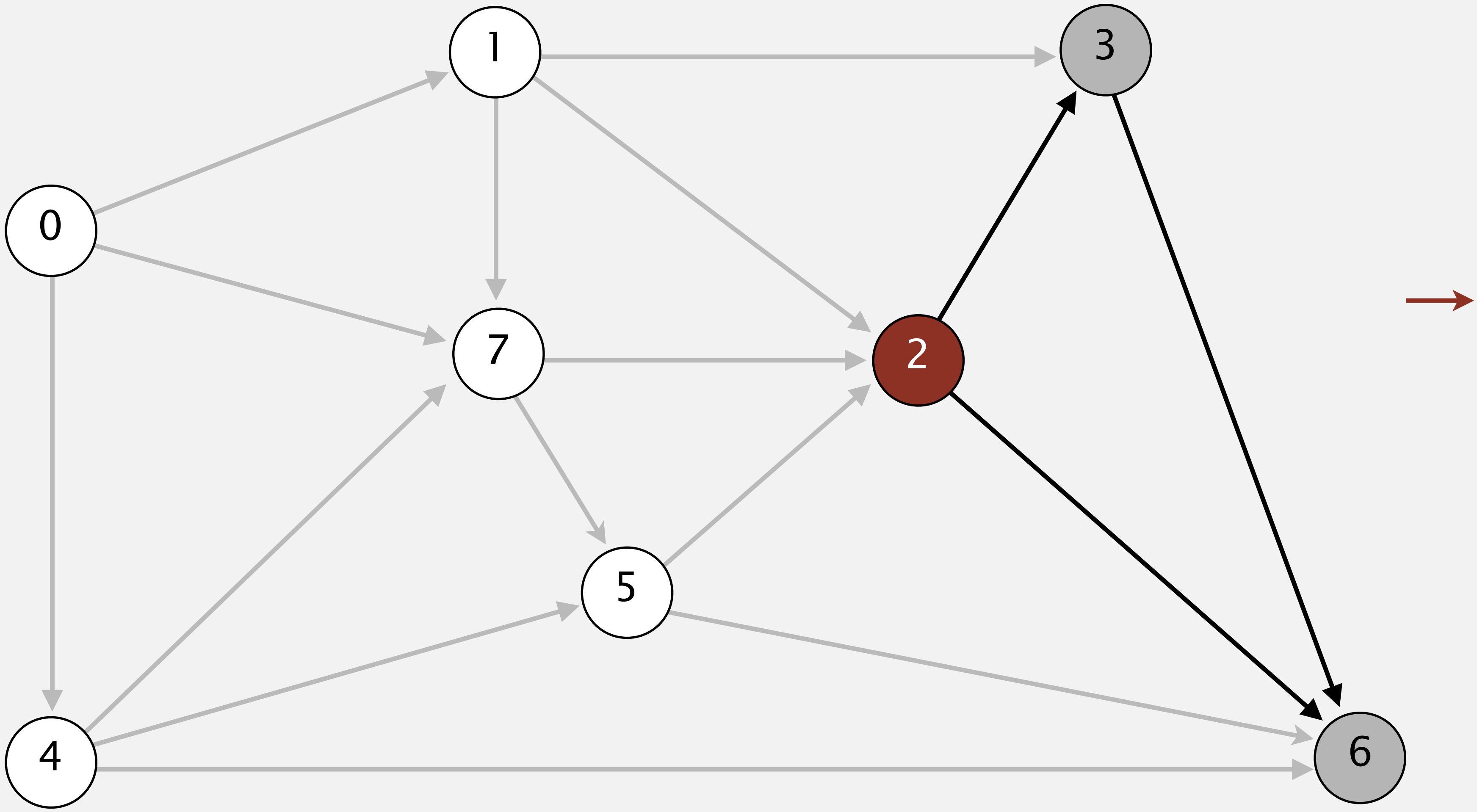
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
→ 5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

relax all edges adjacent from 5

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges adjacent from that vertex.



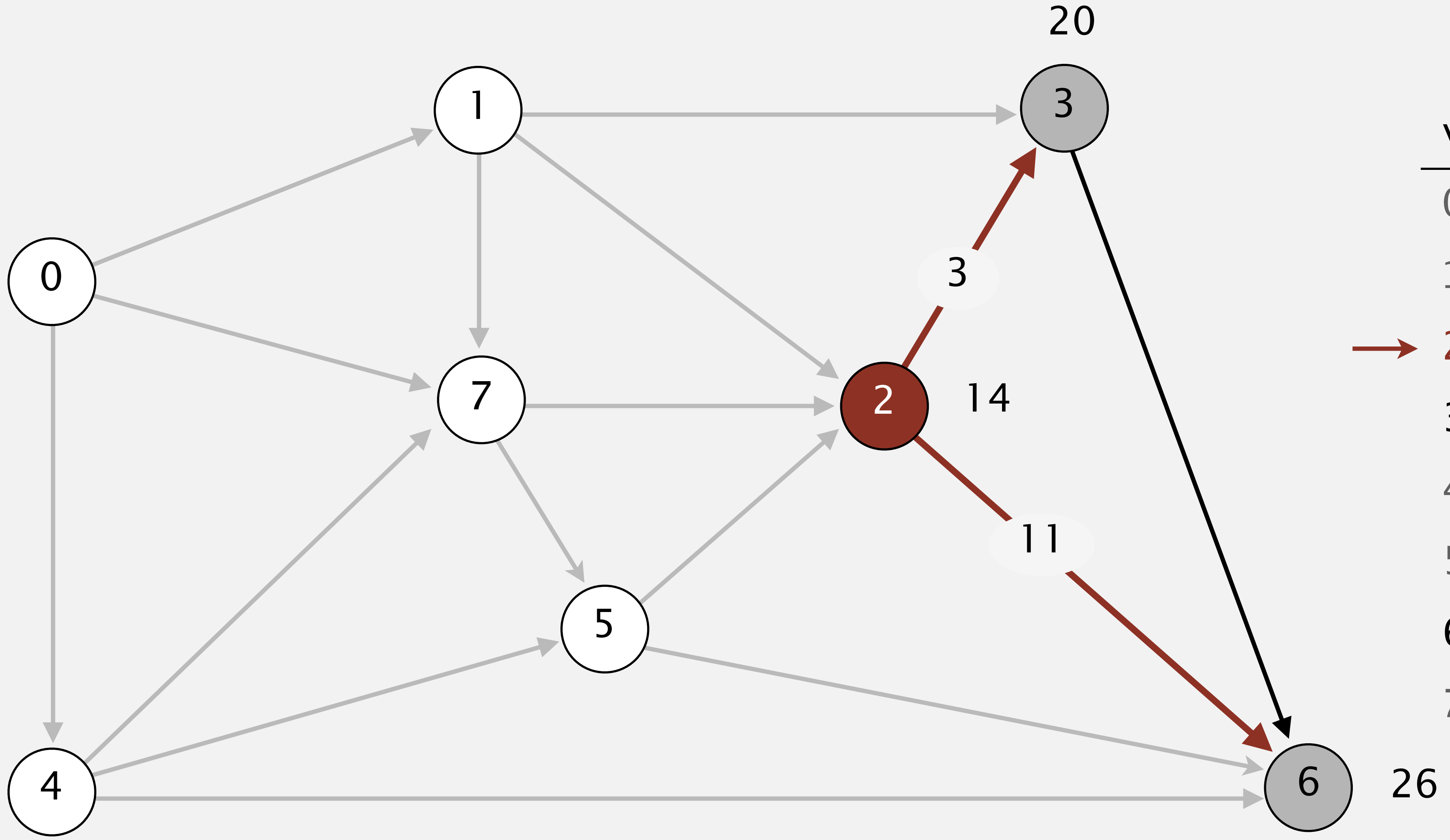
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

select vertex 2

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges adjacent from that vertex.



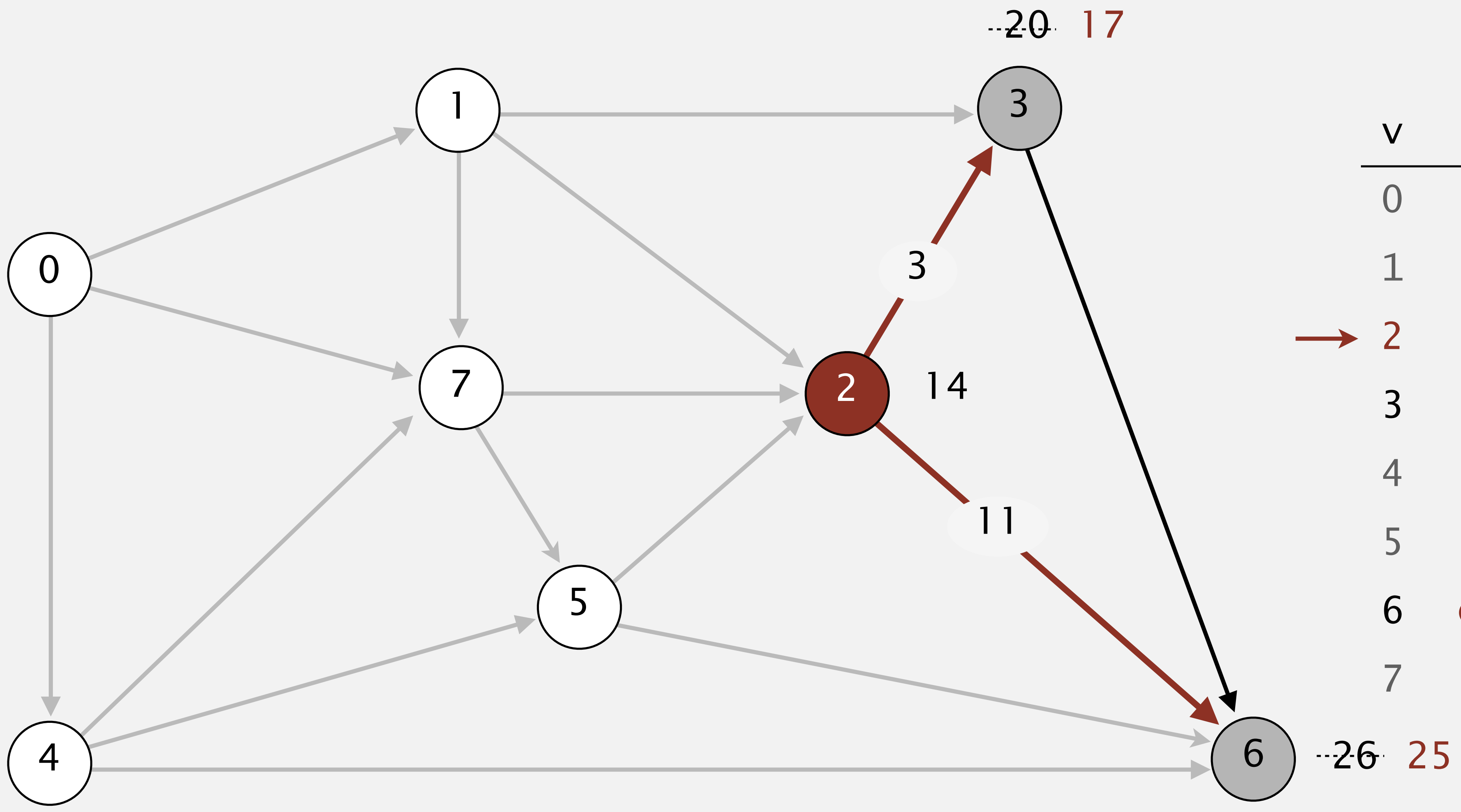
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	20.0	1→3
4	9.0	0→4
5	13.0	4→5
6	26.0	5→6
7	8.0	0→7

relax all edges adjacent from 2

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges adjacent from that vertex.

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0



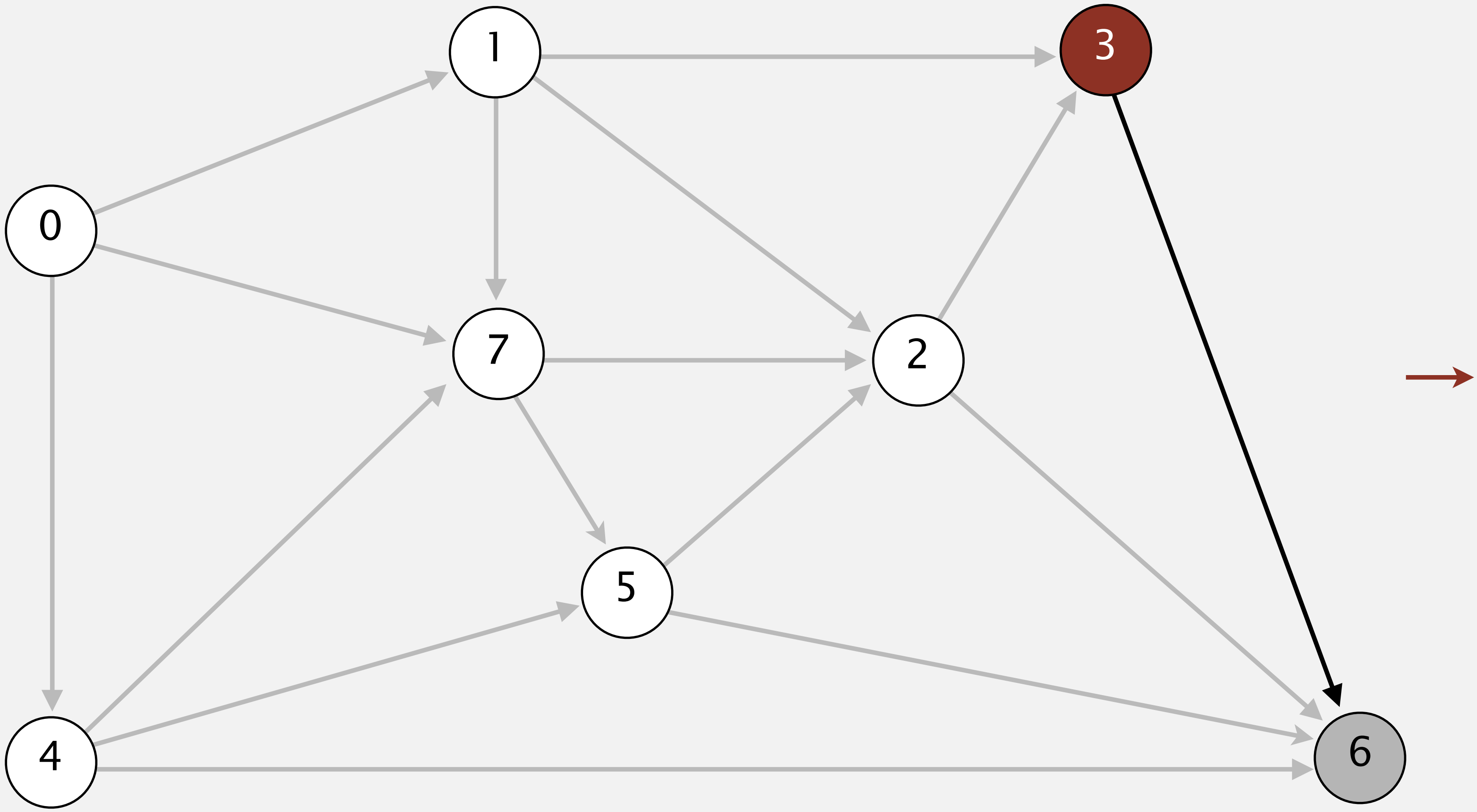
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

relax all edges adjacent from 2

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges adjacent from that vertex.



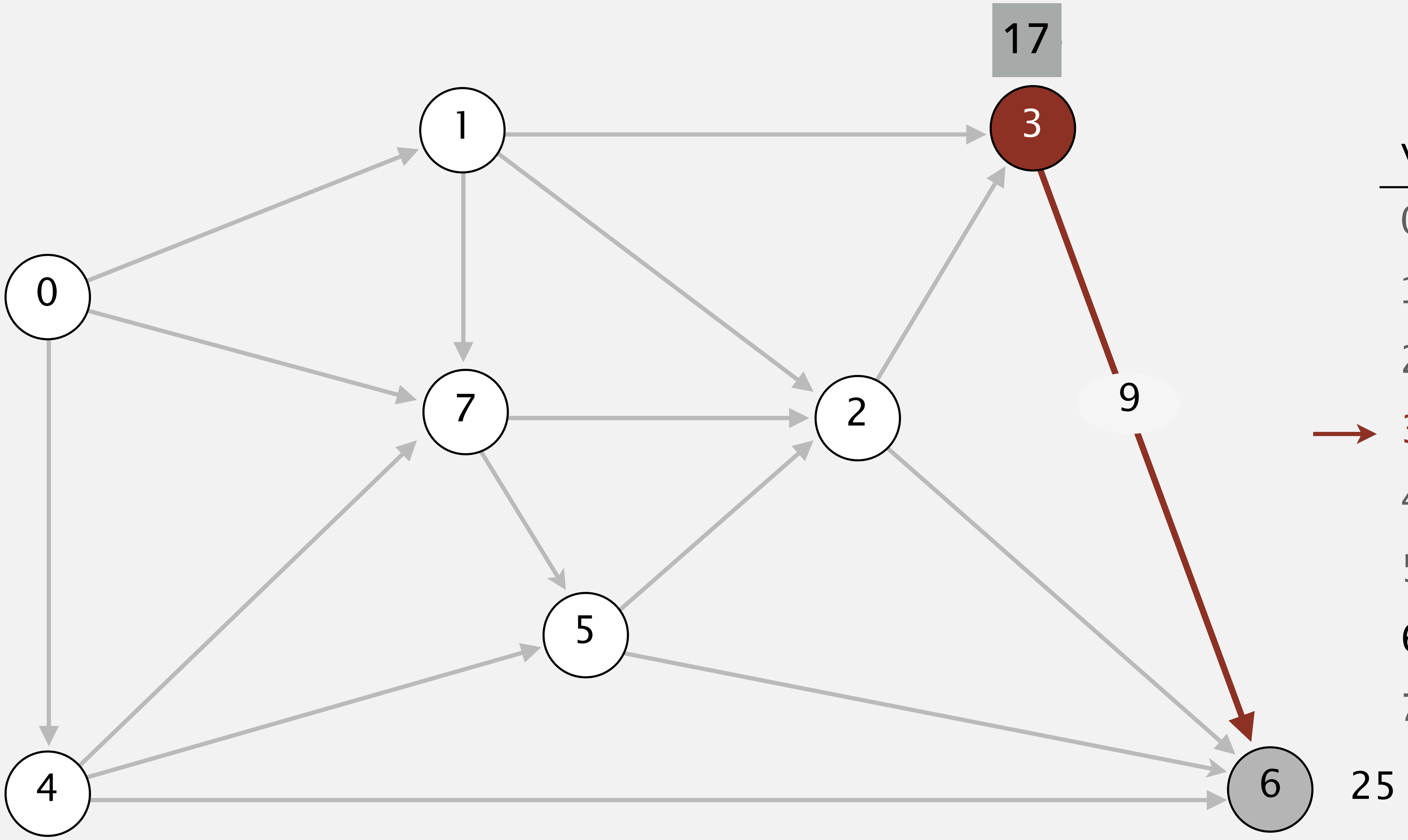
	v	distTo[]	edgeTo[]
	0	0.0	-
	1	5.0	0→1
	2	14.0	5→2
→	3	17.0	2→3
	4	9.0	0→4
	5	13.0	4→5
	6	25.0	2→6
	7	8.0	0→7

select vertex 3

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges adjacent from that vertex.



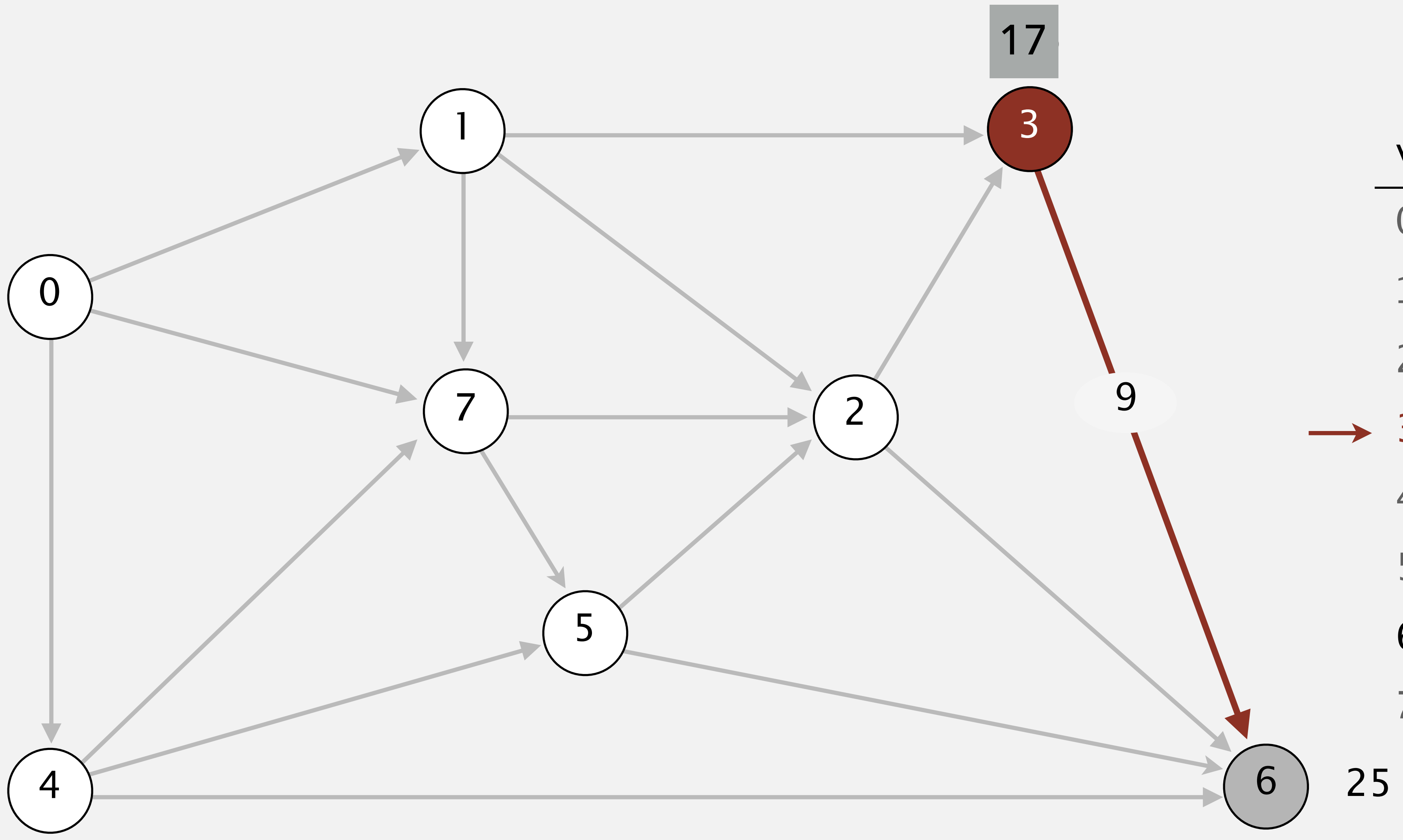
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
→ 3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

relax all edges adjacent from 3

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges adjacent from that vertex.



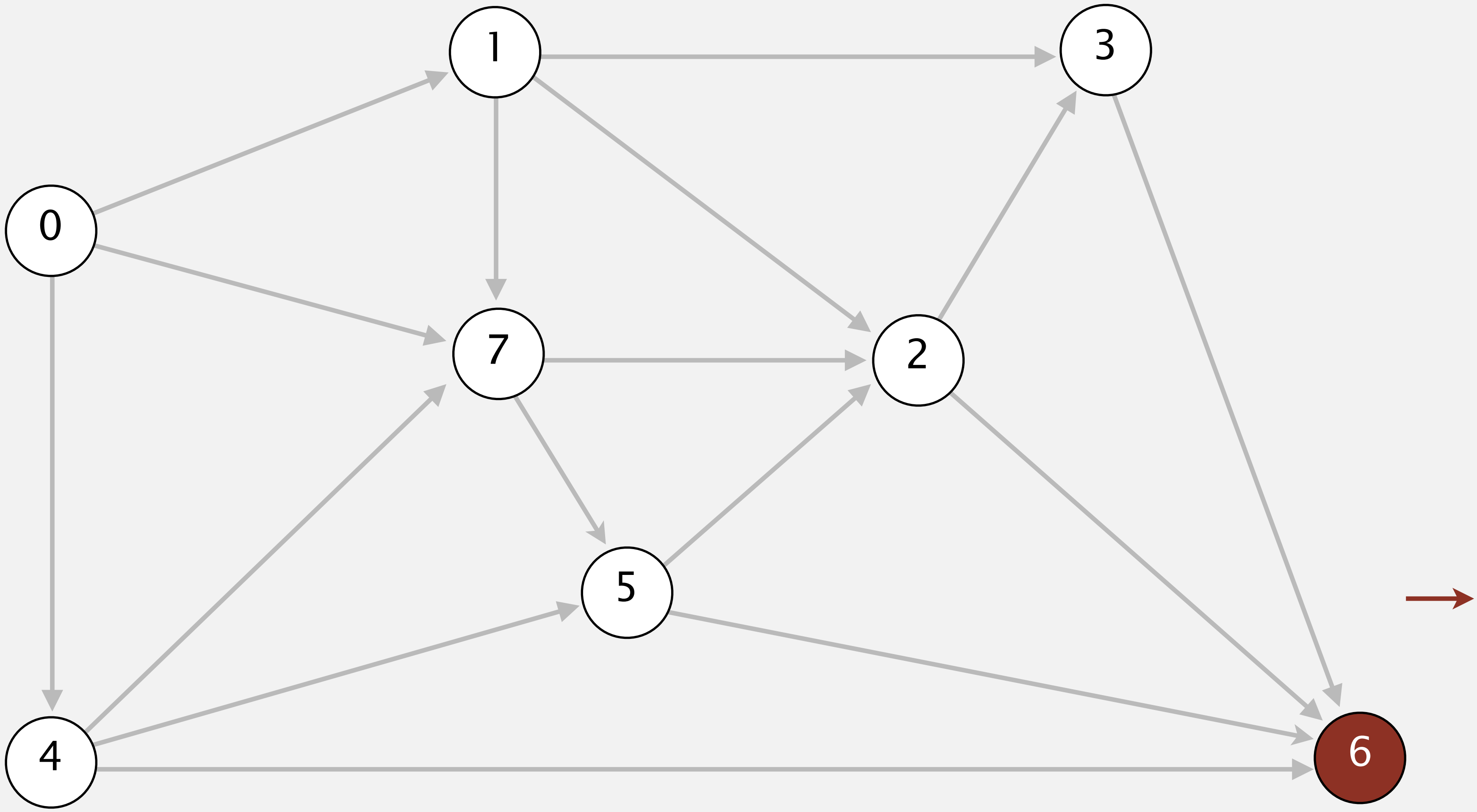
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
→ 3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0 ✔	2→6
7	8.0	0→7

relax all edges adjacent from 3

Dijkstra's algorithm demo

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges adjacent from that vertex.



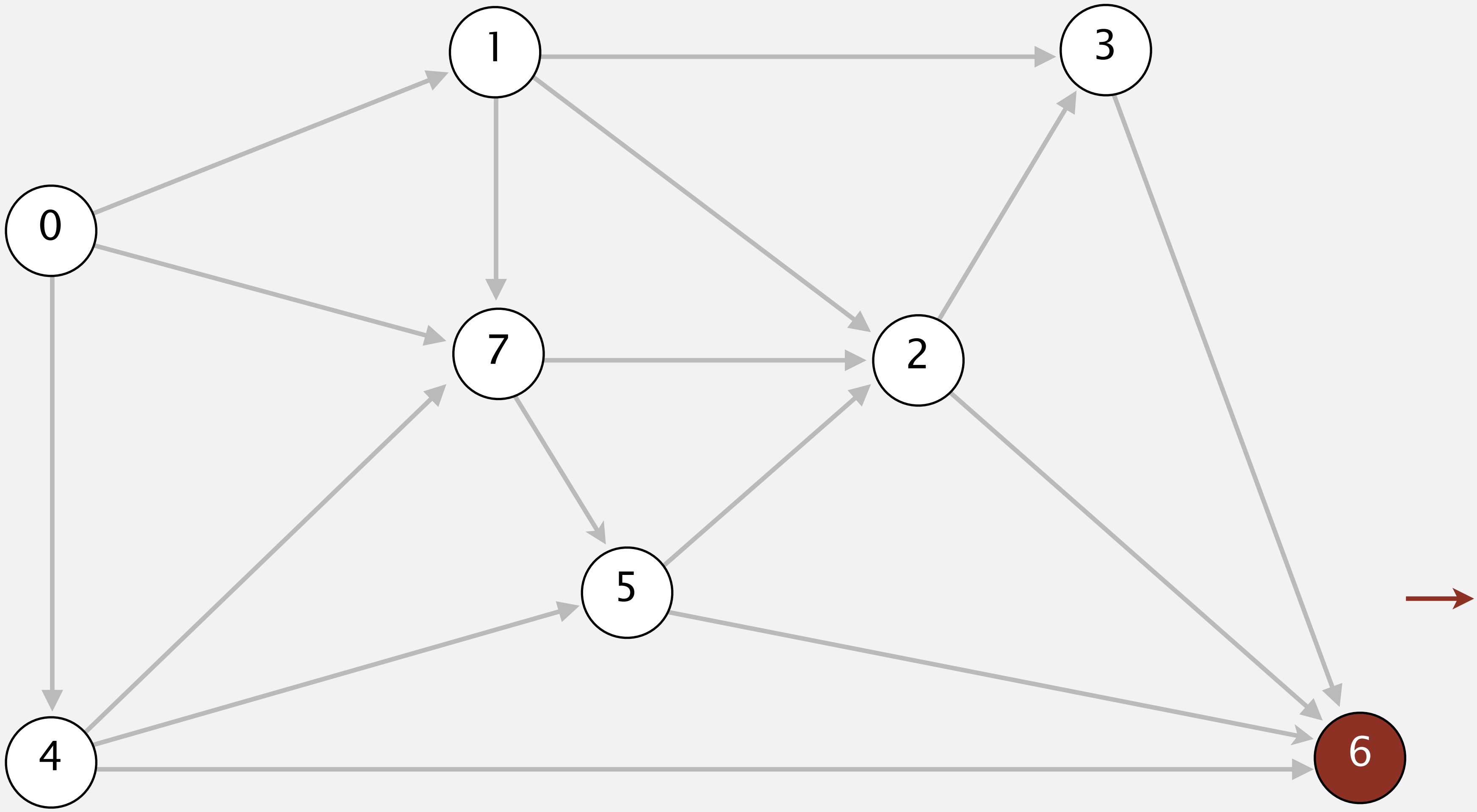
	v	distTo[]	edgeTo[]
	0	0.0	-
	1	5.0	0→1
	2	14.0	5→2
	3	17.0	2→3
	4	9.0	0→4
	5	13.0	4→5
→	6	25.0	2→6
	7	8.0	0→7

select vertex 6

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from s (non-tree vertex with the lowest distTo[] value).
- Add vertex to tree and relax all edges adjacent from that vertex.



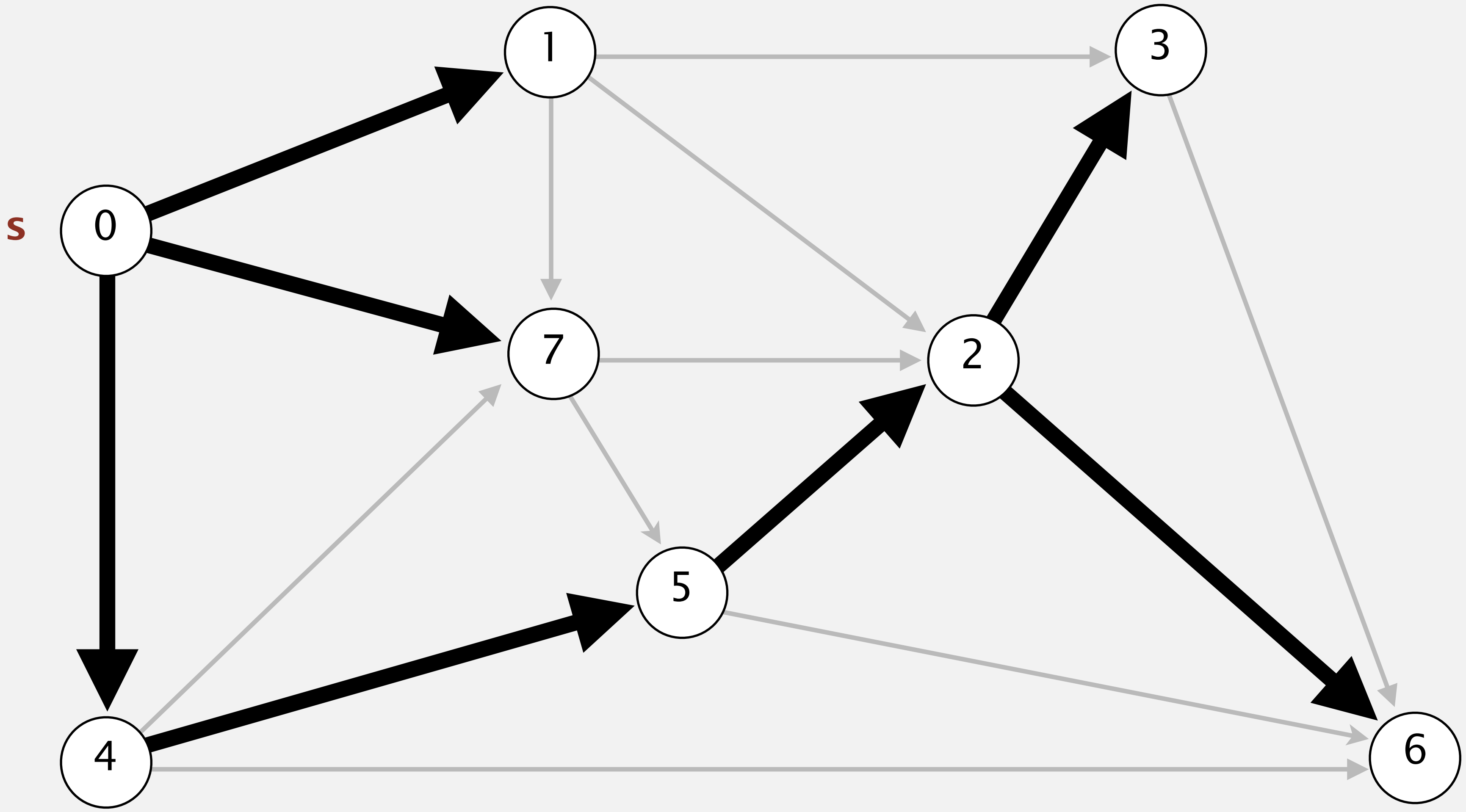
v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
→ 6	25.0	2→6
7	8.0	0→7

relax all edges adjacent from 6

Dijkstra's algorithm demo

0→1	5.0
0→4	9.0
0→7	8.0
1→2	12.0
1→3	15.0
1→7	4.0
2→3	3.0
2→6	11.0
3→6	9.0
4→5	4.0
4→6	20.0
4→7	5.0
5→2	1.0
5→6	13.0
7→5	6.0
7→2	7.0

- Consider vertices in increasing order of distance from `s` (non-tree vertex with the lowest `distTo[]` value).
- Add vertex to tree and relax all edges adjacent from that vertex.



v	distTo[]	edgeTo[]
0	0.0	-
1	5.0	0→1
2	14.0	5→2
3	17.0	2→3
4	9.0	0→4
5	13.0	4→5
6	25.0	2→6
7	8.0	0→7

shortest-paths tree from vertex s

Indexed min-priority queue (Section 2.4 in recommended textbook)

- Associate an index between 0 and n-1 with each key in a priority queue.
    - Insert a key associated with a given index.
    - Delete a minimum key and return associated index.
    - Decrease the key associated with a given index.
- `public class IndexMinPQ<Key extends Comparable<Key>>`
    - `IndexMinPQ(int n)`
        - Create indexed PQ with indices 0,1,…n-1
    - `void insert(int i, Key key)`
        - Associate key with index i.
    - `int delMin()`
        - Remove a minimal key and return its associated index.
    - `void decreaseKey(int i, Key key)`
        - Decrease the key with index i to the specified value.

DIJKSTRA'S ALGORITHM

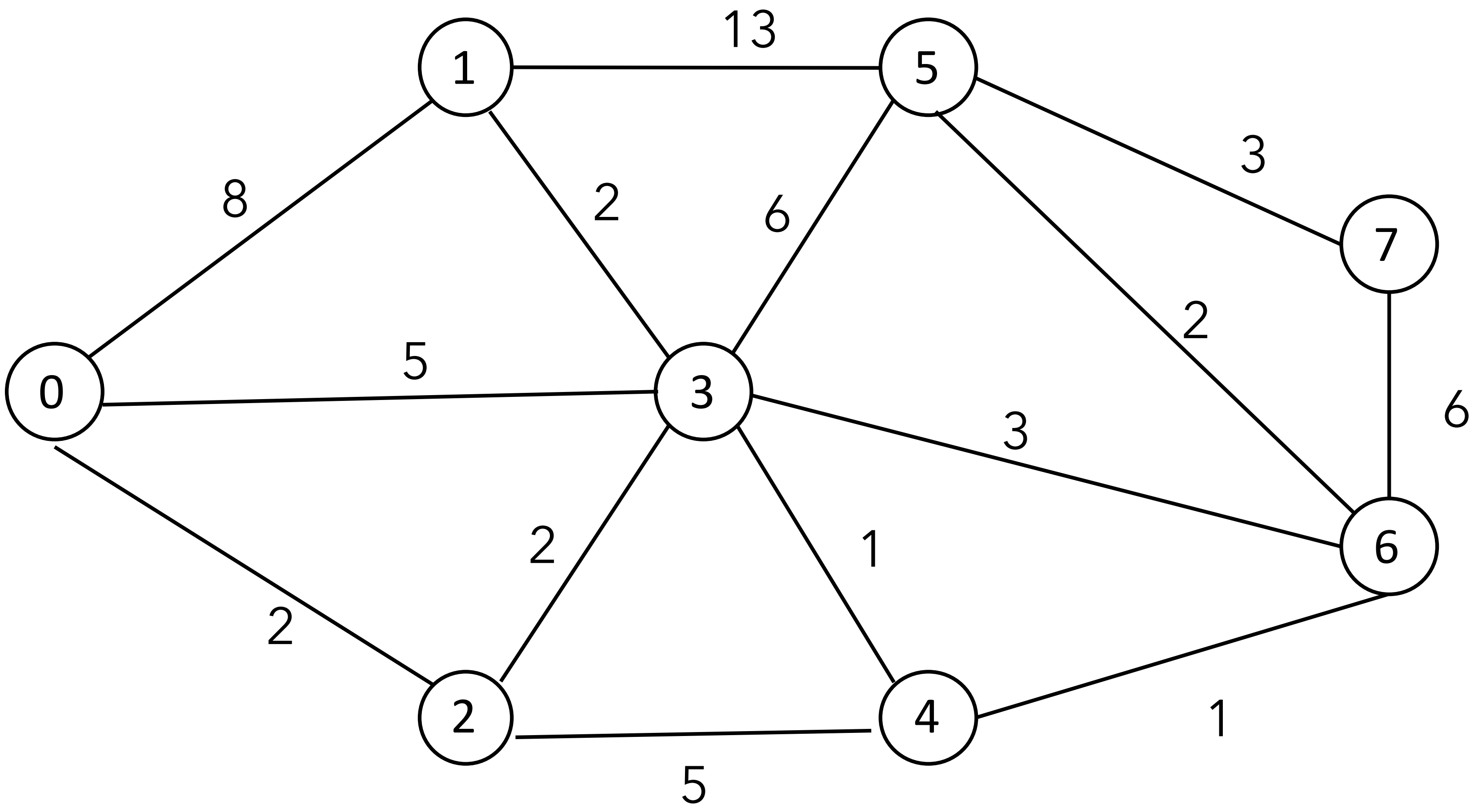
```
public class DijkstraSP {
    private double[] distTo;          // distTo[v] = distance  of shortest s->v path
    private DirectedEdge[] edgeTo;    // edgeTo[v] = last edge on shortest s->v path
    private IndexMinPQ<Double> pq;    // priority queue of vertices


    public DijkstraSP(EdgeWeightedDigraph G, int s) {
        distTo = new double[G.V()];
        edgeTo = new DirectedEdge[G.V()];

        for (int v = 0; v < G.V(); v++)
            distTo[v] = Double.POSITIVE_INFINITY;
        distTo[s] = 0.0;

        // relax vertices in order of distance from s
        pq = new IndexMinPQ<Double>(G.V());
        pq.insert(s, distTo[s]);
        while (!pq.isEmpty()) {
            int v = pq.delMin();
            for (DirectedEdge e : G.adj(v))
                relax(e);
        }
    }

    // relax edge e and update pq if changed
    private void relax(DirectedEdge e) {
        int v = e.from(), w = e.to();
        if (distTo[w] > distTo[v] + e.weight()) {
            distTo[w] = distTo[v] + e.weight();
            edgeTo[w] = e;
            if (pq.contains(w)) pq.decreaseKey(w, distTo[w]);
            else                pq.insert(w, distTo[w]);
        }
    }
```

Running time depends on PQ implementation

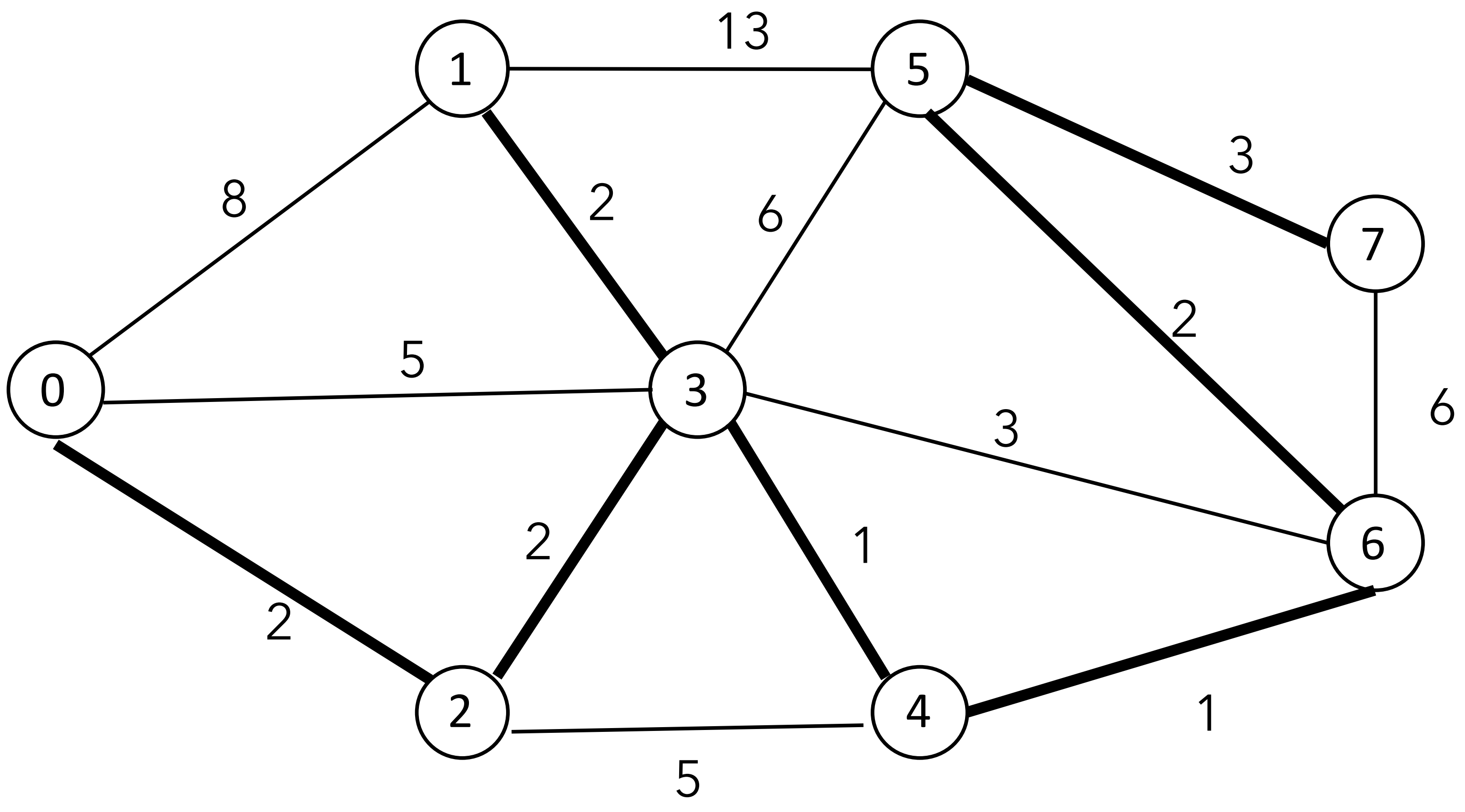
- Many variations. Assuming binary heap, running time is proportional to $|E|\log|V|$ and $|V|$ extra space.
  - Cost of insert, delete-min, decrease-key are all $\log|V|$.
- More complicated version with a Fibonacci heap takes $O(|E|+|V|\log|V|)$ time but in practice it's not worth implementing.

Practice Time

- Run Dijkstra's algorithm on the following graph with 0 being the starting vertex.



Answer



v	distTo[]	edgeTo[]
0	0	-
1	6	3->1
2	2	0->2
3	4	2->3
4	5	3->4
5	8	6->5
6	6	4->6
7	11	5->7

Lecture 23: Shortest Paths

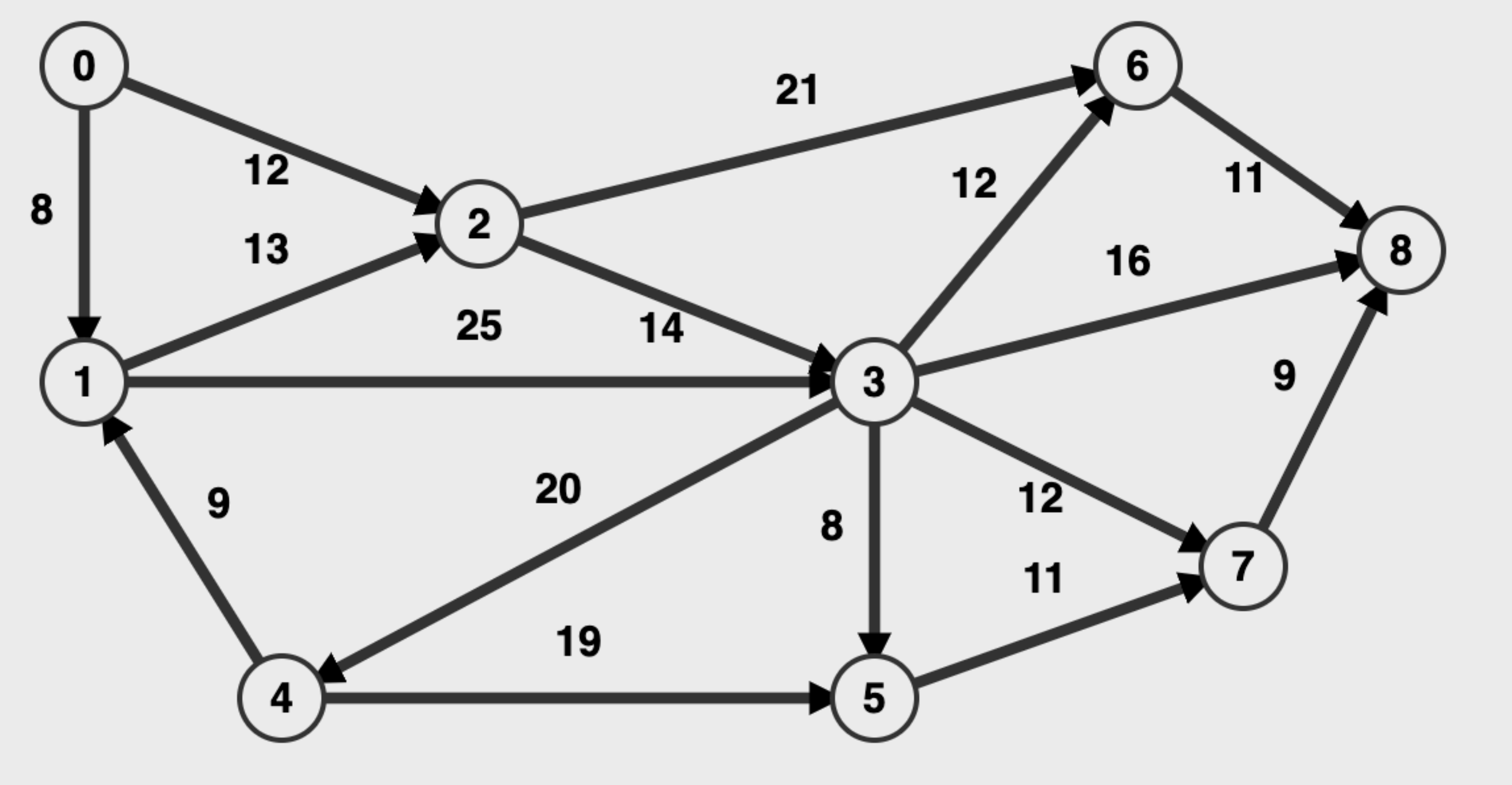
- Introduction to Shortest Paths
- API
- Properties
- Dijkstra's Algorithm

Readings:

- Recommended Textbook: Chapter 4.4 (Pages 638-676)
- Website:
  - https://algs4.cs.princeton.edu/44sp/
- Visualization
  - https://visualgo.net/en/sssp

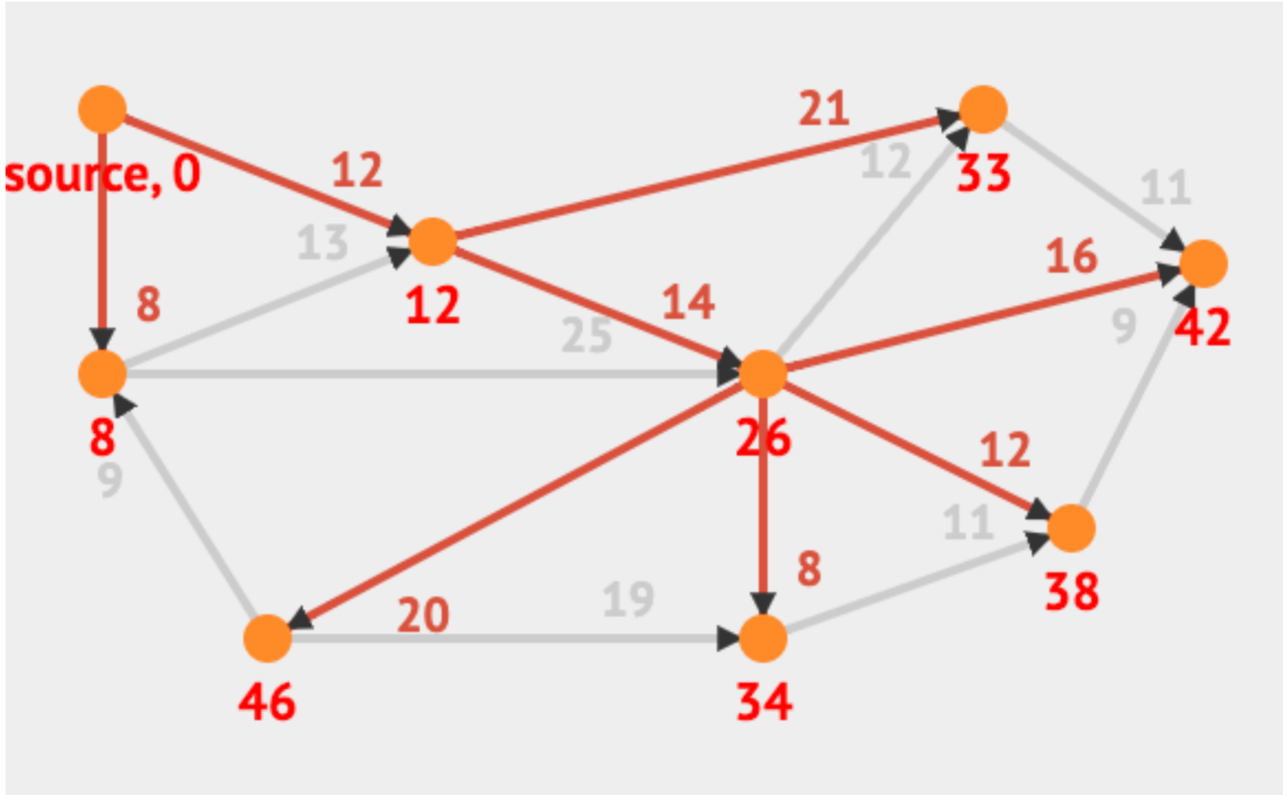
Problem

- Run Dijkstra's algorithm on the following graph with 0 being the starting vertex.



Answer

- Run Dijkstra’s algorithm on the following graph with 0 being the starting vertex.



v	distTo[]	edgeTo[]
0	0	-
1	8	0->1
2	12	0->2
3	26	2->3
4	46	3->4
5	34	3->5
6	33	3->6
7	38	3->7
8	42	3->8