CS062

DATA STRUCTURES AND ADVANCED PROGRAMMING

16: Binary Trees, Binary Search, Heaps, and Priority Queues

Alexandra Papoutsaki
she/her/hers

Lecture 16: Binary Trees, Binary Search, Heaps, and Priority Queues

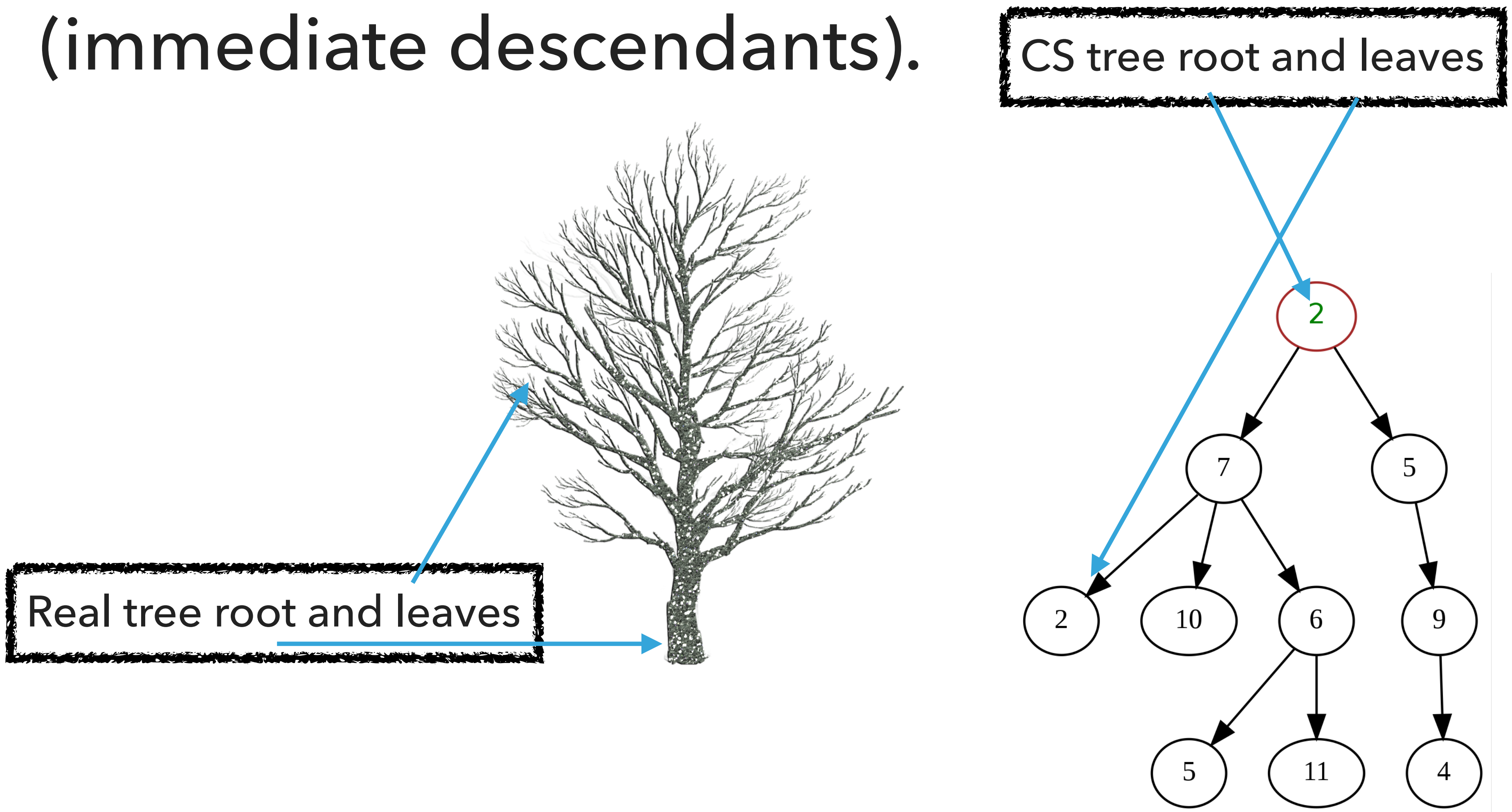
- Binary Trees
- Tree traversals
- Binary Search
- Binary Heaps
- Priority Queues

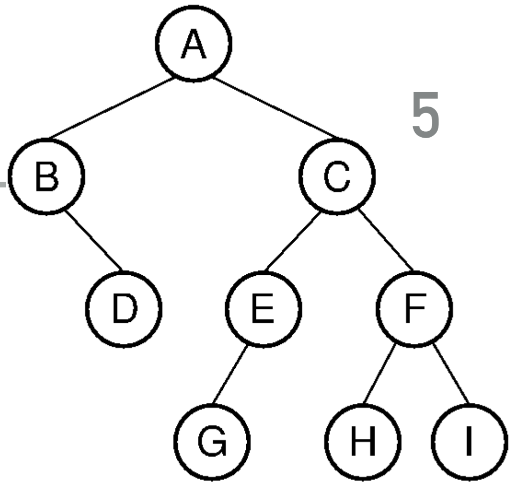
Trees in Computer Science

- Abstract data types that store elements hierarchically rather than linearly.
- Examples of hierarchical structures:
  - Organization charts for
    - Companies (CEO at the top followed by CFO, CMO, COO, CTO, etc).
    - Universities (Board of Trustees at the top, followed by President, then by VPs, etc).
  - Sitemaps (home page links to About, Products, etc. They link to other pages).
  - Computer file systems (user at top followed by Documents, Downloads, Music, etc. Each folder can hold more folders.).

Trees in Computer Science

- Hierarchical: Each element in a tree has a **single** parent (immediate ancestor) and zero or more children (immediate descendants).

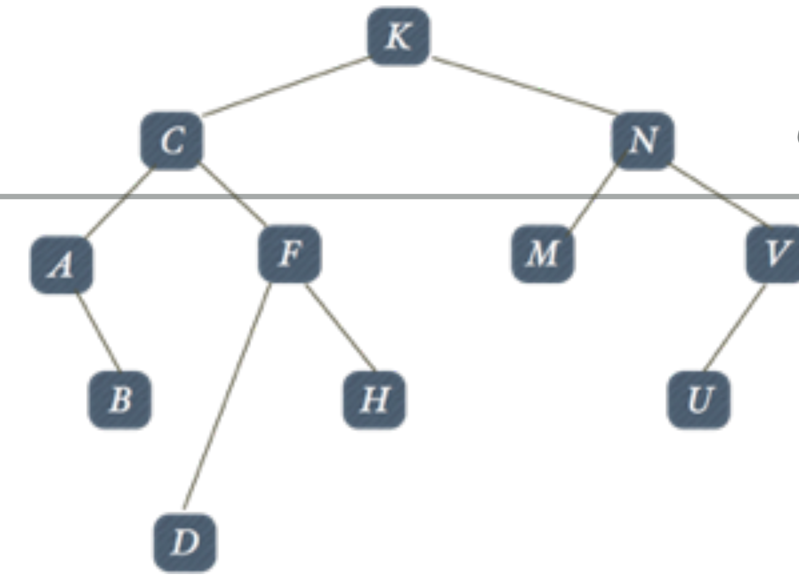




Definition of a tree

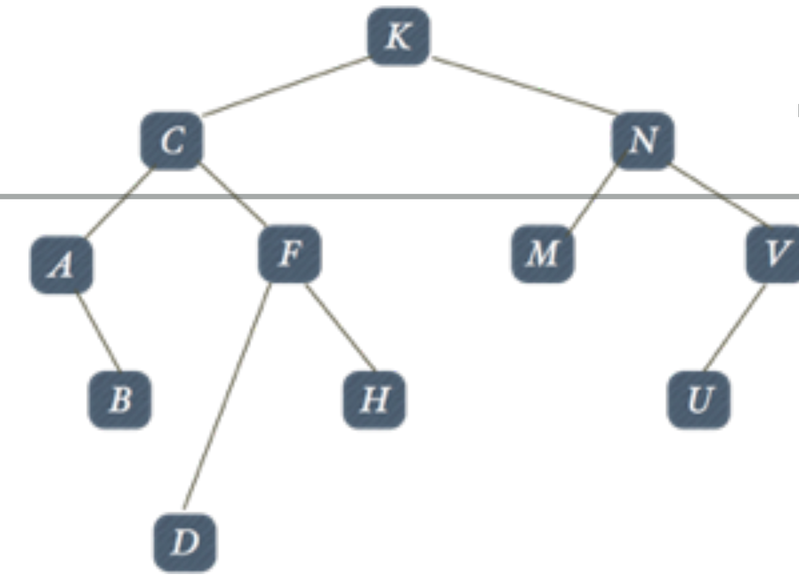
- A tree T is a set of nodes that store elements based on a parent-child relationship:
  - If T is non-empty, it has a node called the root of T, that has no parent.
    - Here, the root is A.
  - Each node v, other than the root, has a unique parent node u. Every node with parent u is a child of u.
    - Here, E's parent is C and F has two children, H and I.

Tree Terminology

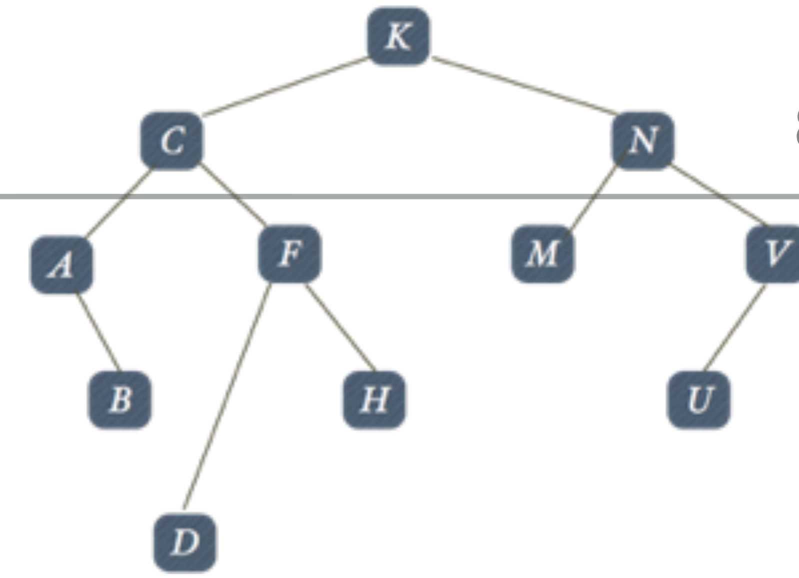


- Edge: a pair of nodes s.t. one is the parent of the other, e.g., (K,C).
- Parent node is directly above child node, e.g., K is parent of C and N.
- Sibling nodes have same parent, e.g., A and F.
- K is ancestor of B.
- B is descendant of K.
- Node plus all descendants gives subtree.
- Nodes without descendants are called leaves or external. The rest are called internal.
- A set of trees is called a forest.

More Terminology

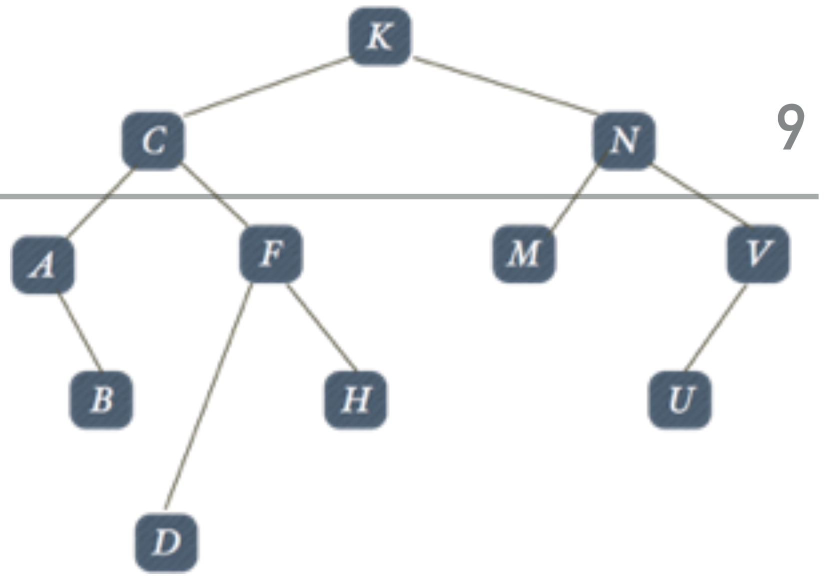


- Simple path: a series of distinct nodes s.t. there are edges between successive nodes, e.g., K-N-V-U.
- Path length: number of edges in path, e.g., path K-C-A has length 2.
- Height of node: length of longest path from the node to a leaf, e.g., N's height is 2 (for path N-V-U).
- Height of tree: length of longest path from the root to a leaf. Here 3.
- Degree of node: number of its children, e.g., F's degree is 2.
- Degree of tree (arity): max degree of any of its nodes. Here is 2.
- Binary tree: a tree with arity of 2, that is any node will have 0-2 children.



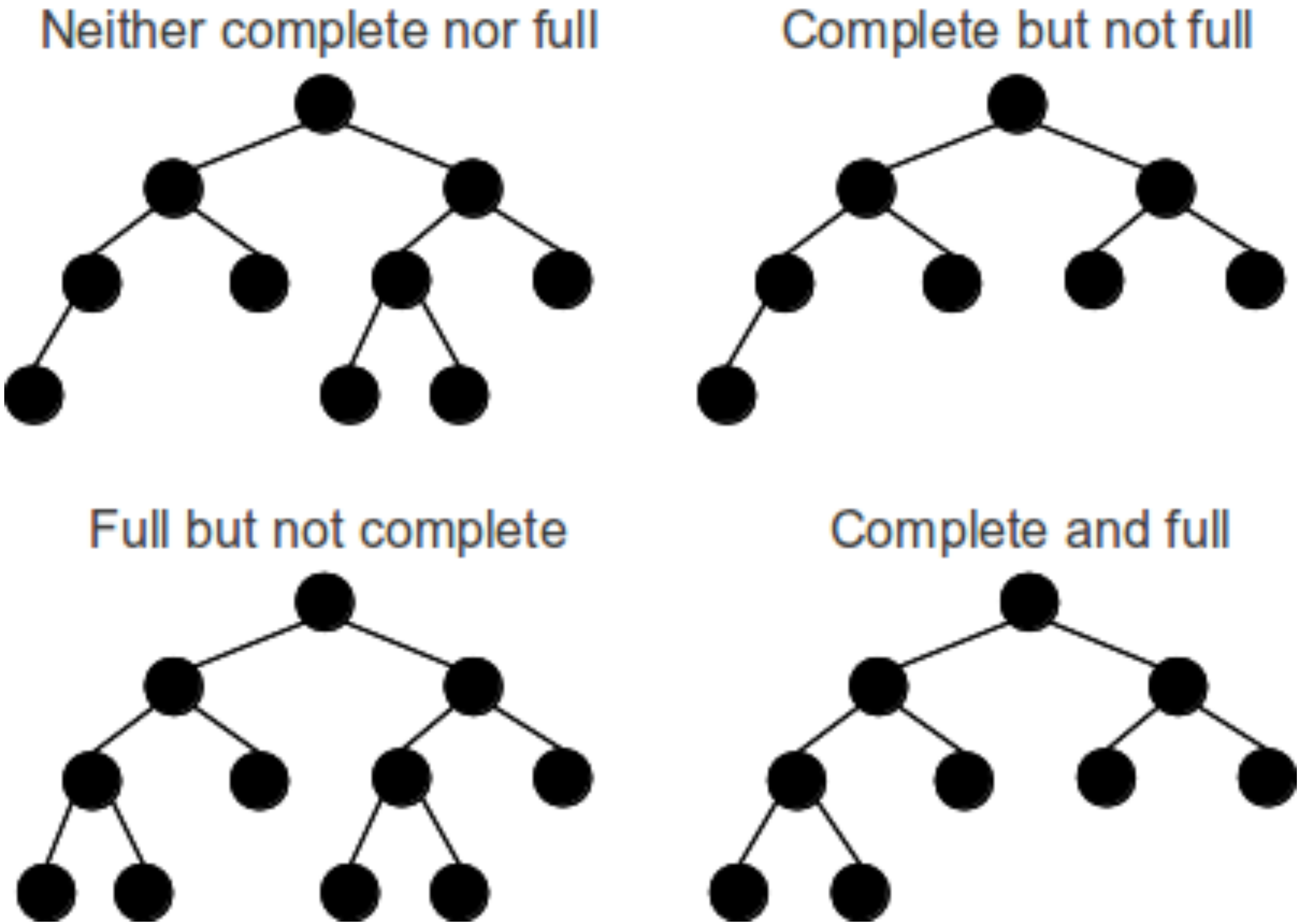
Even More Terminology

- Level/depth of node defined recursively:
  - Root is at level 0.
  - Level of any other node is equal to level of parent + 1.
  - It is also known as the length of path from root or number of ancestors excluding itself.
- Height of node defined recursively:
  - If leaf, height is 0.
  - Else, height is max height of child + 1.



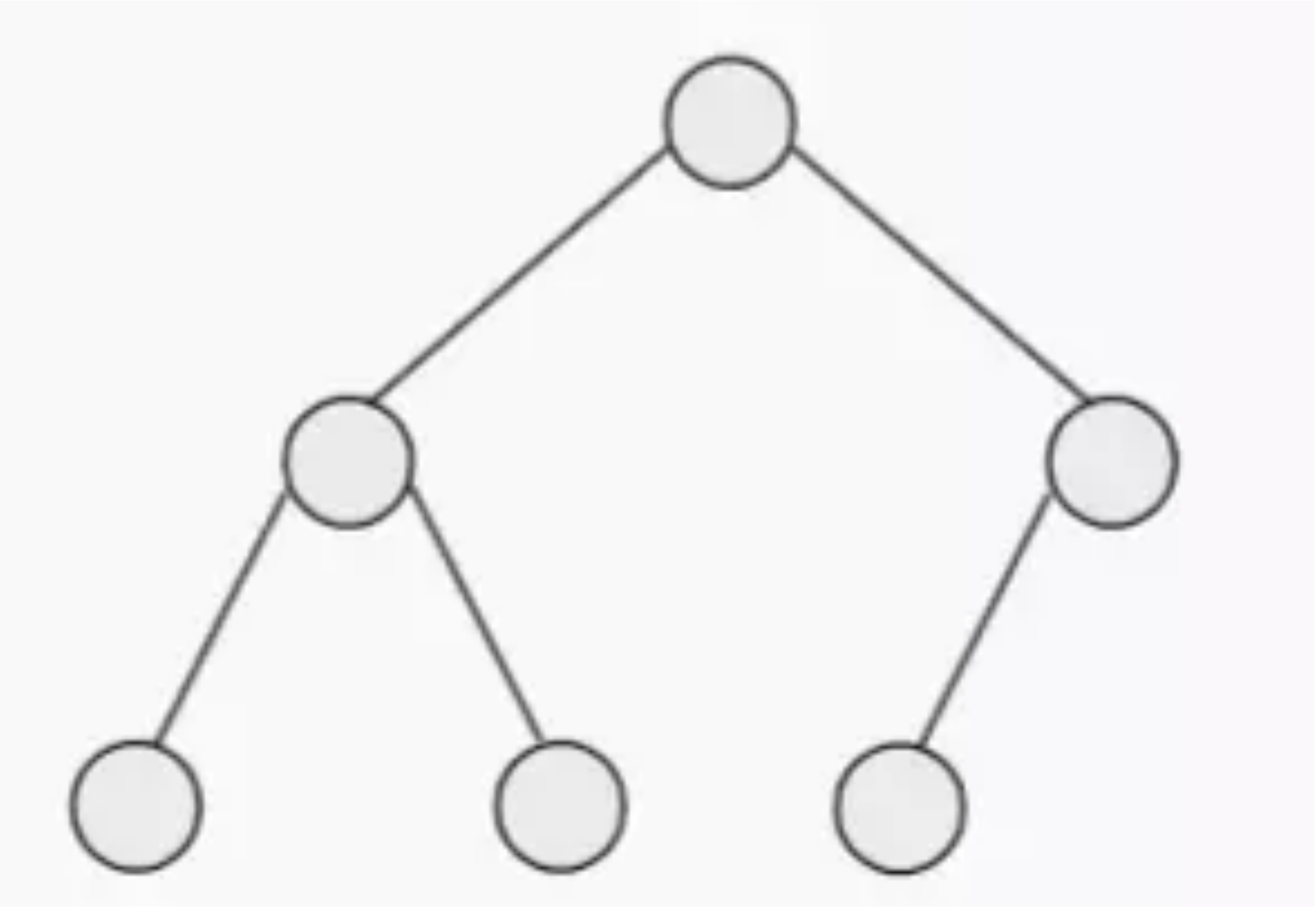
But wait there's more!

- Full (or proper): a binary tree whose every node has 0 or 2 children.
- Complete: a binary tree with minimal height. Any holes in tree would appear at last level to right, i.e., all nodes of last level are as left as possible.



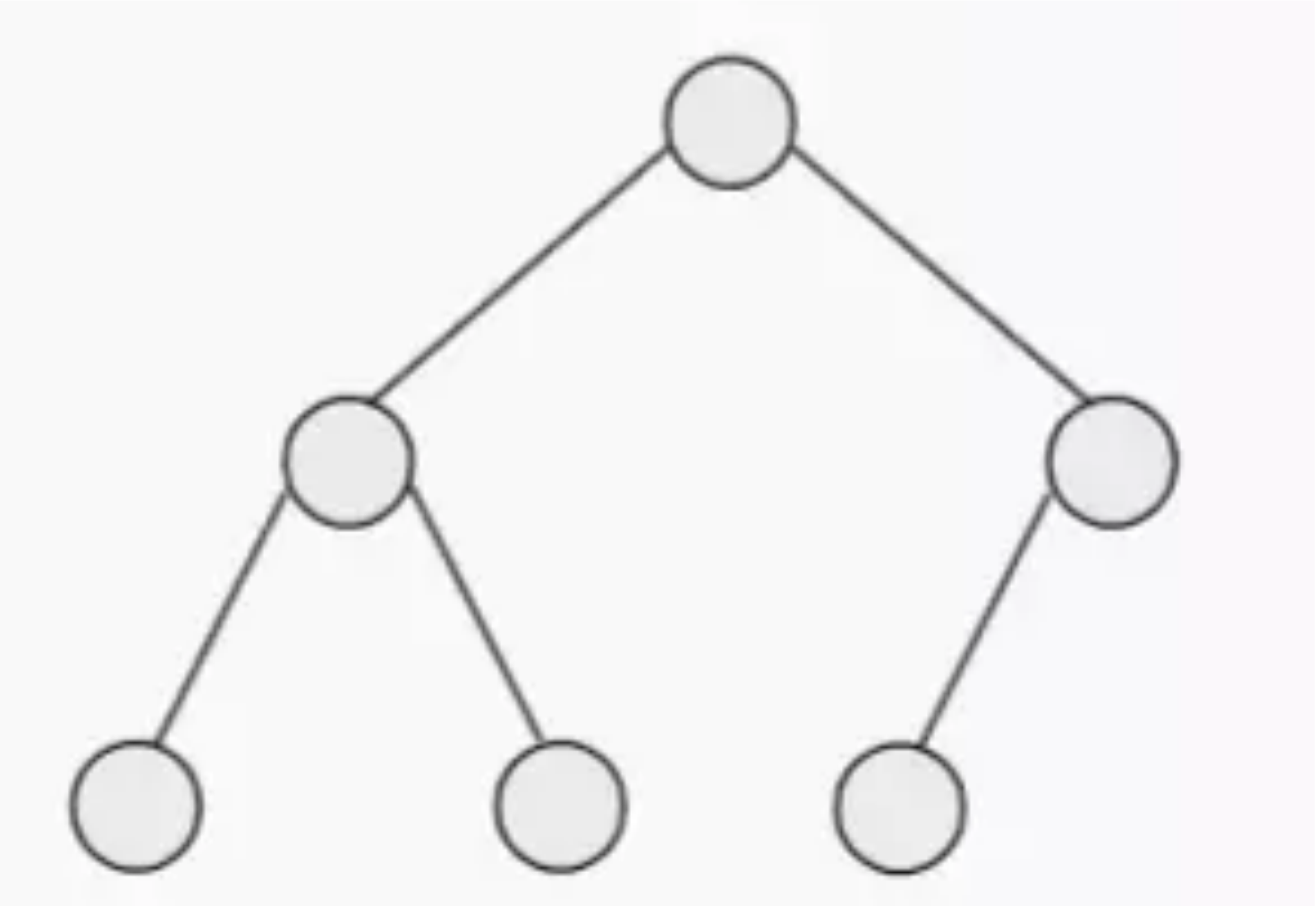
http://code.cloudkaksha.org/binary-tree/types-binary-tree

Practice Time: This tree is



- A: Full
- B: Complete
- C: Full and Complete
- D: Neither Full nor Complete

Answer



- A: Full
- **B: Complete**
- C: Full and Complete
- D: Neither Full nor Complete

Practice Time: This tree is

- A: Full
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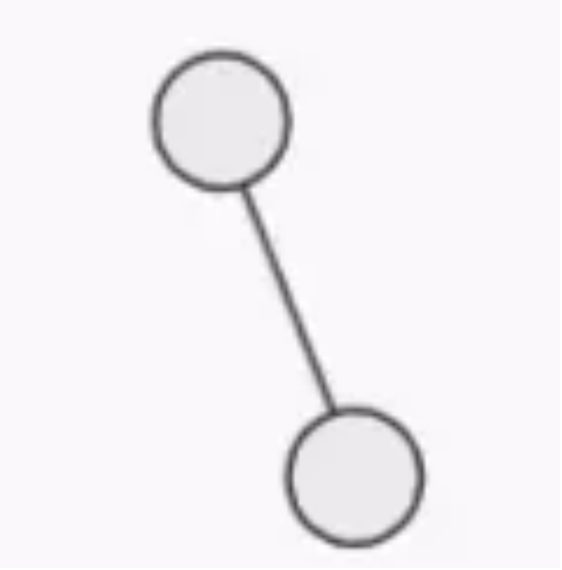
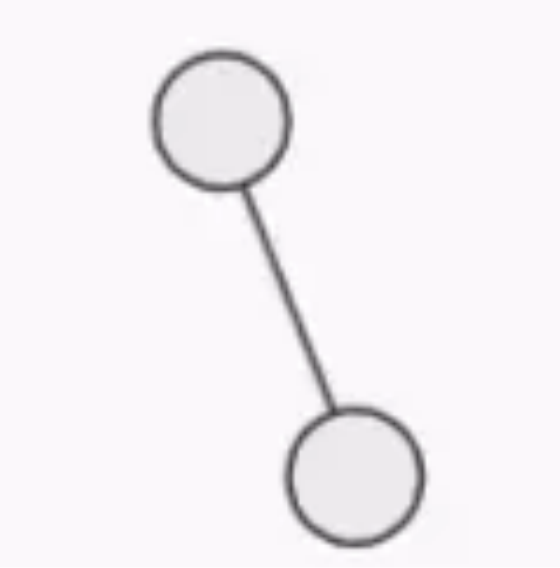
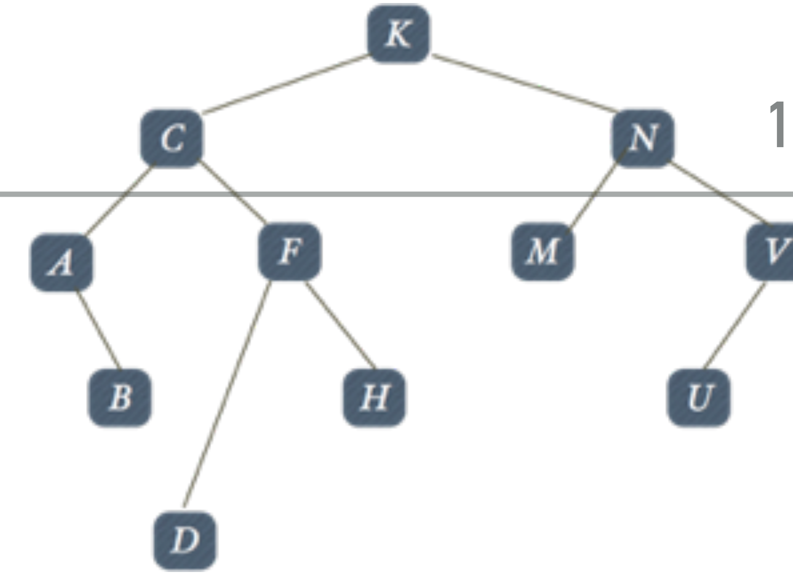


Answer

- A: Full
- B: Complete
- C: Full and Complete
- **D: Neither Full nor Complete**



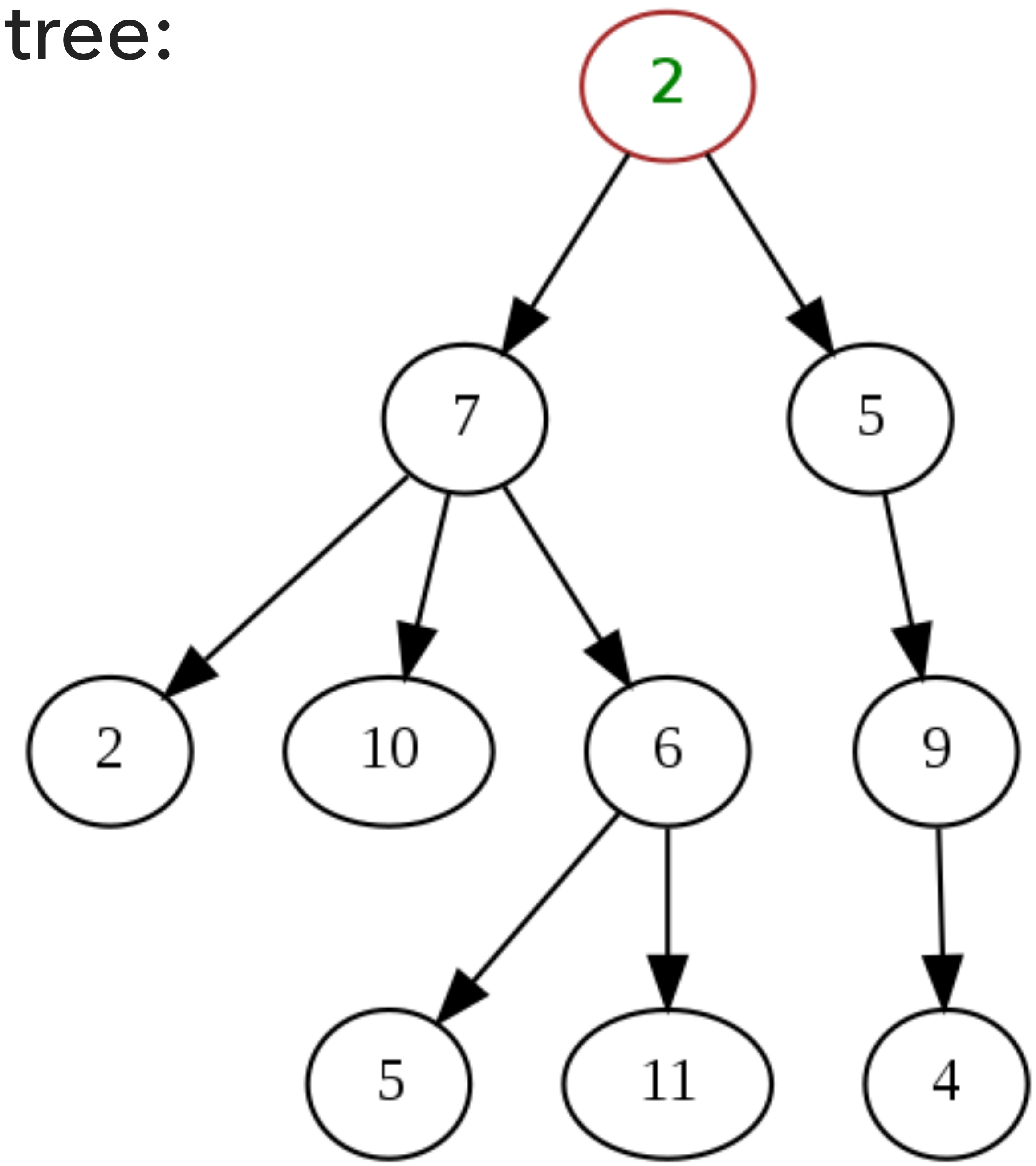
Counting in binary trees



- Lemma: if T is a binary tree, then at level k, T has $\leq 2^k$ nodes.
  - E.g., at level 2, at most 4 nodes (A, F, M, V)
- Theorem: If T has height h, then # of nodes n in T satisfy: $h + 1 \leq n \leq 2^{h+1} - 1$.
- Equivalently, if T has n nodes, then $log(n + 1) - 1 \leq h \leq n - 1$.
  - Worst case: When $h = n - 1$ or $O(n)$, the tree looks like a left or right-leaning "stick".
  - Best case: When a tree is as compact as possible (e.g., complete) it has $O(\log n)$ height.

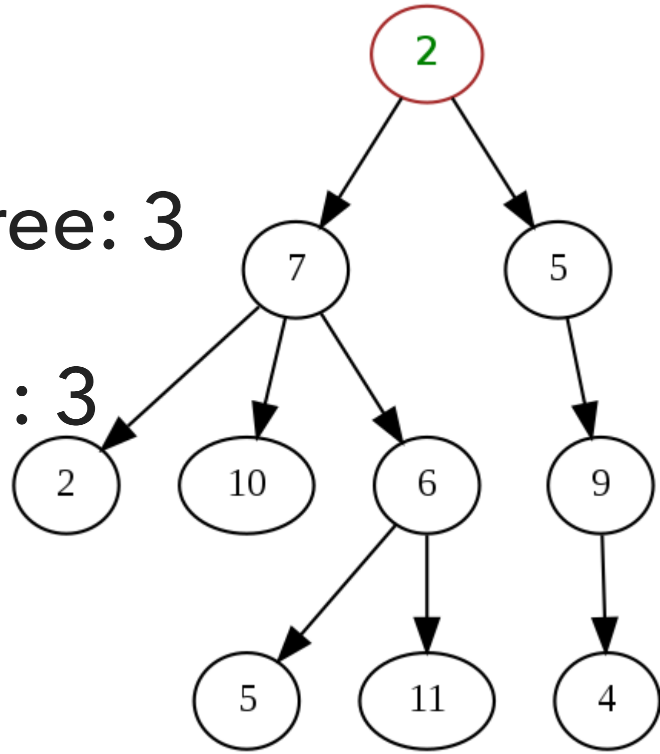
Practice Time - Problem 1 Worksheet #16

- Follow the instructions in the worksheet about the following tree:



ANSWER 1 - Worksheet #16

- Root: 2
- Leaves: 2 (in black), 10, 5, 11, 4
- Internal nodes: 7, 5, 6, 9
- Siblings of 10: 2, 6
- Parent of 6: 7
- Children of 2 (in red): 7, 5
- Ancestors of 10: 7 and 2 (in red)
- Descendants of 7: 2, 10, 6, 5, 11
- Length of path 2-5-9-4: 3
- Height of 7: 2
- Height of tree: 3
- Degree of 7: 3
- Arity/Degree of tree: 3
- Level/depth of 11: 3



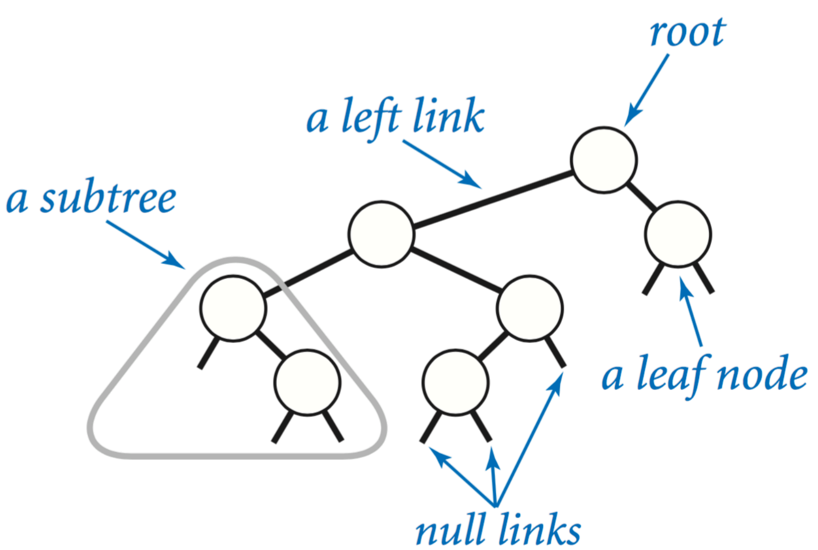
Basic idea behind a simple implementation

```
public class BinaryTree<E> {
    private Node root;

    /**
     * A node subclass which contains various recursive methods
     *
     * @param <E>  The type of the contents of nodes
     */
    private class Node {
        private E element;

        private Node left;
        private Node right;

        /**
         * Node constructor with subtrees
         *
         * @param left   the left node child
         * @param right      the right node child
         * @param E      the element contained in the node
         */
        public Node(Node left, Node right, E element) {
            this.left = left;
            this.right = right;
            this.element = item;
        }
```

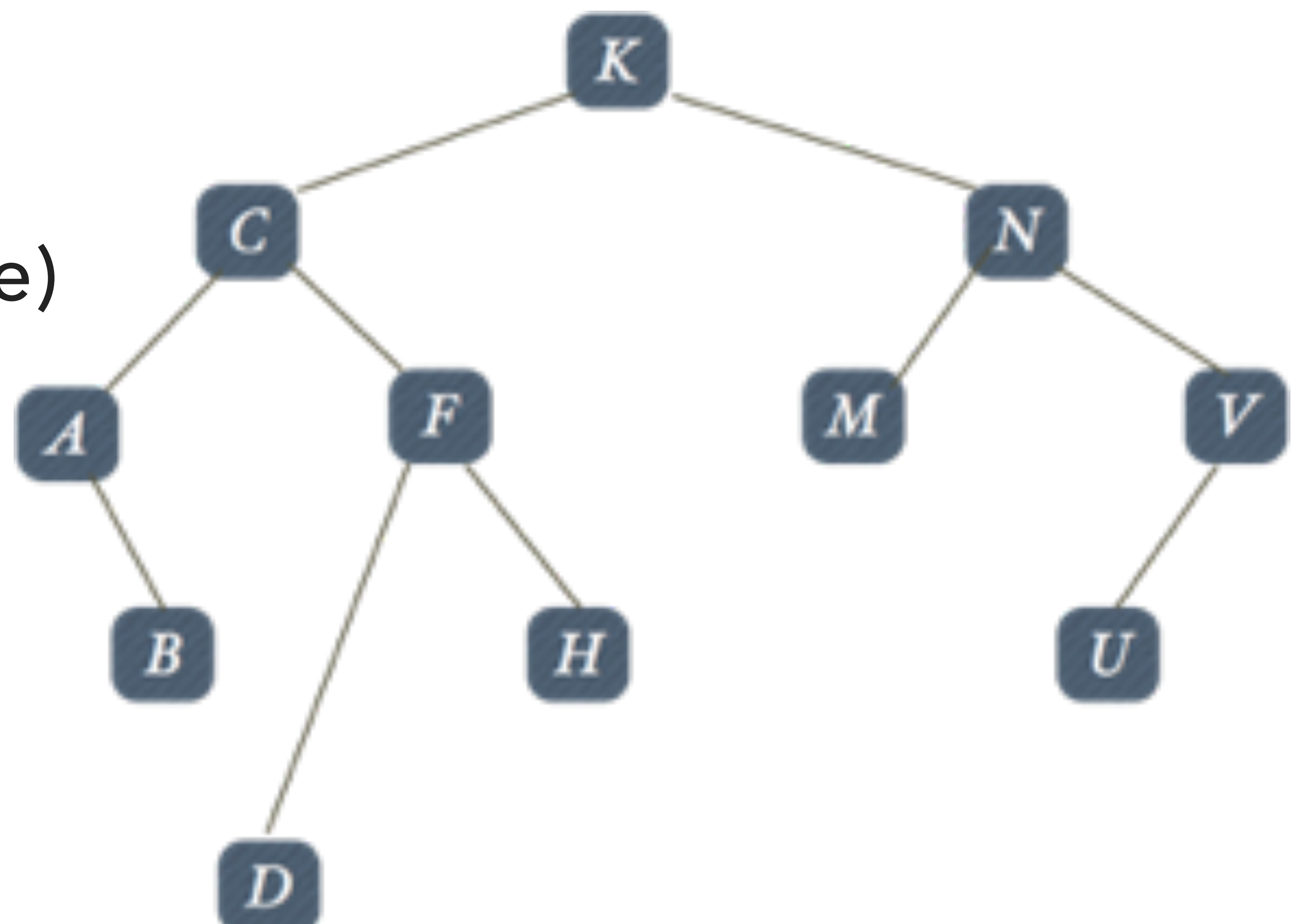


Lecture 16: Binary Trees, Binary Search, Heaps, and Priority Queues

- Binary Trees
- Tree traversals
- Binary Search
- Binary Heaps
- Priority Queues

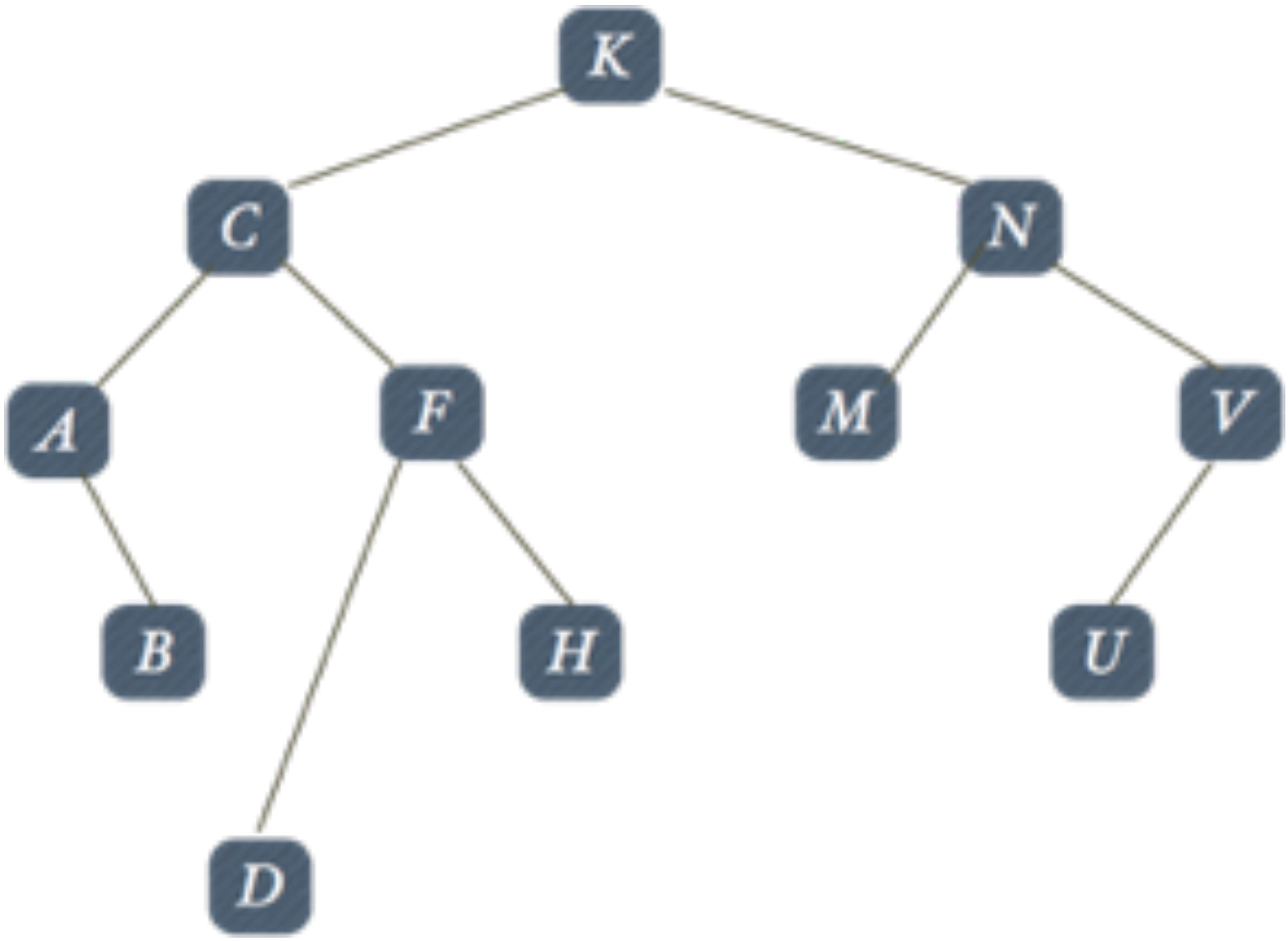
Pre-order traversal

- Preorder(Tree)
  - Mark root as visited
  - Preorder(Left Subtree)
  - Preorder(Right Subtree)
- K C A B F D H N M V U



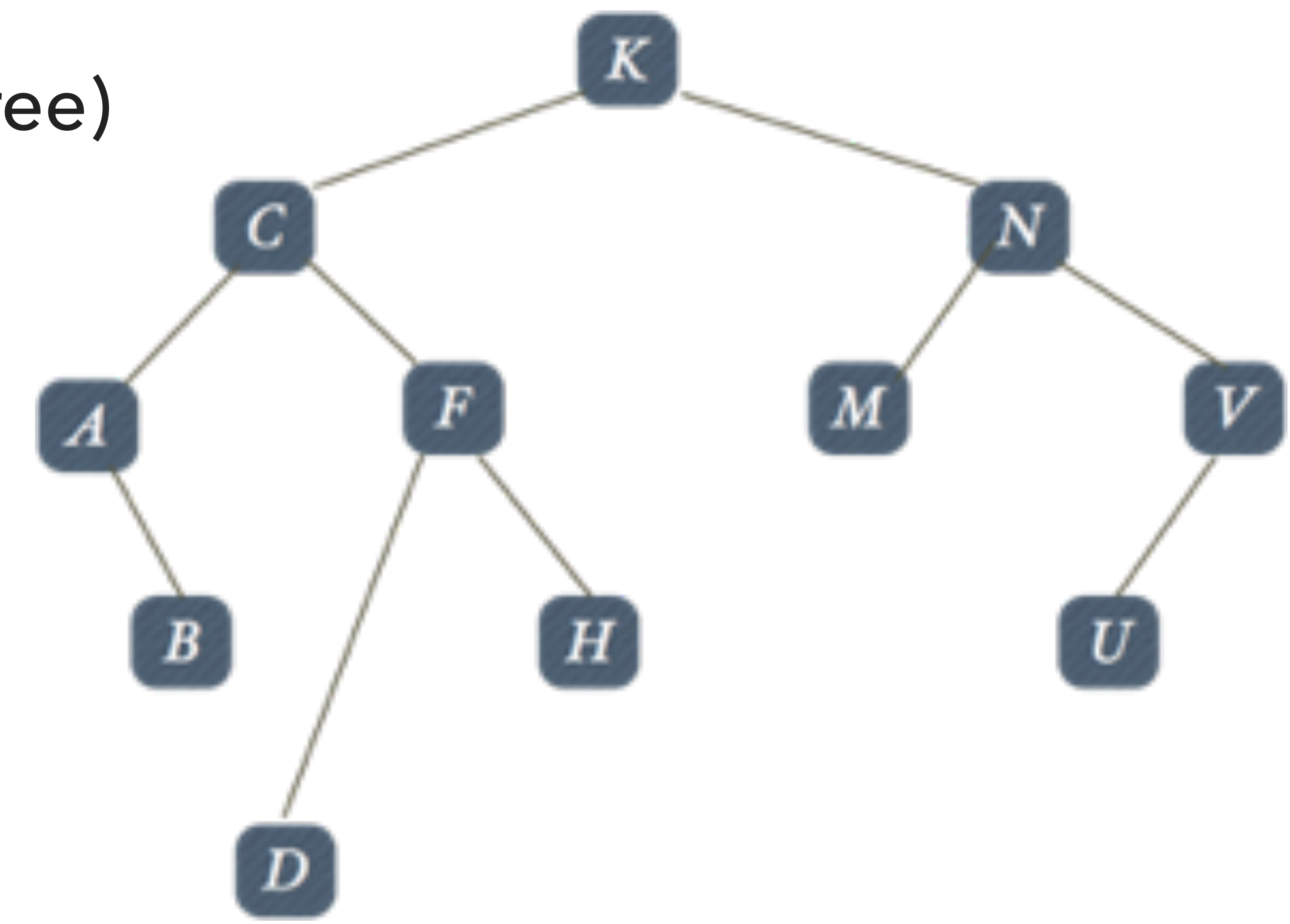
In-order traversal

- Inorder(Tree)
  - Inorder(Left Subtree)
  - Mark root as visited
  - Inorder(Right Subtree)
- A B C D F H K M N U V



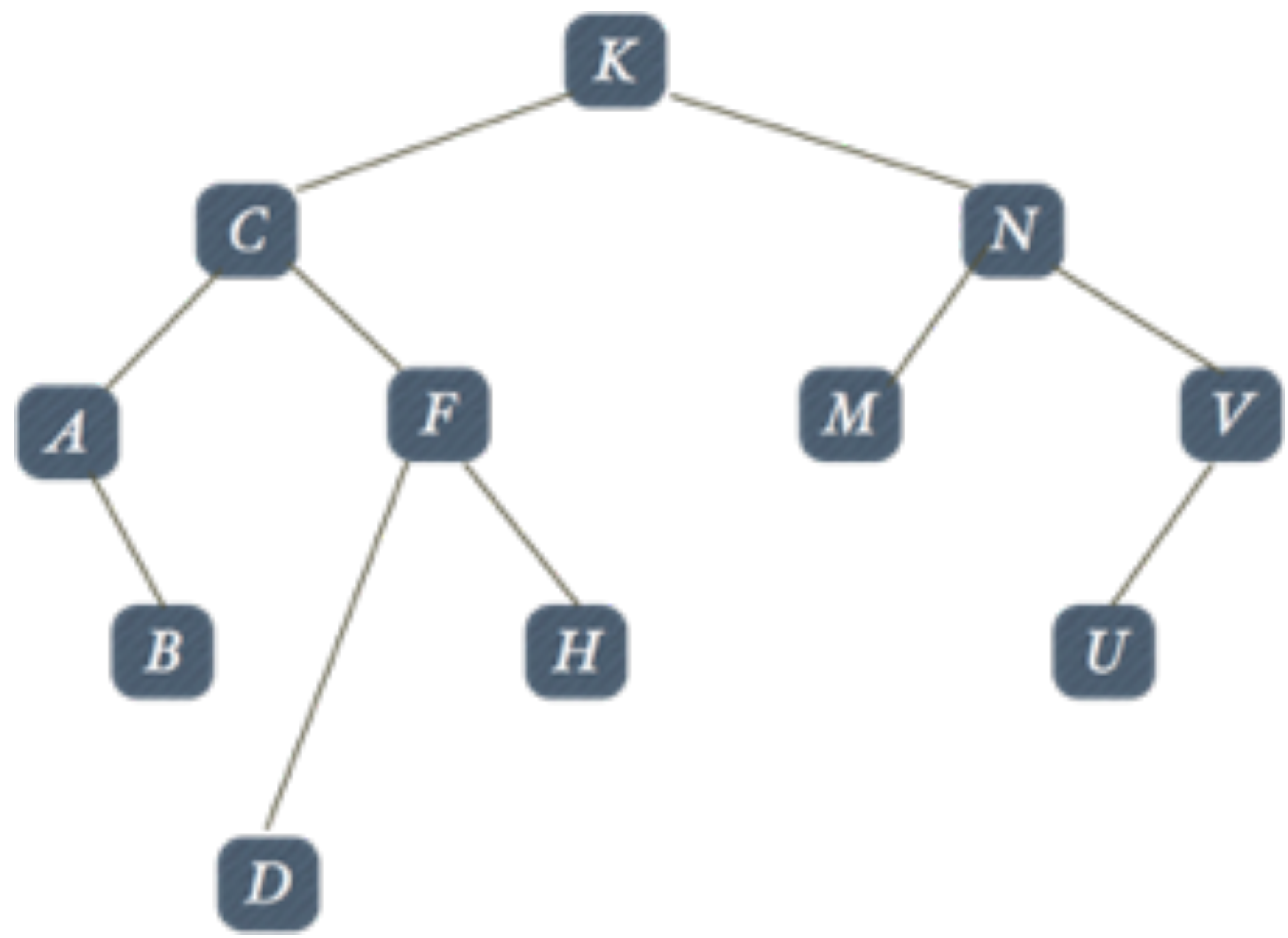
Post-order traversal

- Postorder(Tree)
  - Postorder(Left Subtree)
  - Postorder(Right Subtree)
  - Mark root as visited
- B A D H F C M U V N K



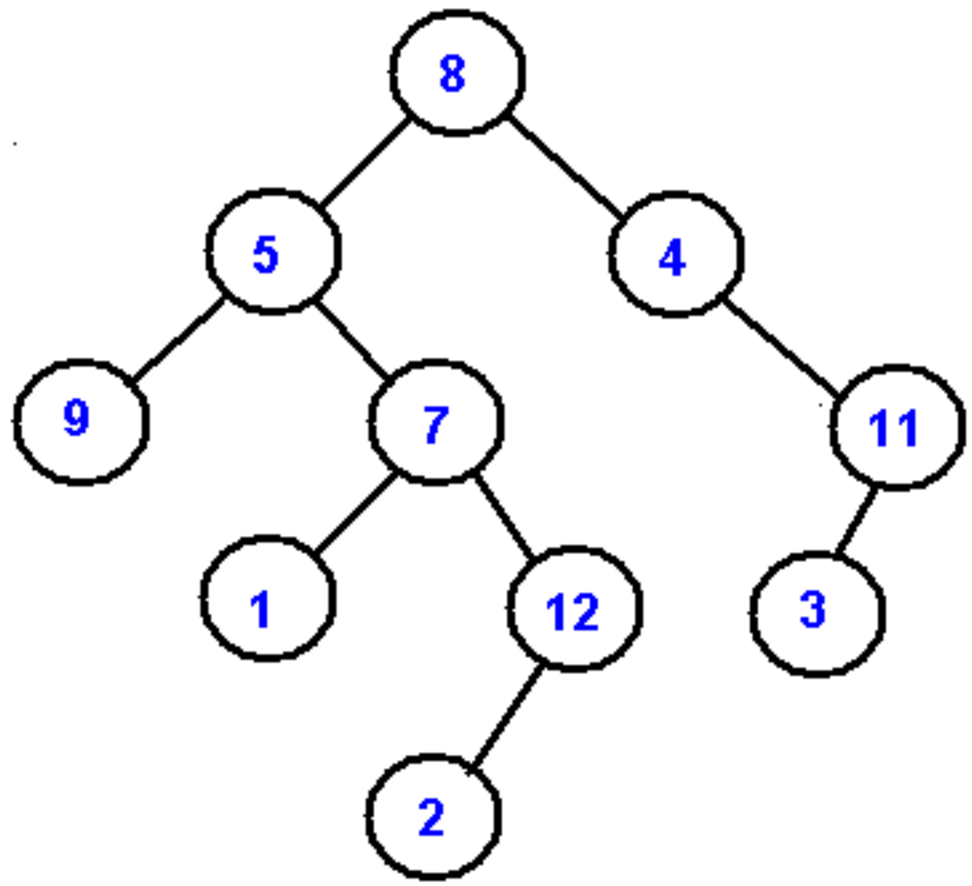
Level-order traversal

- From left to right, mark nodes of level i as visited before nodes in level $i + 1$. Start at level 0.
- K C N A F M V B D H U

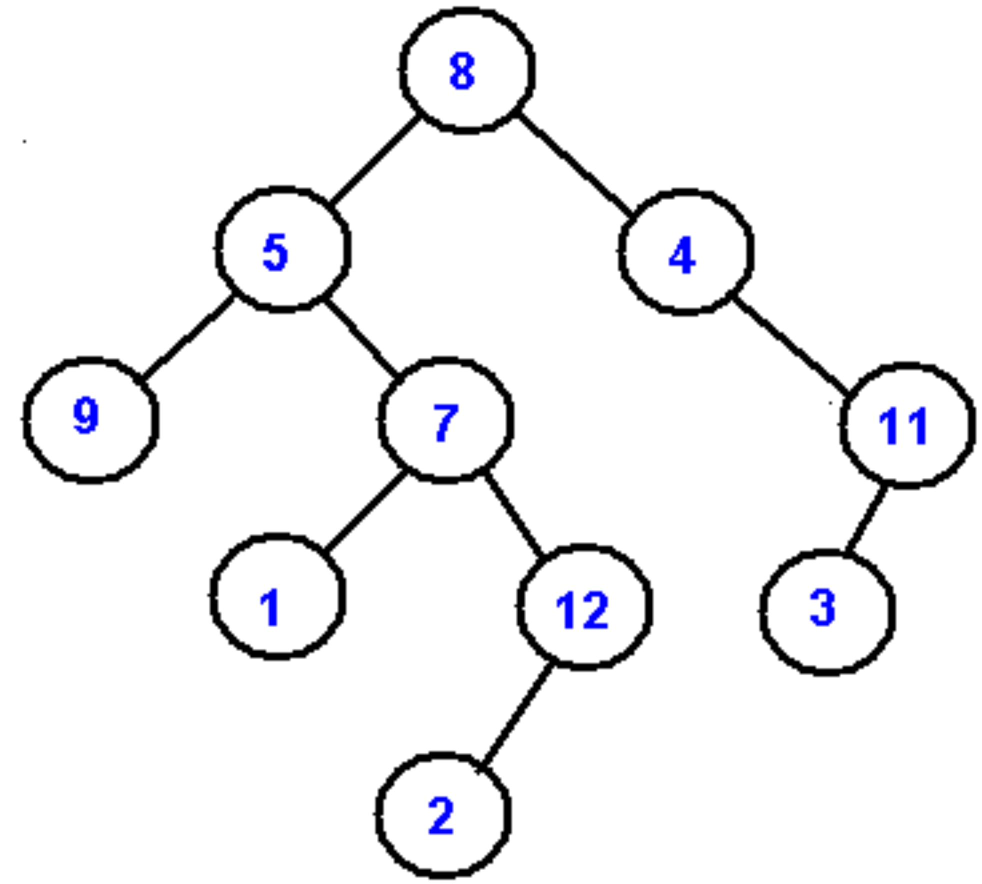


Practice Time - Problem 2 Worksheet #16

- List the nodes in pre-order, in-order, post-order, and level order:



ANSWER Problem 2 Worksheet #16



- Pre-order: 8, 5, 9, 7, 1, 12, 2, 4, 11, 3
- In-order: 9, 5, 1, 7, 2, 12, 8, 4, 3, 11
- Post-order: 9, 1, 2, 12, 7, 5, 3, 11, 4, 8
- Level-order: 8, 5, 4, 9, 7, 11, 1, 12, 3, 2

Lecture 16: Binary Trees, Binary Search, Heaps, and Priority Queues

- Binary Trees
- Tree traversals
- Binary Search
- Binary Heaps
- Priority Queues

Binary search

- Goal: Given a sorted array and a key, find index of the key in the array.
- Basic mechanism: Compare key against middle entry.
  - If too small, repeat in left half.
  - If too large, repeat in right half.
  - If equal, you are done.

Binary search implementation

- First binary search published in 1946.
- First bug-free one in 1962.
- Bug in Java's `Arrays.binarySearch()` discovered in 2006 https://ai.googleblog.com/2006/06/extra-extra-read-all-about-it-nearly.html

```
public static int binarySearch(int[] a, int key) {
    int lo = 0, hi = a.length-1;
    while (lo <= hi) {
         int mid = lo + (hi - lo) / 2;
         if (key < a[mid])
              hi = mid - 1;
         else if (key > a[mid])
              lo = mid + 1;
         else return mid; }
    return -1;
}
```

- Uses at most $1 + \log n$ key compares to search in a sorted array of size n, that is it is $O(\log n)$.

Lecture 16: Binary Trees, Binary Search, Heaps, and Priority Queues

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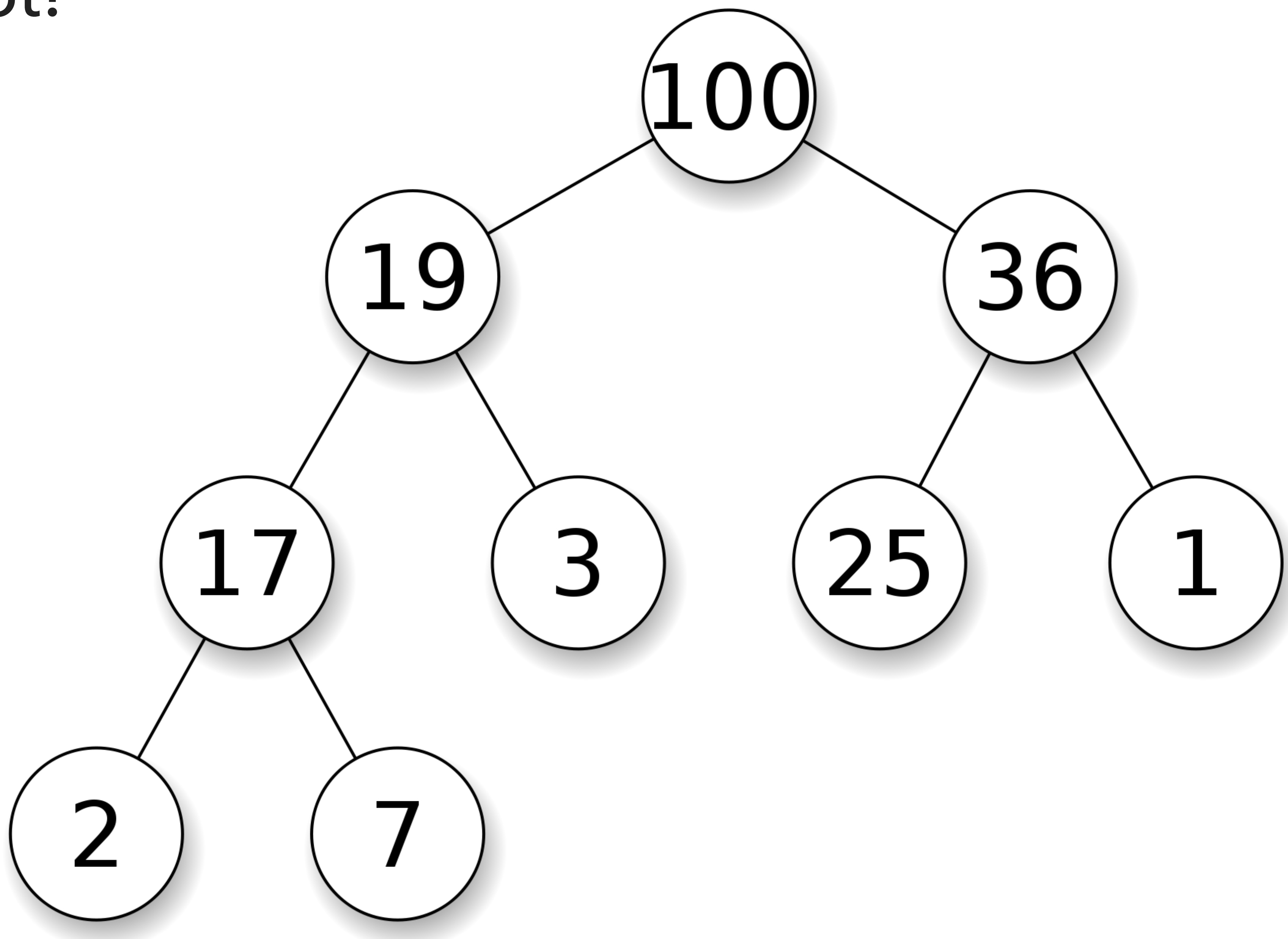
Heap-ordered binary trees

- A binary tree is heap-ordered if the key in each node is larger than or equal to the keys in that node's two children (if any).
- Equivalently, the key in each node of a heap-ordered binary tree is smaller than or equal to the key in that node's parent (if any).
- No assumption of which child is smaller.
- Moving up from any node, we get a non-decreasing sequence of keys.
- Moving down from any node we get a non-increasing sequence of keys.

Heap-ordered binary trees

- The largest key in a heap-ordered binary tree is found at the root!



Binary heap representation

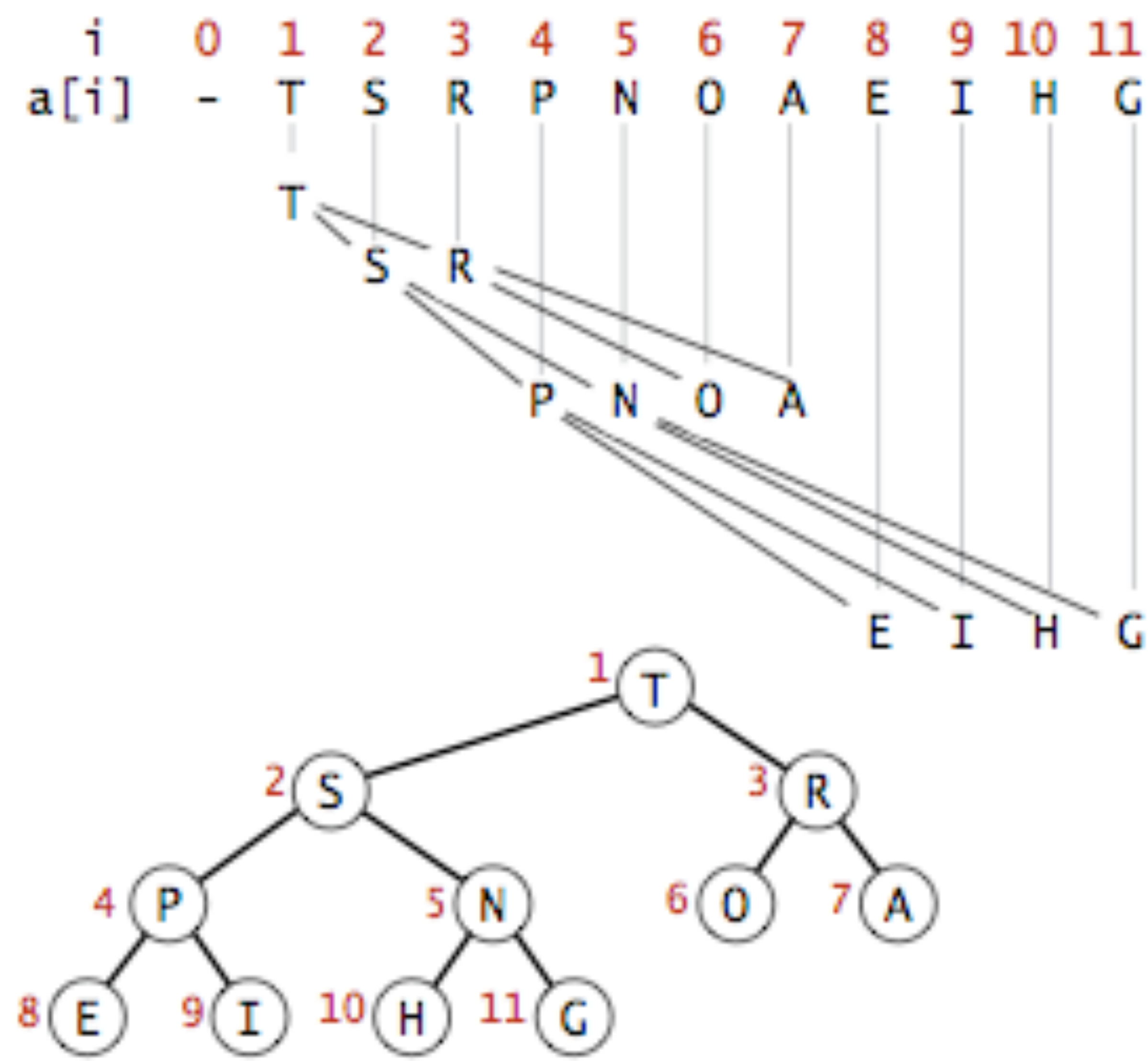
- We could use a linked representation but we would need three links for every node (one for parent, one for left subtree, one for right subtree).
- If we use complete binary trees, we can use instead an array.
  - Compact arrays vs explicit links means memory savings!

Binary heaps

- Binary heap: the array representation of a complete heap-ordered binary tree.
    - Items are stored in an array such that each key is guaranteed to be larger (or equal to) than the keys at two other specific positions (children).
- Max-heap but there are min-heaps, too.

Array representation of heaps

- Nothing is placed at index 0.
- Root is placed at index 1.
- Rest of nodes are placed in level order.
- No unnecessary indices and no wasted space because it's complete.



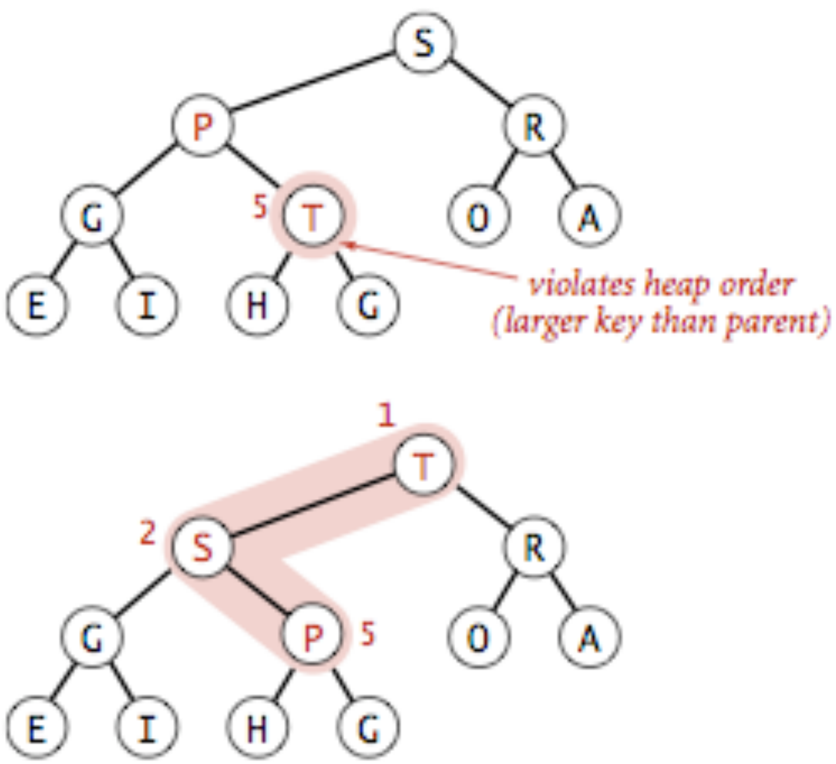
Heap representations

Reuniting immediate family members.

- For every node at index k, its parent is at index $\lfloor k/2 \rfloor$.
- Its two children are at indices $2k$ and $2k+1$.
- We can travel up and down the heap by using this simple arithmetic on array indices.

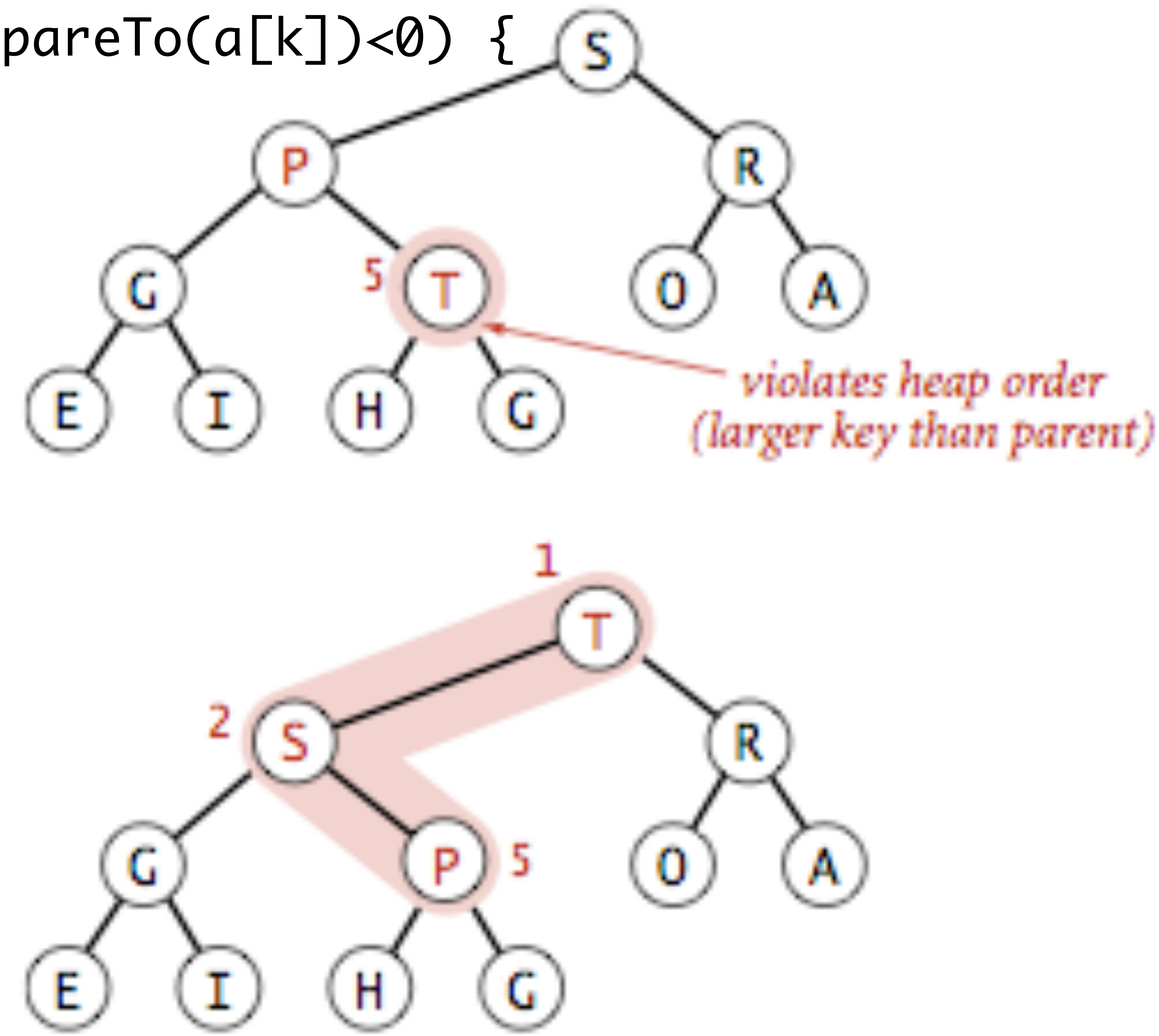
Swim/promote/percolate up/bottom up reheapify

- Scenario: a key becomes larger than its parent therefore it violates the heap-ordered property.
- To eliminate the violation:
  - Exchange key in child with key in parent.
  - Repeat until heap order restored.



Swim/promote/percolate up

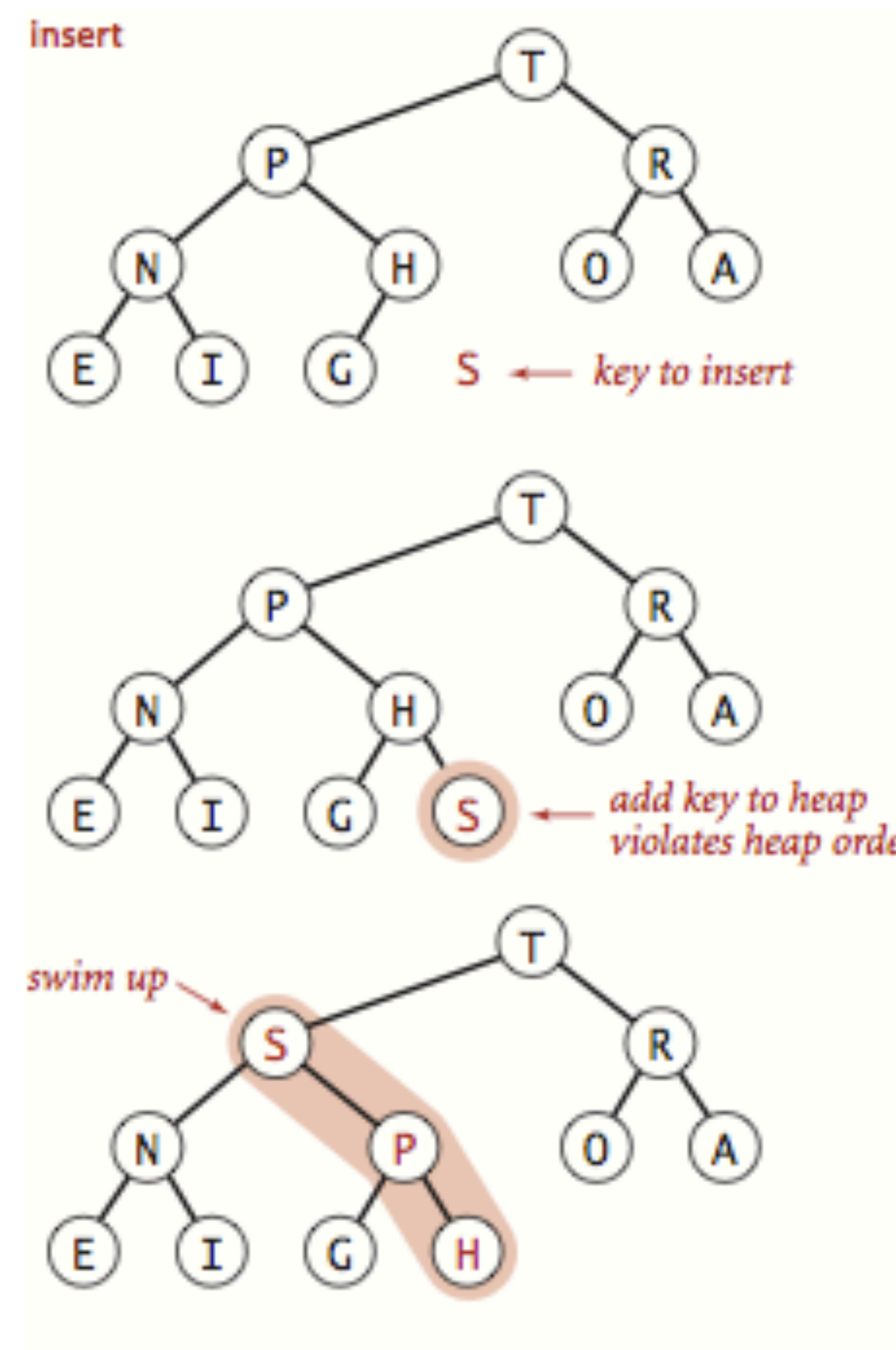
```
private void swim(int k) {
   while (k > 1 && a[k/2].compareTo(a[k])<0) {
      E temp = a[k];
      a[k] = a[k/2];
      a[k/2] = temp;
      k = k/2;
   }
}
```



Binary heap: insertion

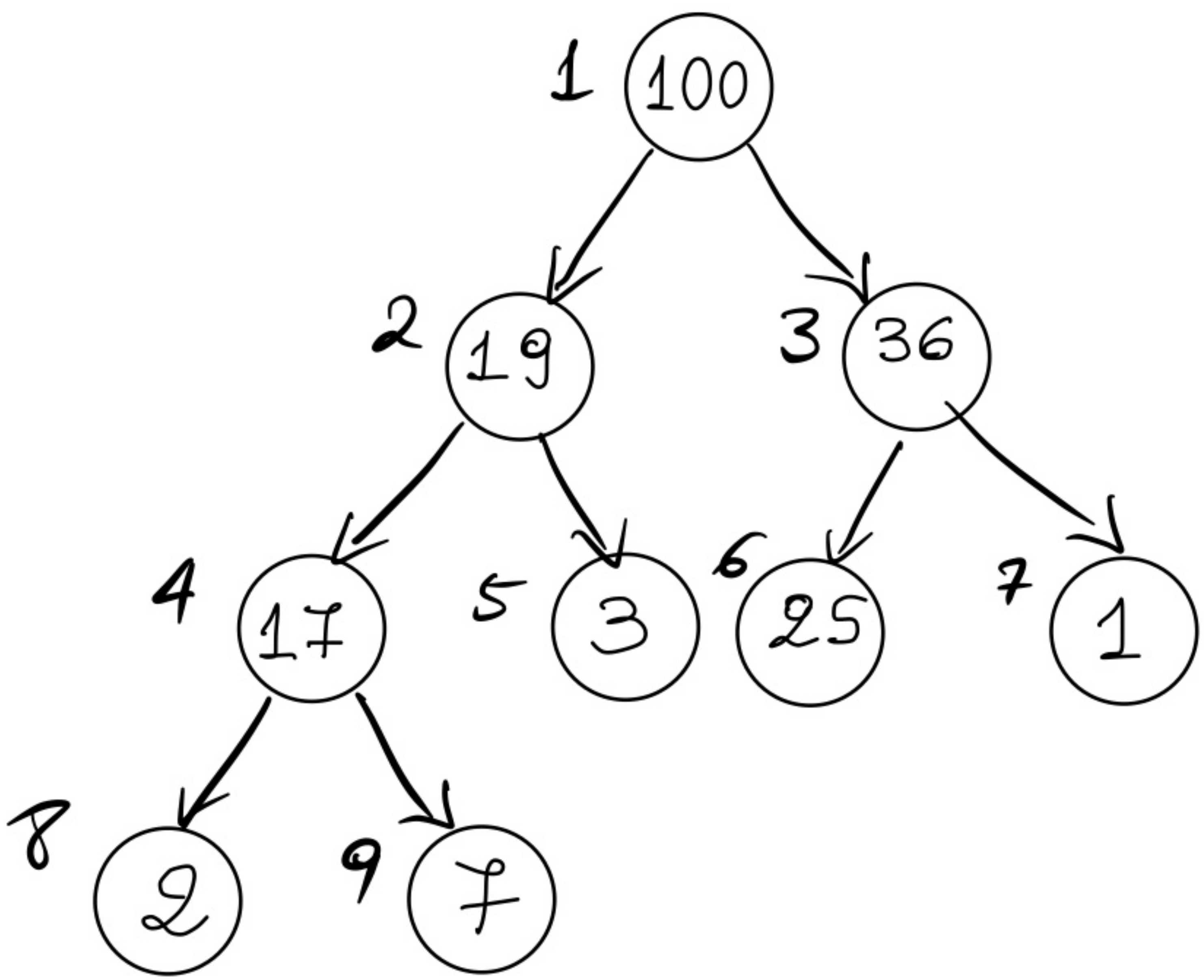
- Insert: Add node at end in bottom level, then swim it up.
- Cost: At most $\log n + 1$ compares.

```
public void insert(E x) {
    a[++n] = x;
    swim(n);
}
```

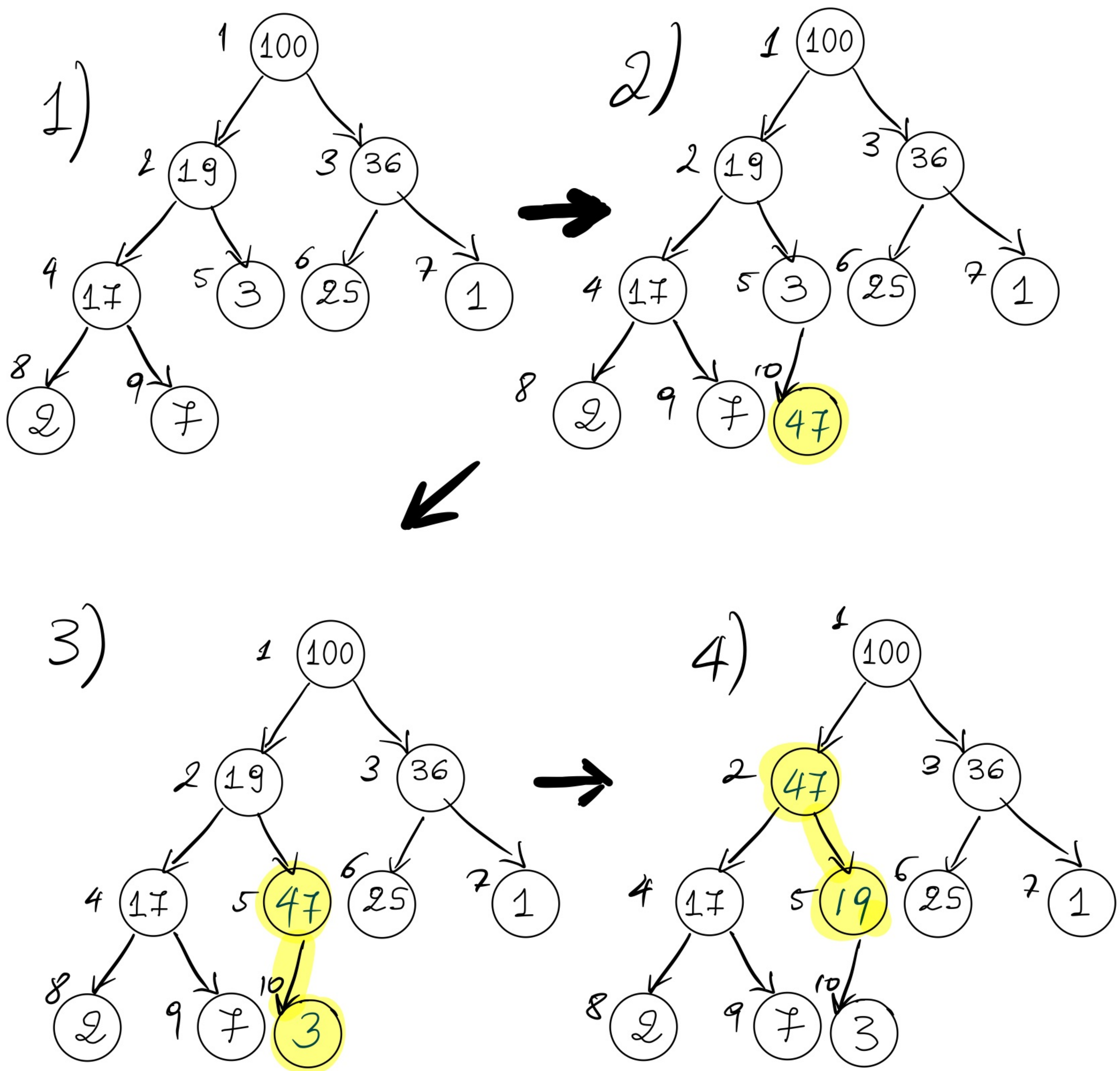


Practice Time - Problem 3 Worksheet #16

- Insert 47 in this binary heap.

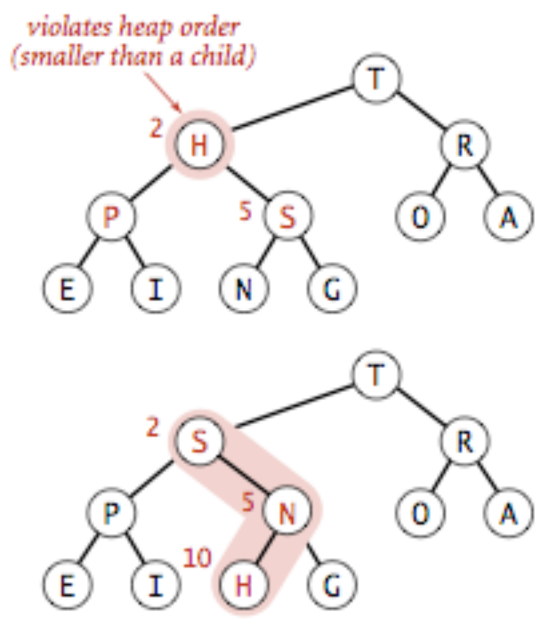


ANSWER 3



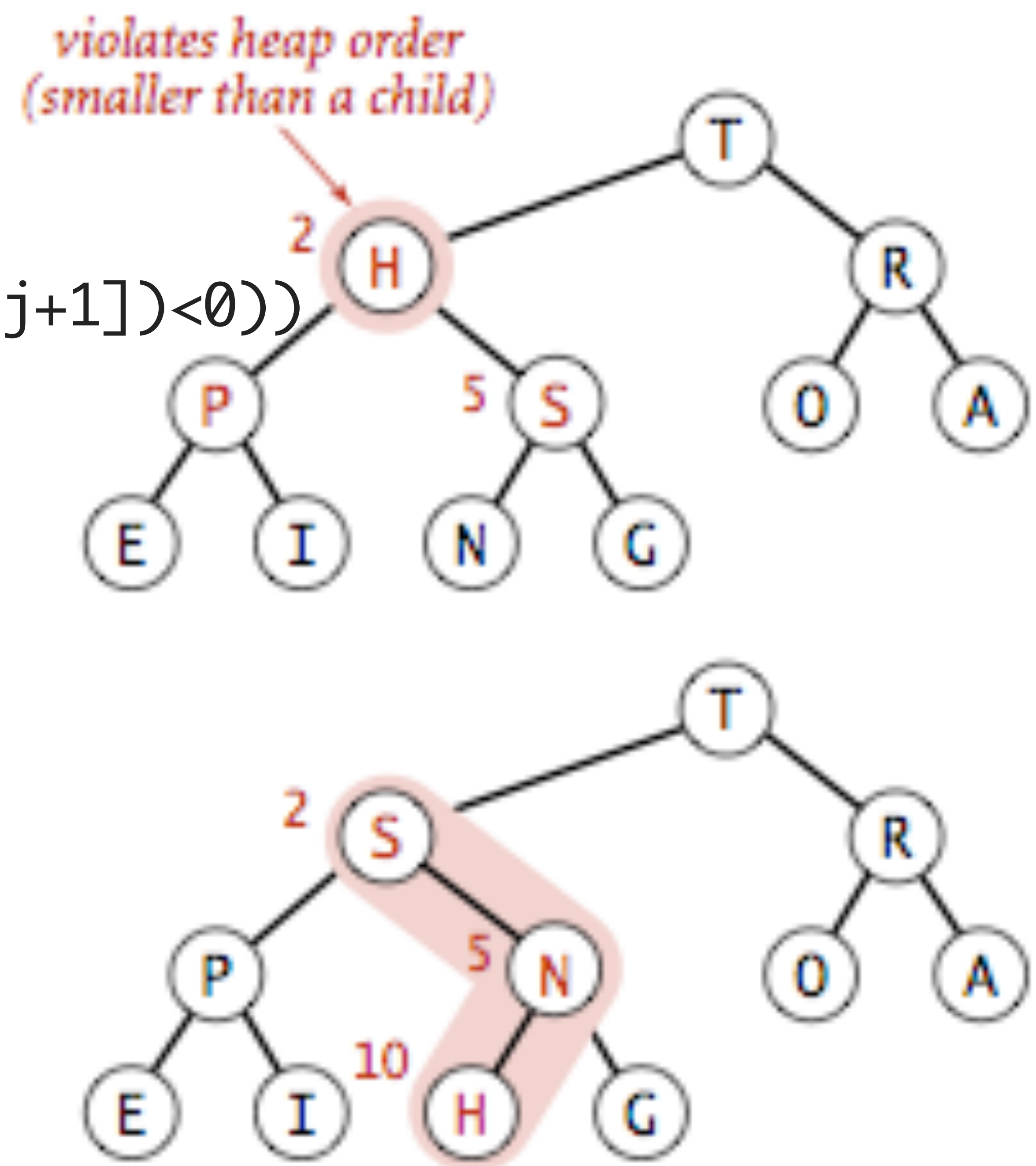
Sink/demote/top down heapify

- Scenario: a key becomes smaller than one (or both) of its children's keys.
- To eliminate the violation:
  - Exchange key in parent with key in **larger** child.
  - Repeat until heap order is restored.



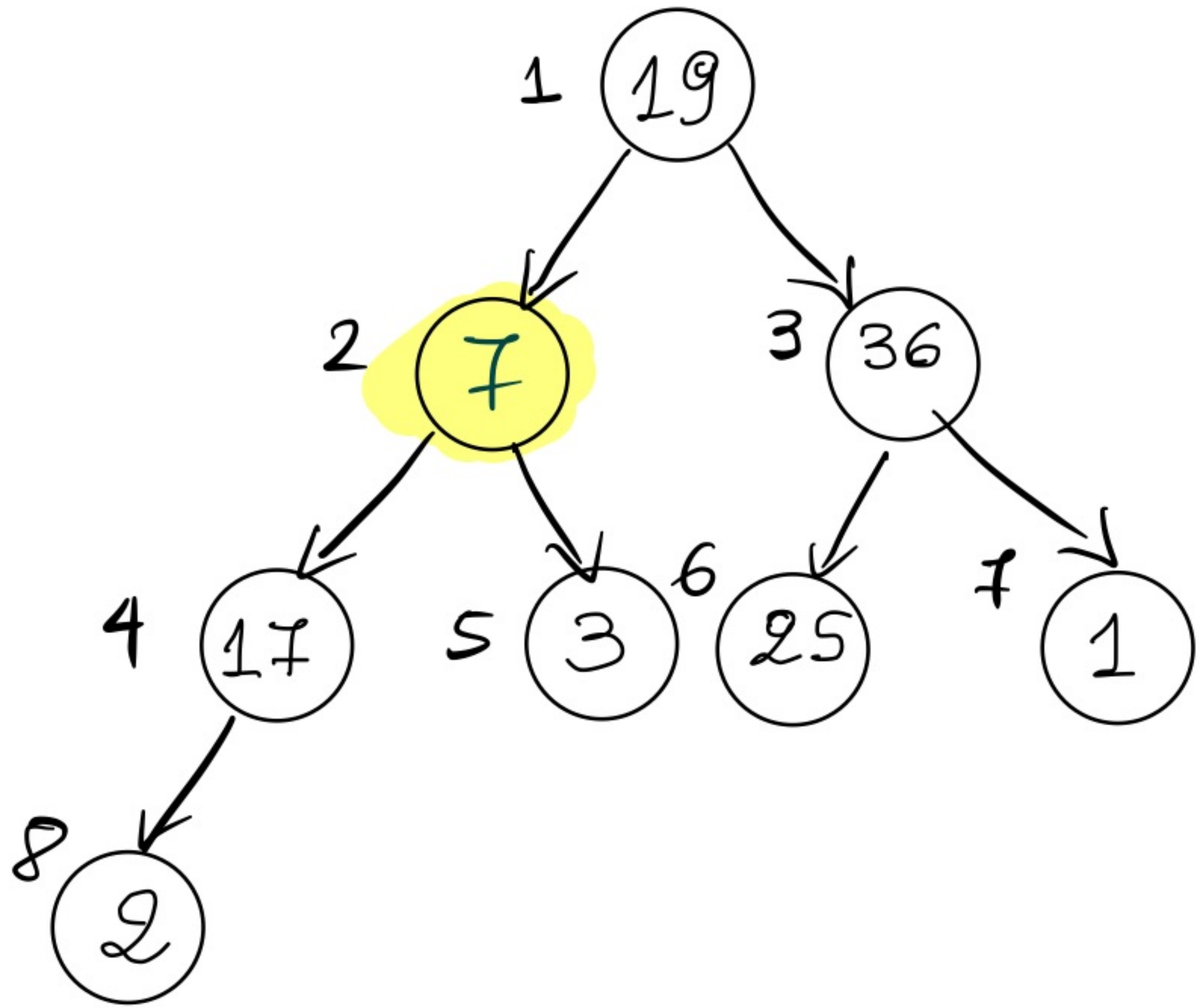
Sink/demote/top down heapify

```
private void sink(int k) {
    while (2*k <= n) {
        int j = 2*k;
        if (j < n && a[j].compareTo(a[j+1])<0))
            j++;
        if (a[k].compareTo(a[j])>=0))
            break;
        E temp = a[k];
        a[k] = a[j];
        a[j] = temp;
        k = j;
    }
}
```

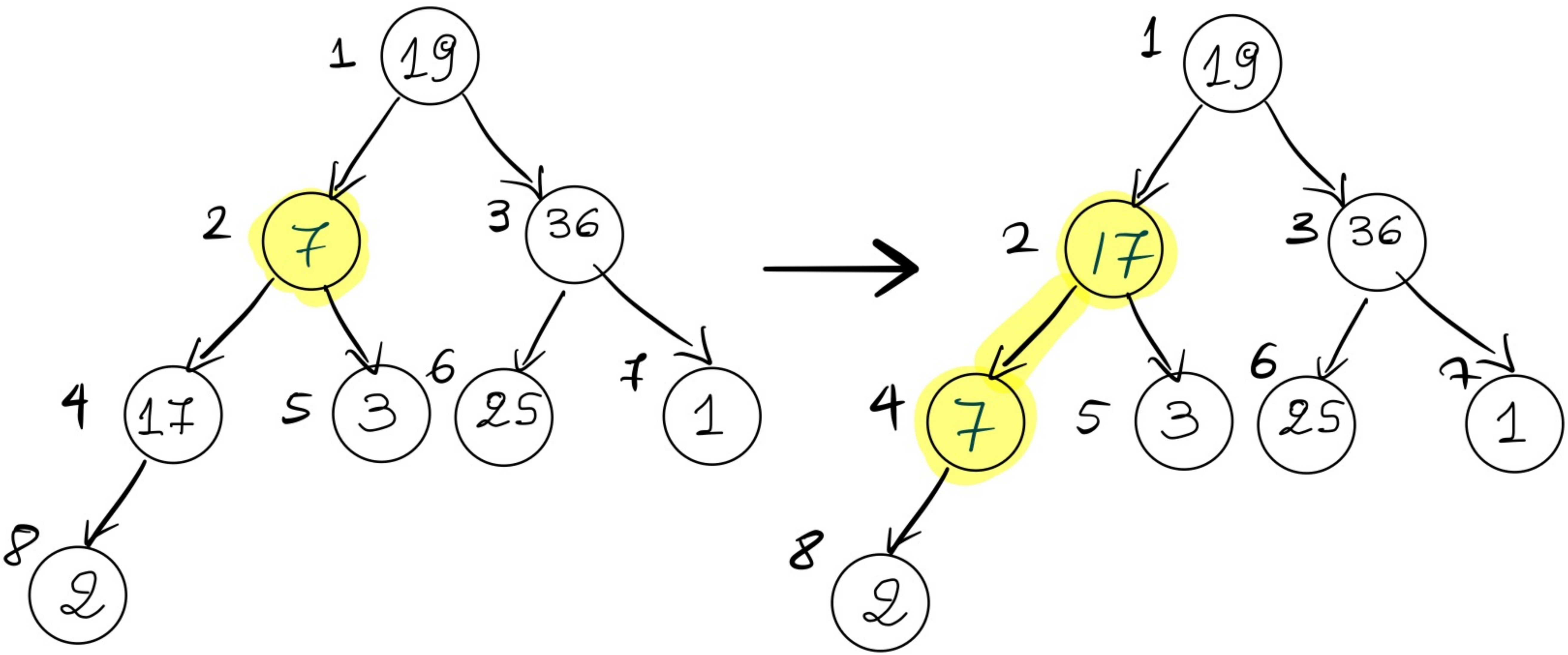


Practice Time - Problem 4 Worksheet #16

- Sink 7 to its appropriate place in this binary heap.



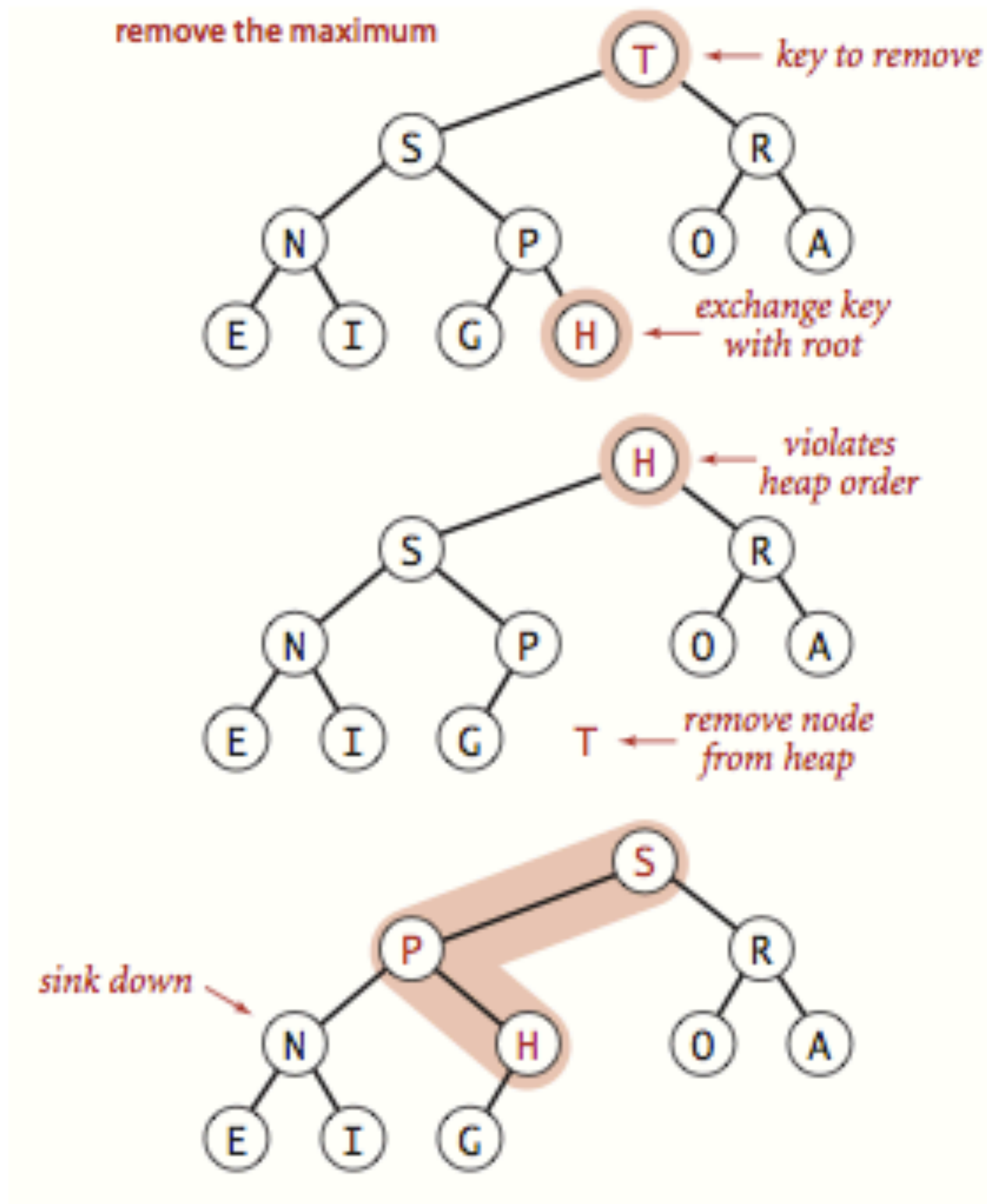
ANSWER 4



Binary heap: return (and delete) the maximum

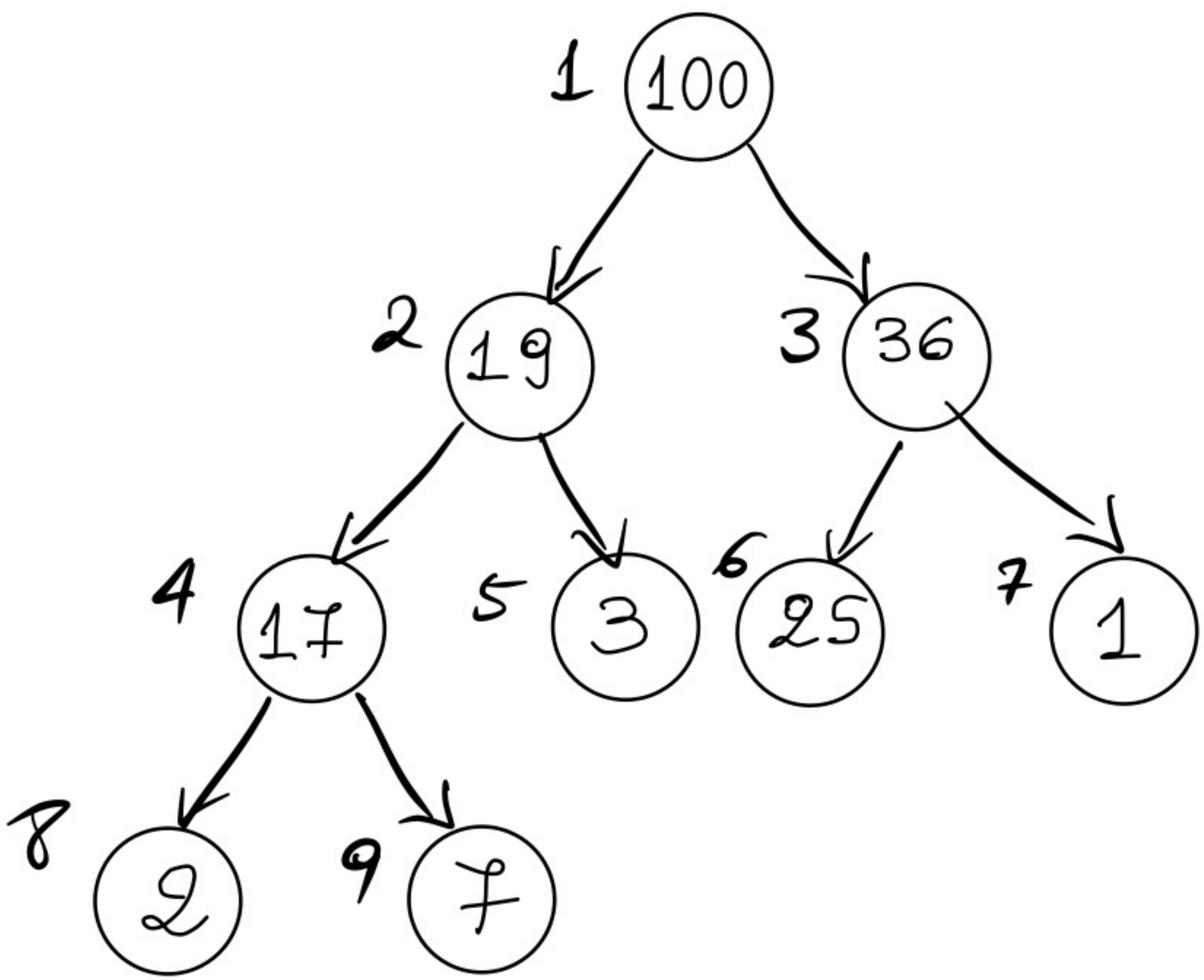
- Delete max: Exchange root with node at end. Return it and delete it. Sink the new root down.
- Cost: At most $2 \log n$ compares.

Binary heap: delete and return maximum

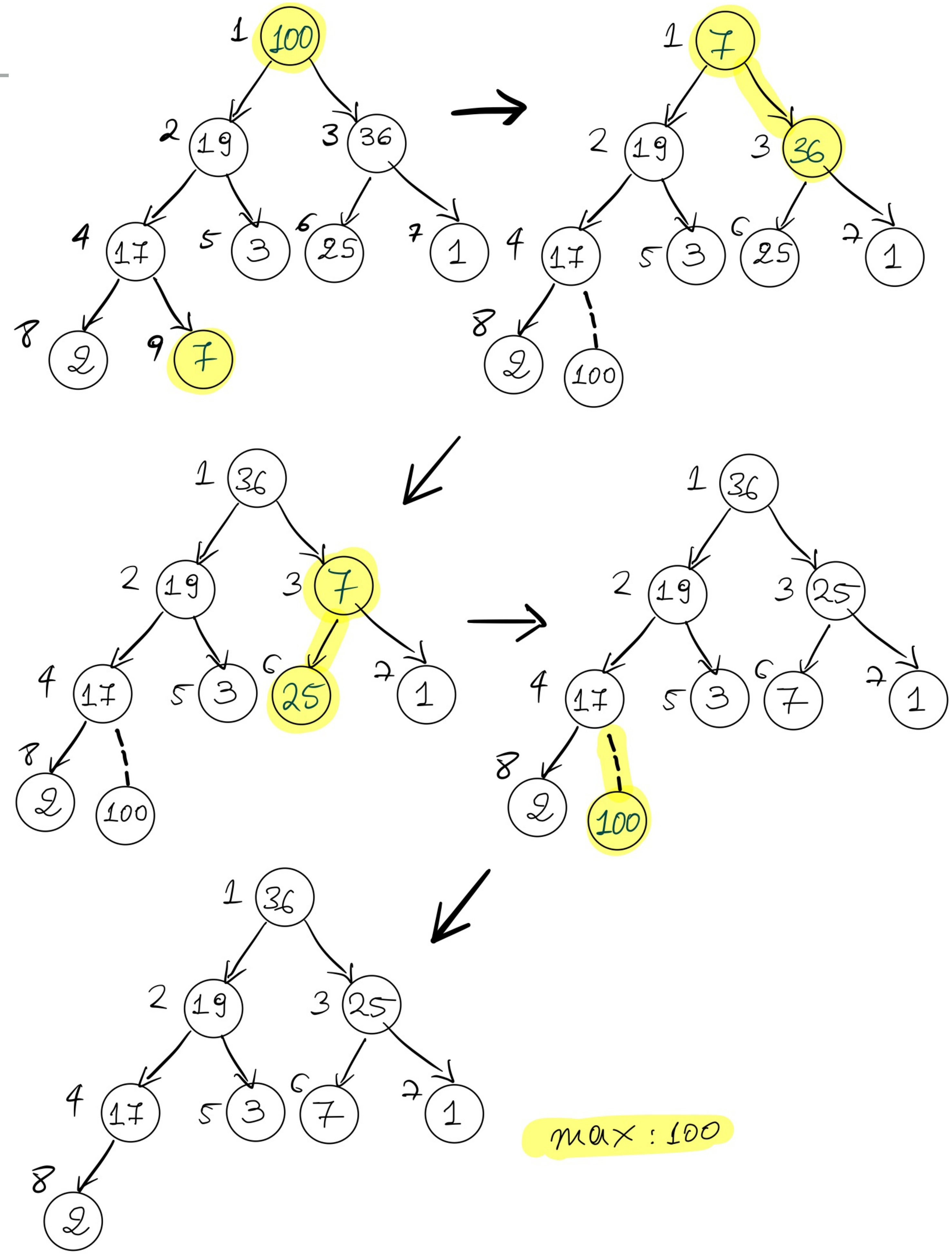


Practice Time - Problem 5 Worksheet #16

- Delete max (and return it!)



ANSWER 5



Things to remember about running time complexity of heaps

- Insertion is $O(\log n)$.
- Delete max is $O(\log n)$.
- Space efficiency is $O(n)$.

http://algs4.cs.princeton.edu

2.4 BINARY HEAP DEMO

Lecture 16: Binary Trees, Binary Search, Heaps, and Priority Queues

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- Priority Queues

Priority Queue ADT

- Two operations:
  - Delete the maximum
  - Insert
- Applications: load balancing and interruption handling in OS, Huffman codes for compression, A* search for AI, Dijkstra’s and Prim's algorithm for graph search, etc.
- How can we implement a priority queue efficiently?

Option 1: Unordered array

- The *lazy* approach where we defer doing work (deleting the maximum) until necessary.
- Insert is $O(1)$ and assumes we have the space in the array.
- Delete maximum is $O(n)$ (have to traverse the entire array to find the maximum element and exchange it with the last element).

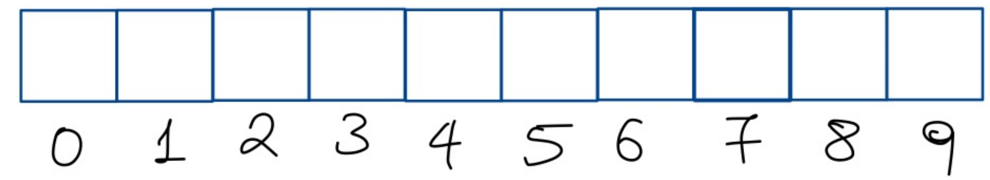
```
public class UnorderedArrayMaxPQ<Key extends Comparable<Key>> {
    private Key[] pq;      // elements
    private int n;         // number of elements

    // set inititial size of heap to hold size elements
    public UnorderedArrayMaxPQ(int capacity) {
        pq = (Key[]) new Comparable[capacity];
        n = 0;
    }

    public boolean isEmpty()   { return n == 0; }
    public int size()          { return n;      }
    public void insert(Key x)  { pq[n++] = x;   }

    public Key delMax() {
        int max = 0;
        for (int i = 1; i < n; i++){
            if (pq[max].compareTo(pq[i]) < 0) {
                 max = i;
            }
        }
        Key temp = pq[max];
        pq[max] = pq[n-1];
        pq[n-1] = temp;

        return pq[--n];
    }
}
```



Practice Time

- Given an empty array of capacity 10, perform the following operations in a priority queue based on an unordered array (lazy approach):

1. Insert P
2. Insert Q
3. Insert E
4. Delete max
5. Insert X
6. Insert A
7. Insert M
8. Delete max
9. Insert P
10. Insert L
11. Insert E
12. Delete max

Answer

0	1	2	3	4	5	6	7	8	9	Operation
P										insert P
P	Q									insert Q
P	Q	E								insert E
P	E	~~Q~~								delete-max → Q
P	E	X								insert X
P	E	X	A							insert A
P	E	X	A	M						insert M
P	E	M	A	~~X~~						delete-max → X
P	E	M	A	P						insert P
P	E	M	A	P	L					insert L
P	E	M	A	P	L	E				insert E
E	E	M	A	P	L	~~P~~				delete-max → P

Option 2: Ordered array

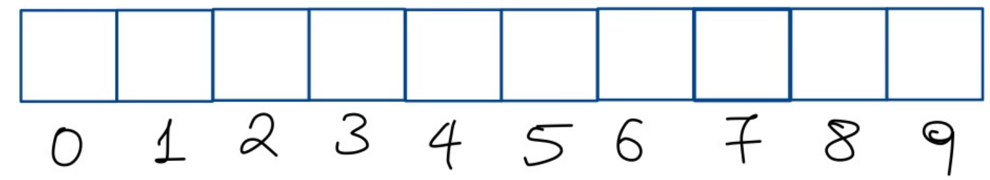
- The *eager* approach where we do the work (keeping the array sorted) up front to make later operations efficient.
- Insert is $O(n)$ (we have to find the index to insert and shift elements to perform insertion).
- Delete maximum is $O(1)$ (just take the last element which will be the maximum).

```
public class OrderedArrayMaxPQ<Key extends Comparable<Key>> {
    private Key[] pq;          // elements
    private int n;             // number of elements

    // set inititial size of heap to hold size elements
    public OrderedArrayMaxPQ(int capacity) {
        pq = (Key[]) (new Comparable[capacity]);
        n = 0;
    }


    public boolean isEmpty() { return n == 0;  }
    public int size()        { return n;       }
    public Key delMax()      { return pq[--n]; }

    public void insert(Key key) {
        int i = n-1;
        while (i >= 0 && key.compareTo(pq[i]) < 0) {
            pq[i+1] = pq[i];
            i--;
        }
        pq[i+1] = key;
        n++;
    }
}
```



Practice Time

- Given an empty array of capacity 10, perform the following operations in a priority queue based on an ordered array (eager approach):

1. Insert P
2. Insert Q
3. Insert E
4. Delete max
5. Insert X
6. Insert A
7. Insert M
8. Delete max
9. Insert P
10. Insert L
11. Insert E
12. Delete max

Answer

0	1	2	3	4	5	6	7	8	9	Operation
P										insert P
P	Q									insert Q
E	P	Q								insert E
E	P	~~Q~~								delete-max → Q
E	P	X								insert X
A	E	P	X							insert A
A	E	M	P	X						insert M
A	E	M	P	~~X~~						delete-max → X
A	E	M	P	P						insert P
A	E	L	M	P	P					insert L
A	E	E	L	M	P	P				insert E
A	E	E	L	M	P	~~P~~				delete-max → P

Option 3: Binary heap

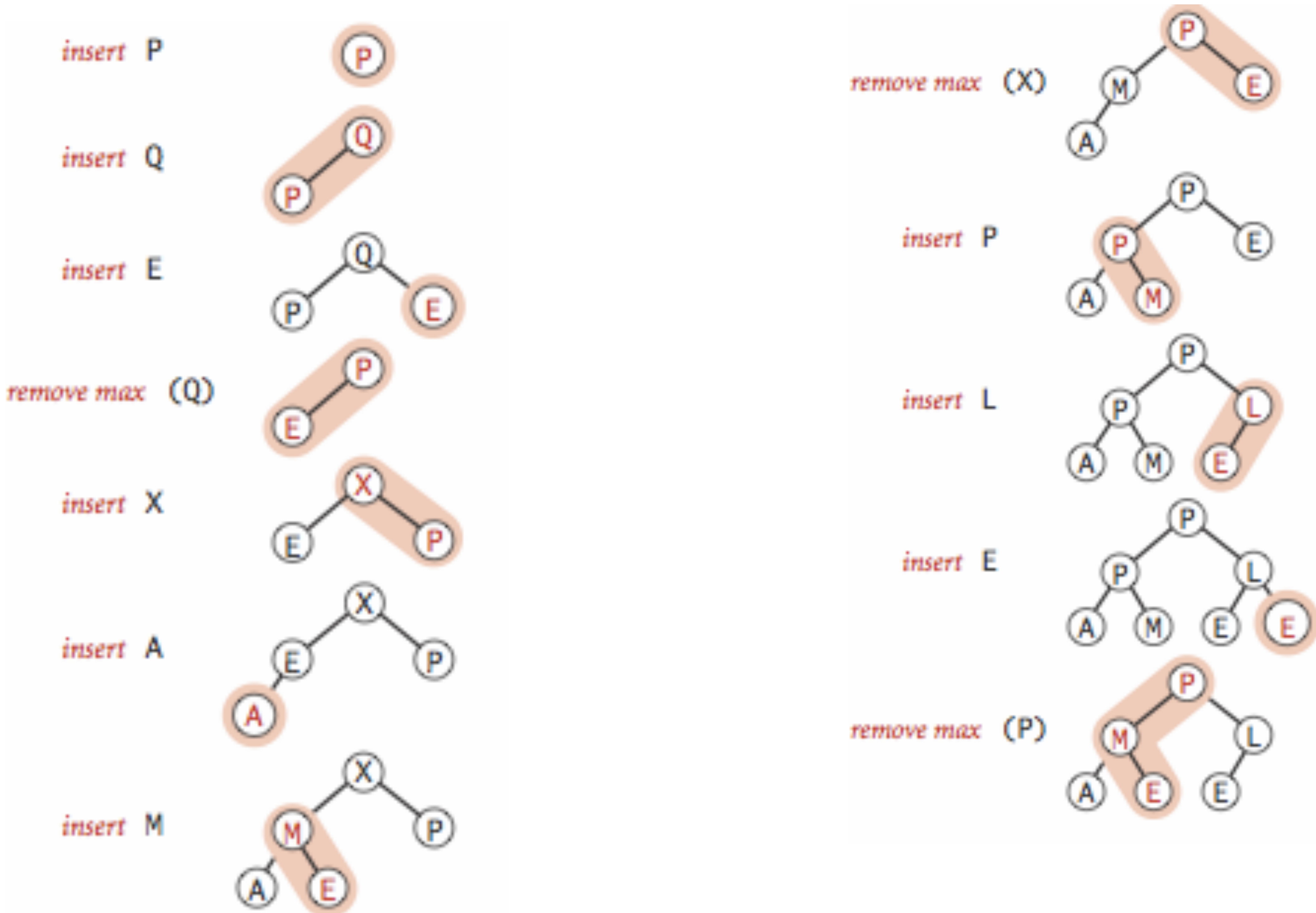
- Will allow us to both insert and delete max in $O(\log n)$ running time.
- There is no way to implement a priority queue in such a way that insert *and* delete max can be achieved in $O(1)$ running time.
- Priority queues are synonyms to binary heaps.

Practice Time

- Given an empty binary heap that represents a priority queue, perform the following operations:

1. Insert P
2. Insert Q
3. Insert E
4. Delete max
5. Insert X
6. Insert A
7. Insert M
8. Delete max
9. Insert P
10. Insert L
11. Insert E
12. Delete max

Answer


insert P
insert Q
insert E
remove max (Q)
insert X
insert A
insert M
remove max (X)
insert P
insert L
insert E
remove max (P)

Lecture 16: Binary Trees, Binary Search, Heaps, and Priority Queues

- Binary Trees
- Tree traversals
- Binary Search
- Binary Heaps
- Priority Queues

Readings:

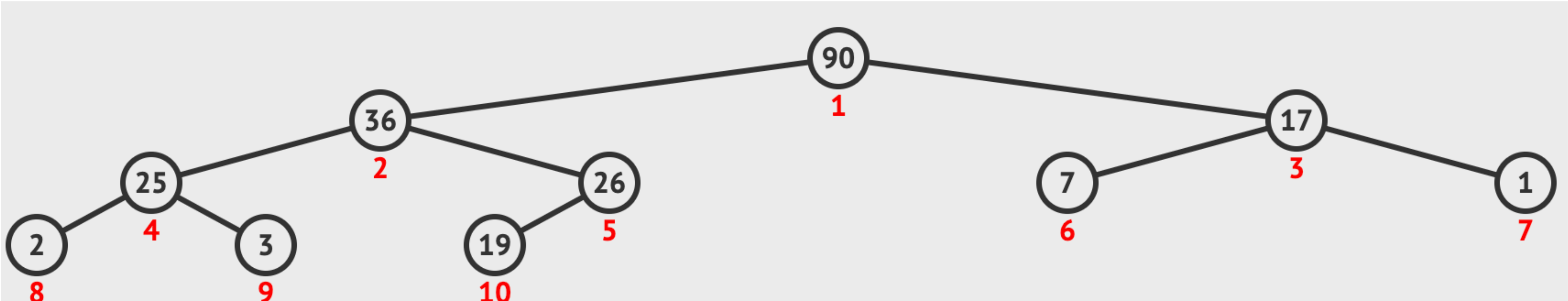
- Recommended Textbook:
  - Chapter 2.4 (Pages 308-327)
- Website:
  - Heaps: https://algs4.cs.princeton.edu/24pq/
- Visualization:
  - Insert and ExtractMax: https://visualgo.net/en/heap

Worksheet

- Lecture 16 worksheet

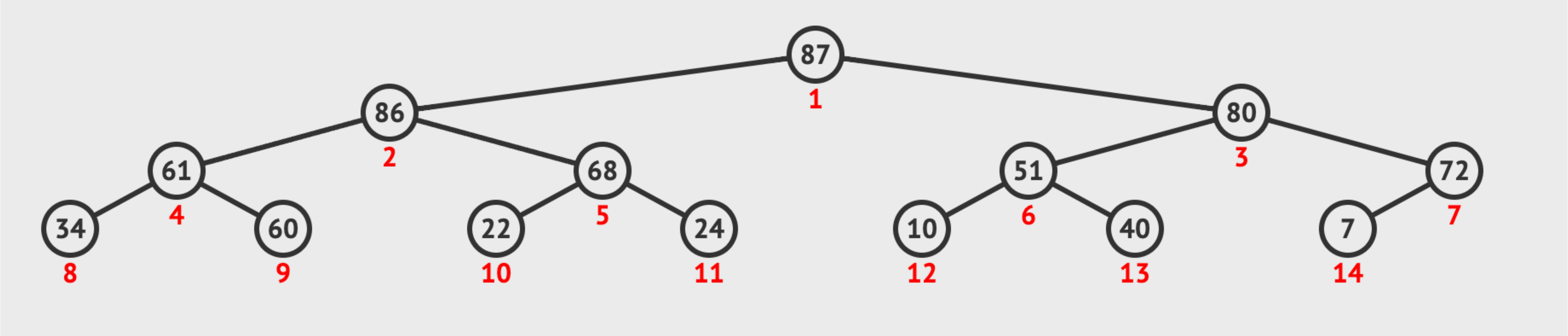
Practice Problem 1

- Given the tree below, list the nodes in order of visit in a:
  - pre-order traversal
  - in-order traversal
  - post-order traversal
  - level-order traversal



Practice Problem 2

- Given the binary heap below, delete and return the max.

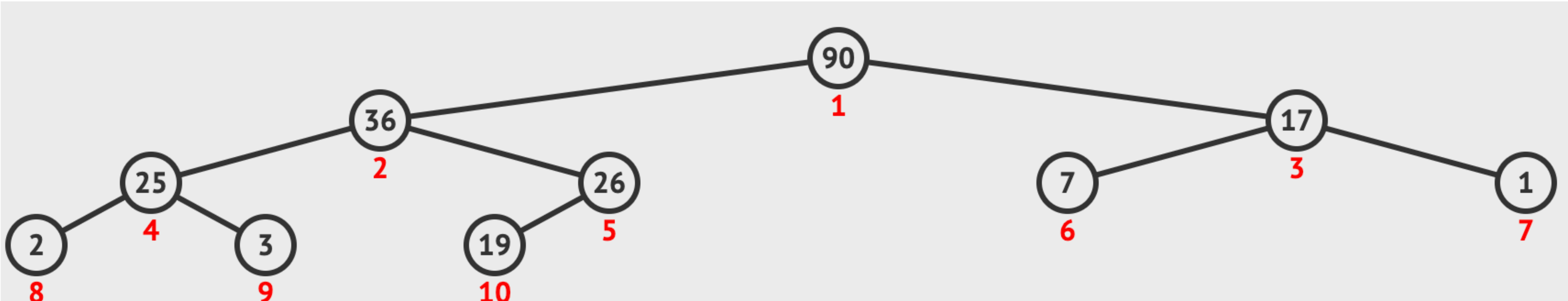


Practice Problem 3

- Suppose that the sequence 16, 18, 9, 15, *, 18, *, *, 9, *, 20, *, 25, *, *, *, 17, 21, 5, *, *, *, 21, *, 5 (where a number means insert and an asterisk means delete the maximum) is applied to an initially empty priority queue. Give the sequence of numbers returned by the delete maximum operations.

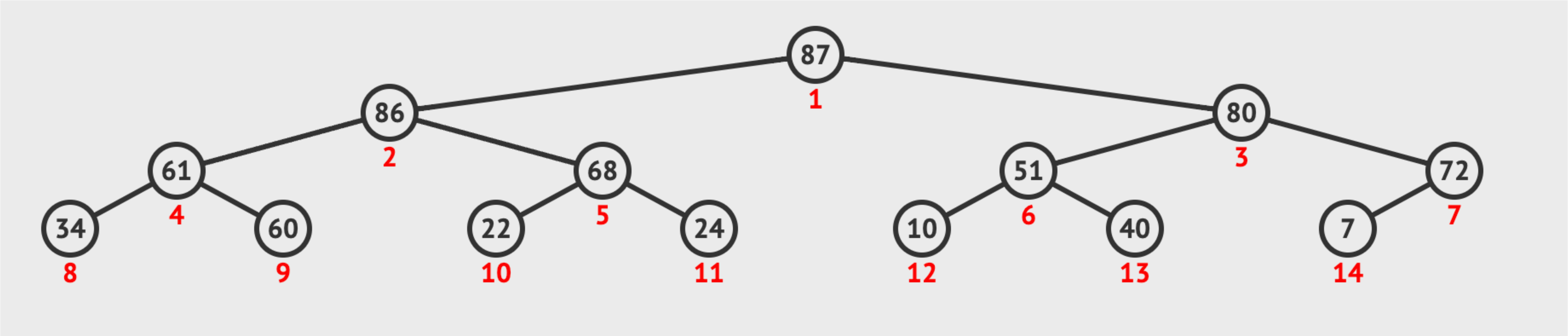
ANSWER 1

- pre-order: 90, 36, 25, 2, 3, 26, 19, 17, 7, 1
- in-order: 2, 25, 3, 36, 19, 26, 90, 7, 17, 1
- post-order: 2, 3, 25, 19, 26, 36, 7, 1, 17, 90
- level-order: 90, 36, 17, 25, 26, 7, 1, 2, 3, 19



ANSWER 2

- Given the binary heap below, delete and return the max.



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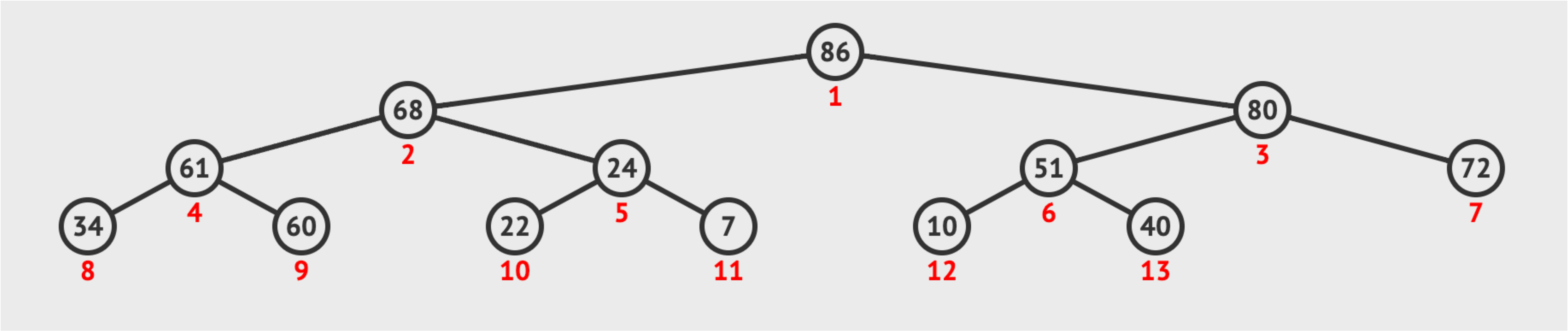

ANSWER 3

- Suppose that the sequence 16, 18, 9, 15, *, 18, *, *, 9, *, 20, *, 25, *, *, *, 17, 21, 5, *, *, *, 21, *, 5 (where a number means insert and an asterisk means delete the maximum) is applied to an initially empty priority queue. Give the sequence of numbers returned by the delete maximum operations.
- 18, 18, 16, 15, 20, 25, 9, 9, 21, 17, 5, 21