SORTING

CS062

DATA STRUCTURES AND ADVANCED PROGRAMMING

18: Binary Search Trees

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2.4 BINARY HEAP DEMO

Things to remember about runtime complexity of heaps

- Insertion is $O(\log n)$. Why?
- Delete max is $O(\log n)$. Why?
- Space efficiency is $O(n)$. Why?
  - Array with complete tree

Lecture 18: Priority Queues, Heapsort, BST

- Binary Heaps
- Priority Queue
- Heapsort

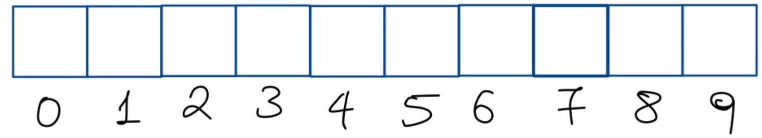
Priority Queue ADT

- Service best element first
  - Compared to FIFO or LIFO
- Two operations:
  - Delete (return) the maximum
  - Insert
- Applications: load balancing and interruption handling in OS, Huffman codes for compression, A* search for AI, Dijkstra’s and Prim's algorithm for graph search, etc.
- How can we implement a priority queue efficiently?
  - Unordered array, Ordered array, Binary Heap

Option 1: Unordered array

- The *lazy* approach where we defer doing work (deleting the maximum) until necessary.
- Insert is $O(1)$ (will be implemented as push in stacks).
- Delete maximum is $O(n)$ (have to traverse the entire array to find the maximum element).

```
public class UnorderedArrayMaxPQ<Key extends Comparable<Key>> {
    private Key[] pq;      // elements
    private int n;         // number of elements

    // set initital size of heap to hold size elements
    public UnorderedArrayMaxPQ(int capacity) {
        pq = (Key[]) new Comparable[capacity];
        n = 0;
    }

    public boolean isEmpty()   { return n == 0; }
    public int size()          { return n;      }
    public void insert(Key x)  { pq[n++] = x;   }   // Insert into index n

    public Key delMax() {
        int max = 0;
        for (int i = 1; i < n; i++)
            if (less(max, i)) max = I;    // Find max element
        exch(max, n-1);                   // Exchange max with last element

        return pq[-n];                    // Return last element
    }
    private boolean less(int i, int j) {
        return pq[i].compareTo(pq[j]) < 0;
    }

    private void exch(int i, int j) {
        Key swap = pq[i];
        pq[i] = pq[j];
        pq[j] = swap;
    }
}
```



Practice Time

- Given an empty array of capacity 10, perform the following operations in a priority queue based on an unordered array (lazy approach):

1. Insert P
2. Insert Q
3. Insert E
4. Delete max
5. Insert X
6. Insert A
7. Insert M
8. Delete max
9. Insert P
10. Insert L
11. Insert E
12. Delete max

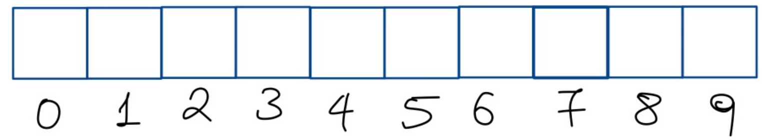
Answer

0	1	2	3	4	5	6	7	8	9	Operation
P										insert P
P	Q									insert Q
P	Q	E								insert E
P	E	~~Q~~								delete-max → Q
P	E	X								insert X
P	E	X	A							insert A
P	E	X	A	M						insert M
P	E	M	A	~~X~~						delete-max → X
P	E	M	A	P						insert P
P	E	M	A	P	L					insert L
P	E	M	A	P	L	E				insert E
E	E	M	A	P	L	~~E~~				delete-max → P

Option 2: Ordered array

- The *eager* approach where we do the work (keeping the list sorted) up front to make later operations efficient.
- Insert is $O(n)$ (we have to find the index to insert and shift elements to perform insertion).
- Delete maximum is $O(1)$ (just take the last element which will the maximum).

```java
public class OrderedArrayMaxPQ<Key extends Comparable<Key>> {
    private Key[] pq;          // elements
    private int n;             // number of elements

    // set inititial size of heap to hold size elements
    public OrderedArrayMaxPQ(int capacity) {
        pq = (Key[]) (new Comparable[capacity]);
        n = 0;
    }


    public boolean isEmpty() { return n == 0;  }
    public int size()        { return n;       }
    public Key delMax()      { return pq[--n]; }

    public void insert(Key key) {
        int i = n-1;
        while (i >= 0 && less(key, pq[i])) {
            pq[i+1] = pq[i];                // Empty element is at index i
            i--;
        }
        pq[i+1] = key;                      // I+1 to get to the empty element
        n++;
    }

   private boolean less(Key v, Key w) {
        return v.compareTo(w) < 0;
    }
```



Practice Time

- Given an empty array of capacity 10, perform the following operations in a priority queue based on an ordered array (eager approach):

1. Insert P
2. Insert Q
3. Insert E
4. Delete max
5. Insert X
6. Insert A
7. Insert M
8. Delete max
9. Insert P
10. Insert L
11. Insert E
12. Delete max

Answer

0	1	2	3	4	5	6	7	8	9	Operation
P										insert P
P	Q									insert Q
E	P	Q								insert E
E	P	~~Q~~								delete-max -> Q
E	P	X								insert X
A	E	P	X							insert A
A	E	M	P	X						insert M
A	E	M	P	~~X~~						delete-max -> X
A	E	M	P	P						insert P
A	E	L	M	P	P					insert L
A	E	E	L	M	P	P				insert E
A	E	E	L	M	P	~~P~~				delete-max -> P

Option 3: Binary heap

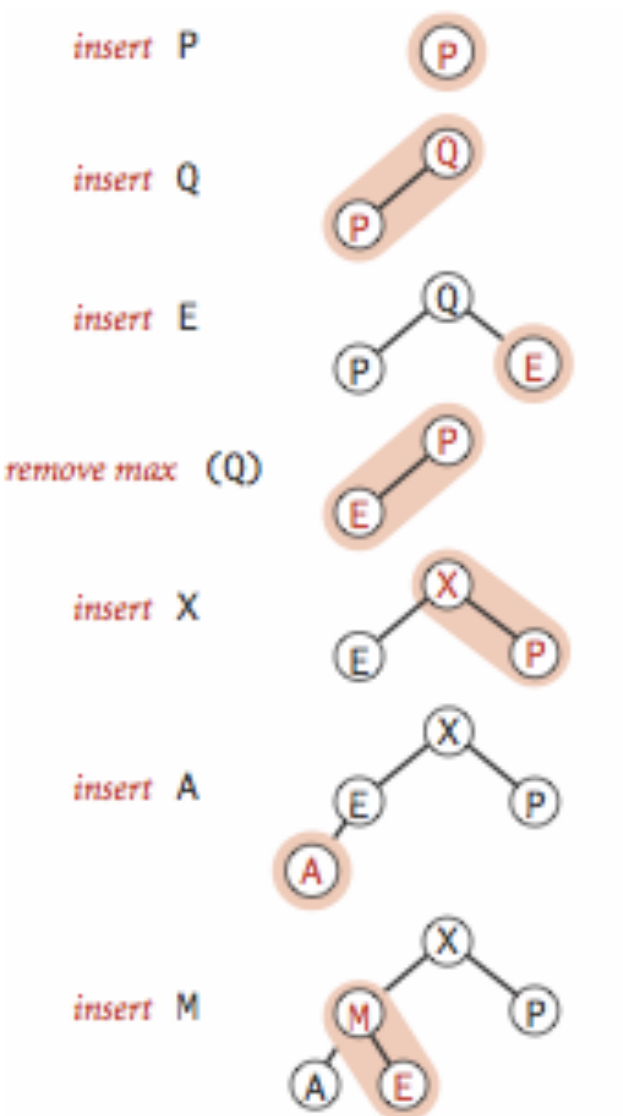
- Will allow us to both insert and delete max in $O(\log n)$ running time.
- There is no way to implement a priority queue in such a way that insert and delete max can be achieved in $O(1)$ running time.
- Priority queues are synonyms to binary heaps.

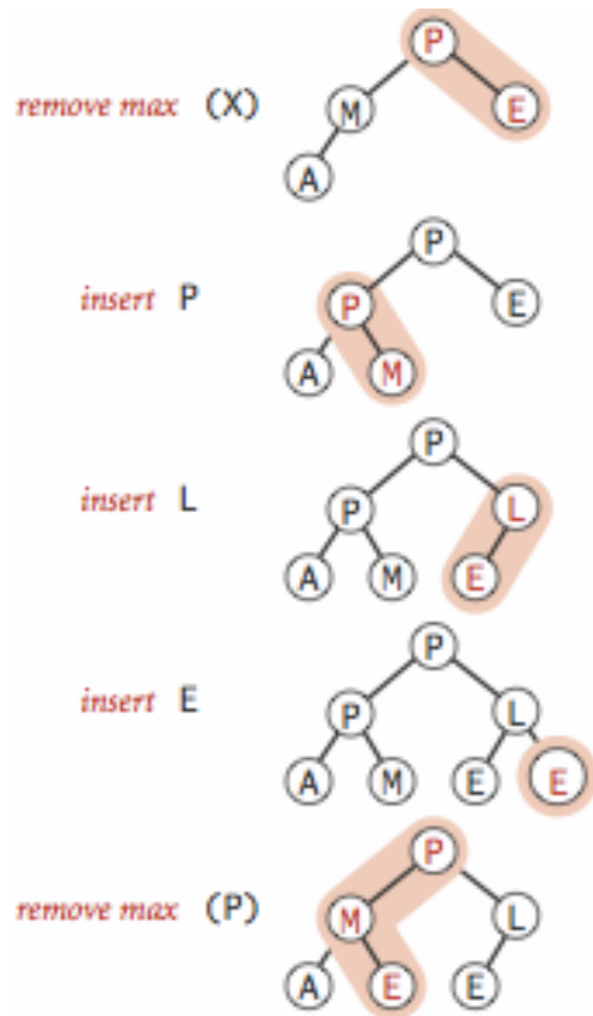
Practice Time

- Given an empty binary heap that represents a priority queue, perform the following operations:

1. Insert P
2. Insert Q
3. Insert E
4. Delete max
5. Insert X
6. Insert A
7. Insert M
8. Delete max
9. Insert P
10. Insert L
11. Insert E
12. Delete max

Answer


insert P
insert Q
insert E
remove max (Q)
insert X
insert A
insert M


remove max (X)
insert P
insert L
insert E
remove max (P)

Lecture 18: Priority Queues and Heapsort

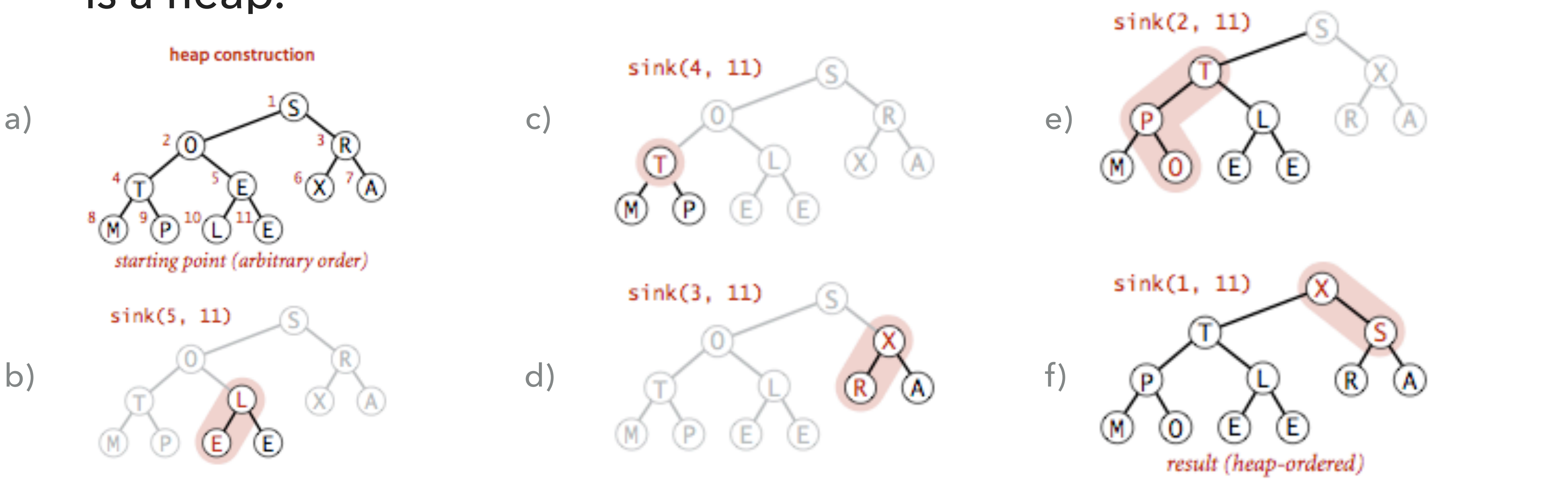
- Priority Queue
- Heapsort

Basic plan for heap sort

- Use a priority queue to develop a sorting method that works in two steps:
- 1) Heap construction: build a binary heap with all n keys that need to be sorted.
- 2) Sortdown: repeatedly remove and return the maximum key.

$O(n)$ Heap construction

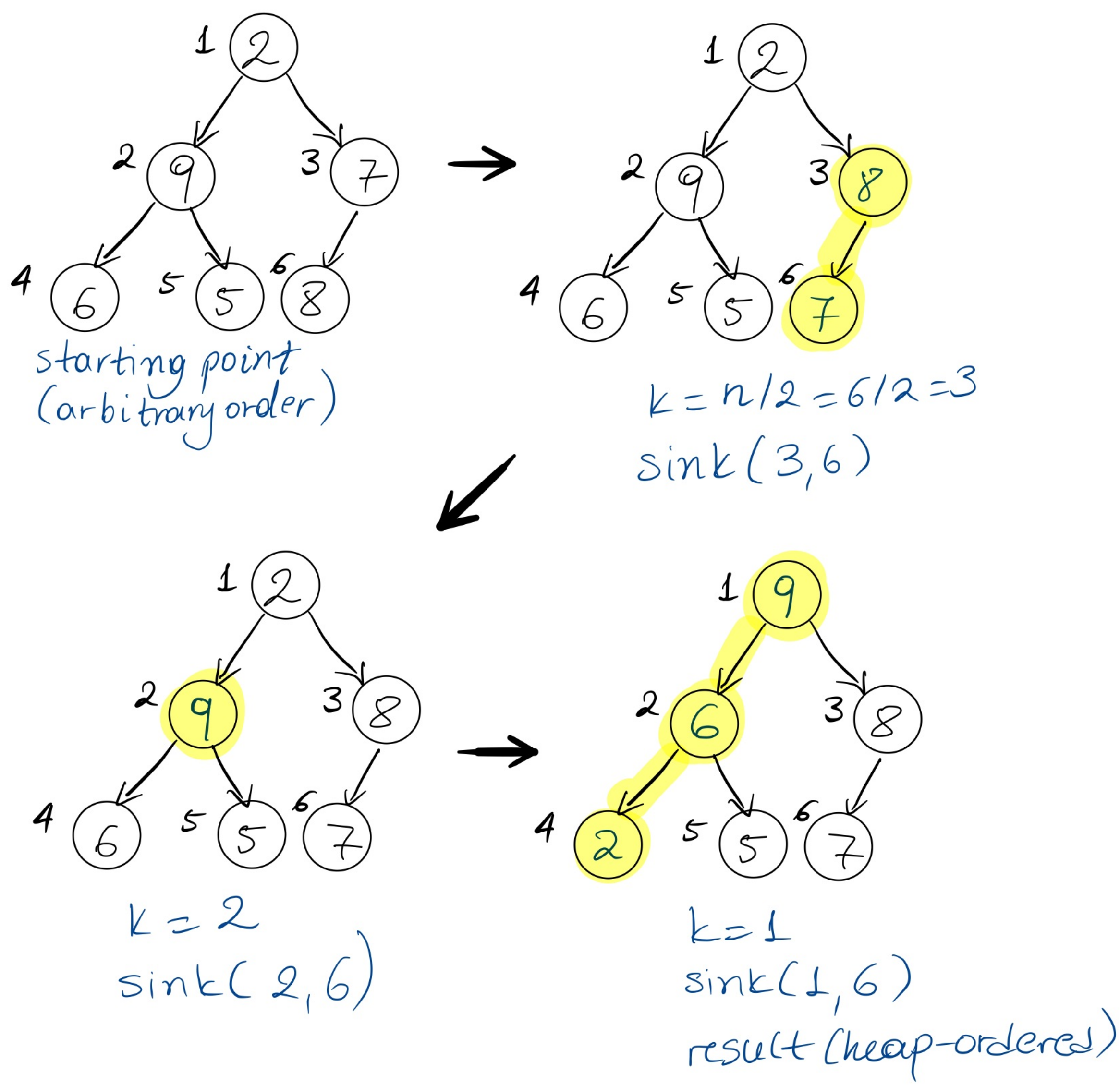
- Construct complete binary tree with elements
- Ignore all leaves (indices n/2+1,...,n).
-
```
for(int k = n/2; k >= 1; k--)
        sink(a, k, n);
```
- Key insight: After `sink(a,k,n)` completes, the subtree rooted at k is a heap.



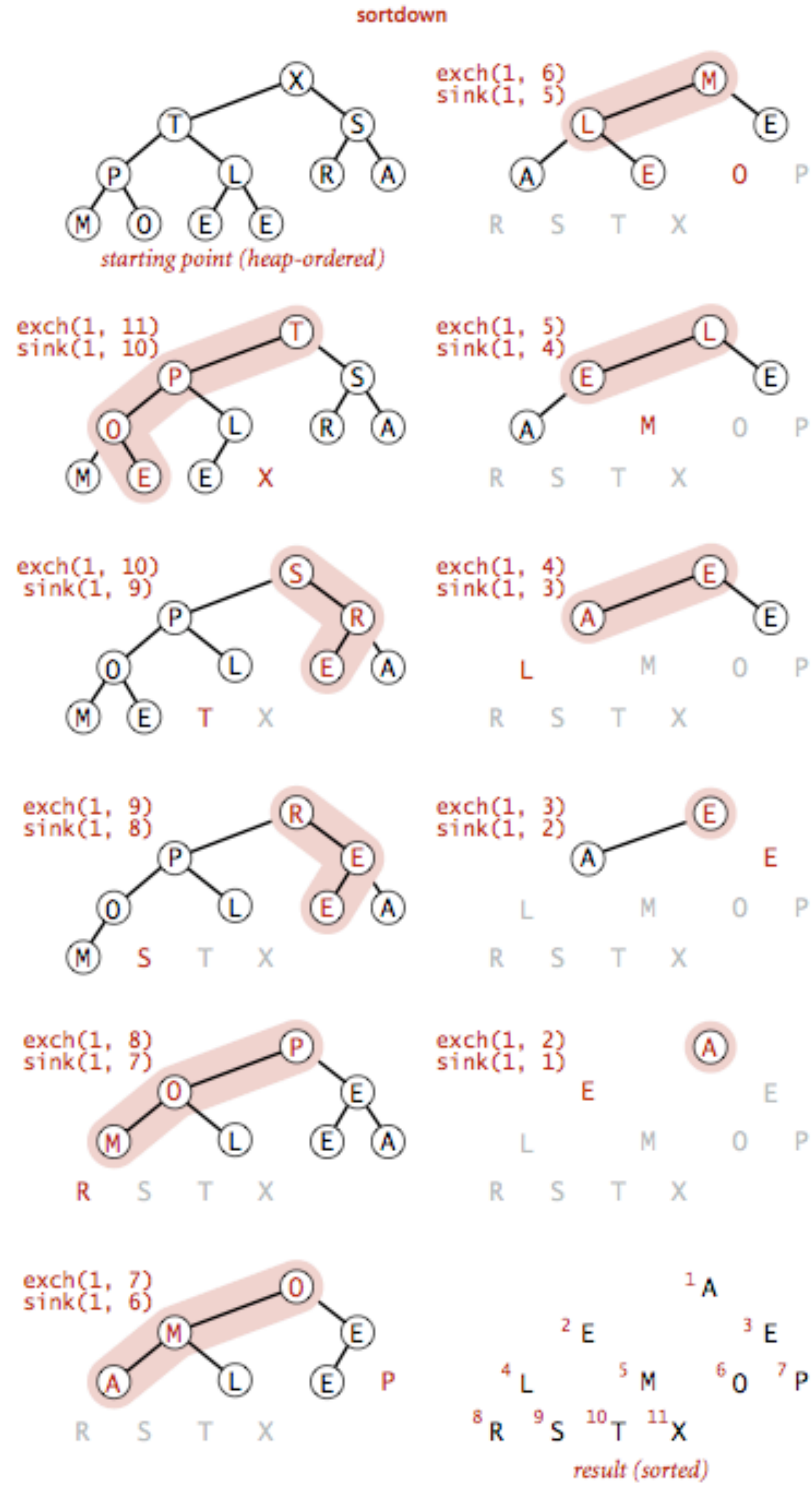
Practice Time

- Run the first step of heapsort, heap construction, on the array [2,9,7,6,5,8].

Answer: Heap construction



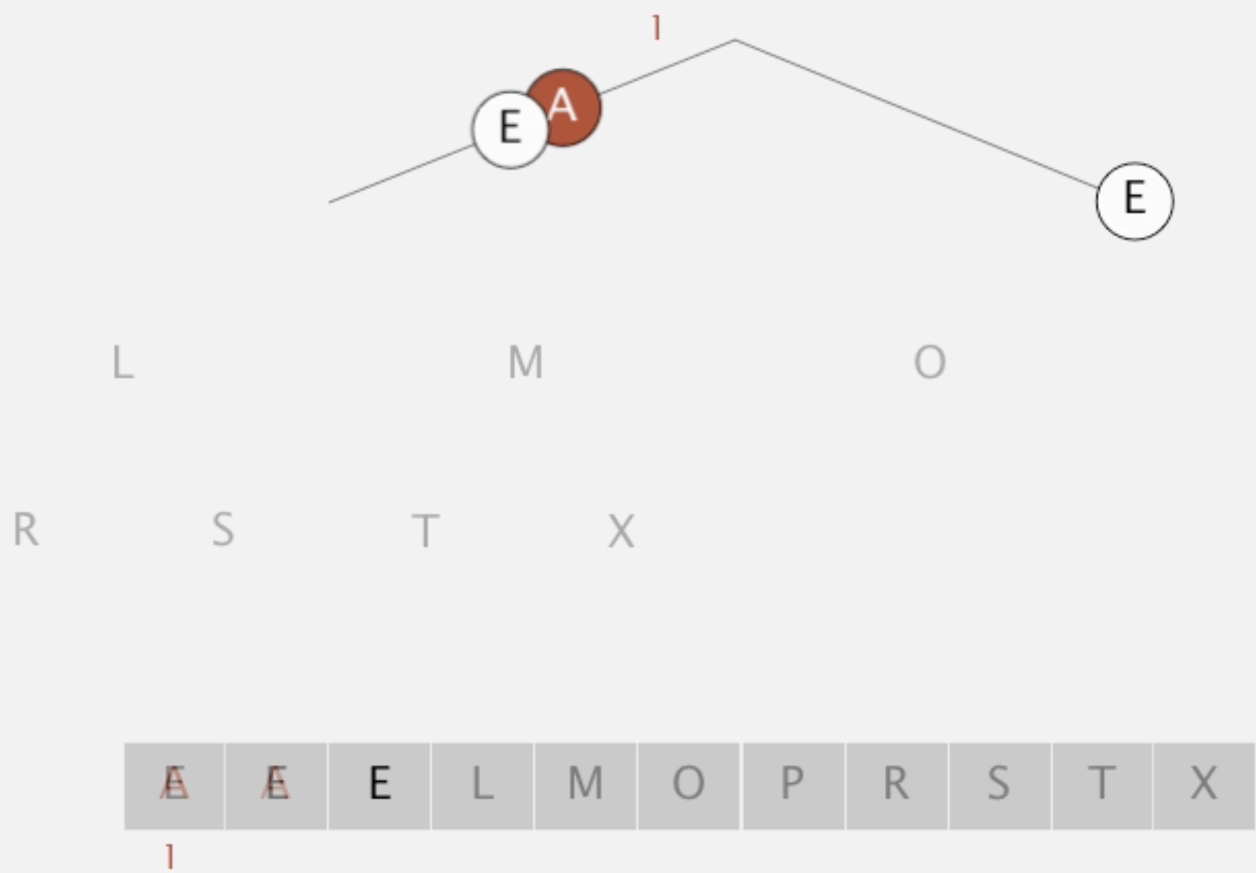
Sortdown

- Remove the maximum, one at a time, but leave in array instead of nulling out.
- ```
  while(n>1){
     exch(a, 1, n--);
     sink(a, 1, n);
  }
  ```
- Key insight: After each iteration the array consists of a heap-ordered subarray followed by a sub-array in final order.
  ```

Sortdown

- ```
  while(n>1){
     exch(a, 1, n--);
     sink(a, 1, n);
  }
  ```


  ```

Heapsort demo

Sortdown. Repeatedly delete the largest remaining item.

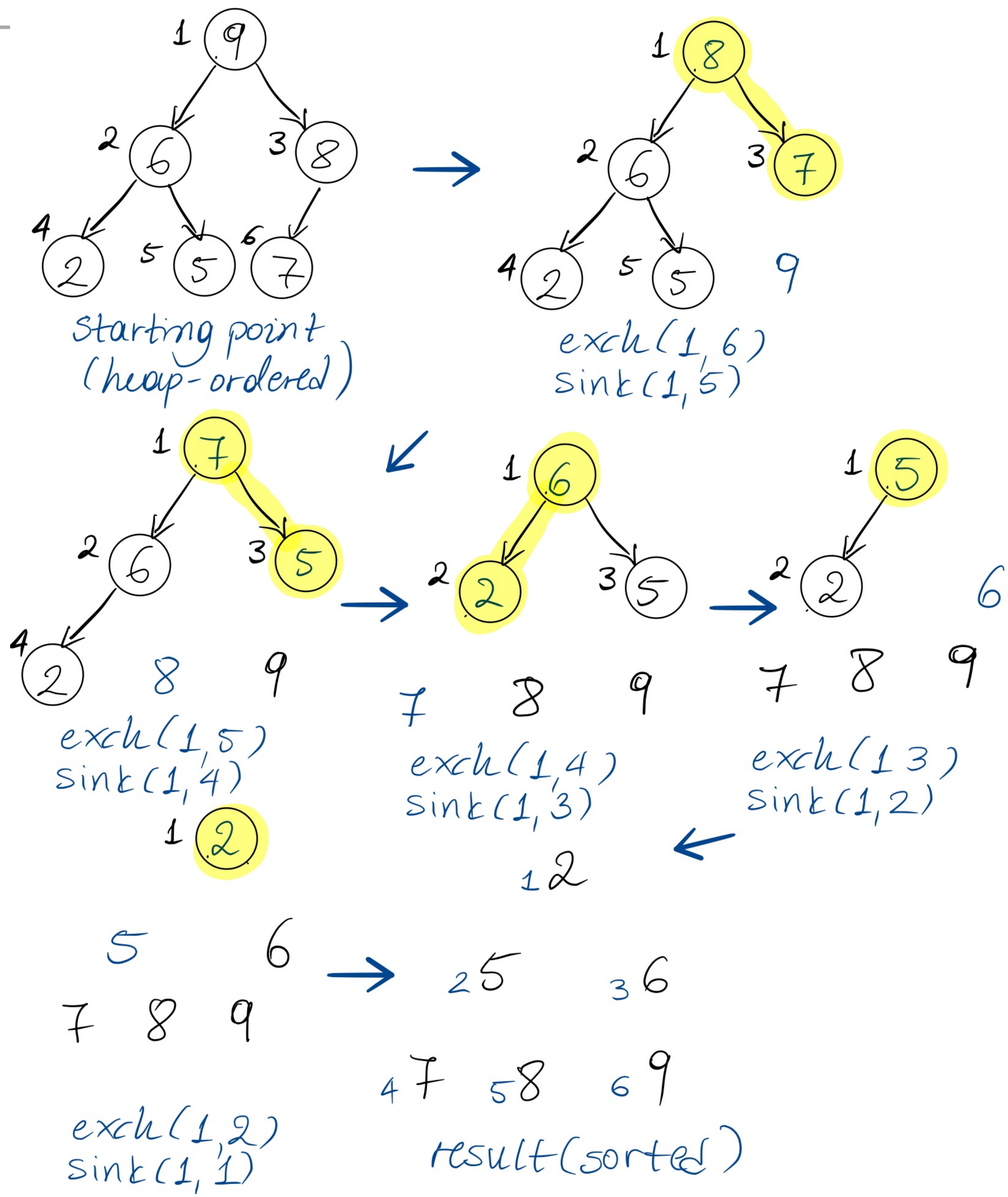
sink 1



Practice Time

- Given the heap you constructed before, run the second step of heapsort, sortdown, to sort the array [2,9,7,6,5,8].

Answer: Sortdown



Heapsort analysis

- Heap construction makes $O(n)$ exchanges and $O(n)$ compares.
- Sortdown and therefore the entire heap sort $O(n \log n)$ exchanges and compares.
- In-place sorting algorithm with $O(n \log n)$ worst-case!
- Remember:
  - mergesort: not in place, requires linear extra space.
  - quicksort: quadratic time in worst case.
- Heapsort is optimal both for time and space in terms of Big-O, but:
  - Inner loop longer than quick sort.
  - Poor use of cache. Why?
  - Not stable.

Sorting: Everything you need to remember about it!

Which Sort	In place	Stable	Best	Average	Worst	Remarks
Selection	X		$O(n^2)$	$O(n^2)$	$O(n^2)$	n exchanges
Insertion	X	X	$O(n)$	$O(n^2)$	$O(n^2)$	Use for small arrays or partially ordered
Merge		X	$O(n \log n)$	$O(n \log n)$	$O(n \log n)$	Guaranteed performance; stable
Quick	X		$O(n \log n)$	$O(n \log n)$	$O(n^2)$	$n \log n$ probabilistic guarantee; fastest!
Heap	X		$O(n \log n)$	$O(n \log n)$	$O(n \log n)$	Guaranteed performance; in place

Lecture 18: Priority Queues and Heapsort

- Priority Queue
- Heapsort

Readings:

- Textbook:
  - Chapter 2.4 (Pages 308-327), 2.5 (336-344)
- Website:
  - Priority Queues: https://algs4.cs.princeton.edu/24pq/
- Visualization:
  - Create (nlogn) and heapsort: https://visualgo.net/en/heap

Practice Problems:

- 2.4.1-2.4.11. Also try some creative problems.

Readings:

- Textbook:
  - Chapter 2.4 (Pages 308-327)
- Website:
  - Priority Queues: https://algs4.cs.princeton.edu/24pq/
- ***Visualization:***
  - Insert and ExtractMax: https://visualgo.net/en/heap

Practice Problems:

- Practice with traversals of trees and insertions and deletions in binary heaps

Lecture 18: Search

- Dictionaries (Symbol Tables)
- Binary Search Trees

Some slides adopted from Algorithms 4th Edition or COS226

Dictionaries

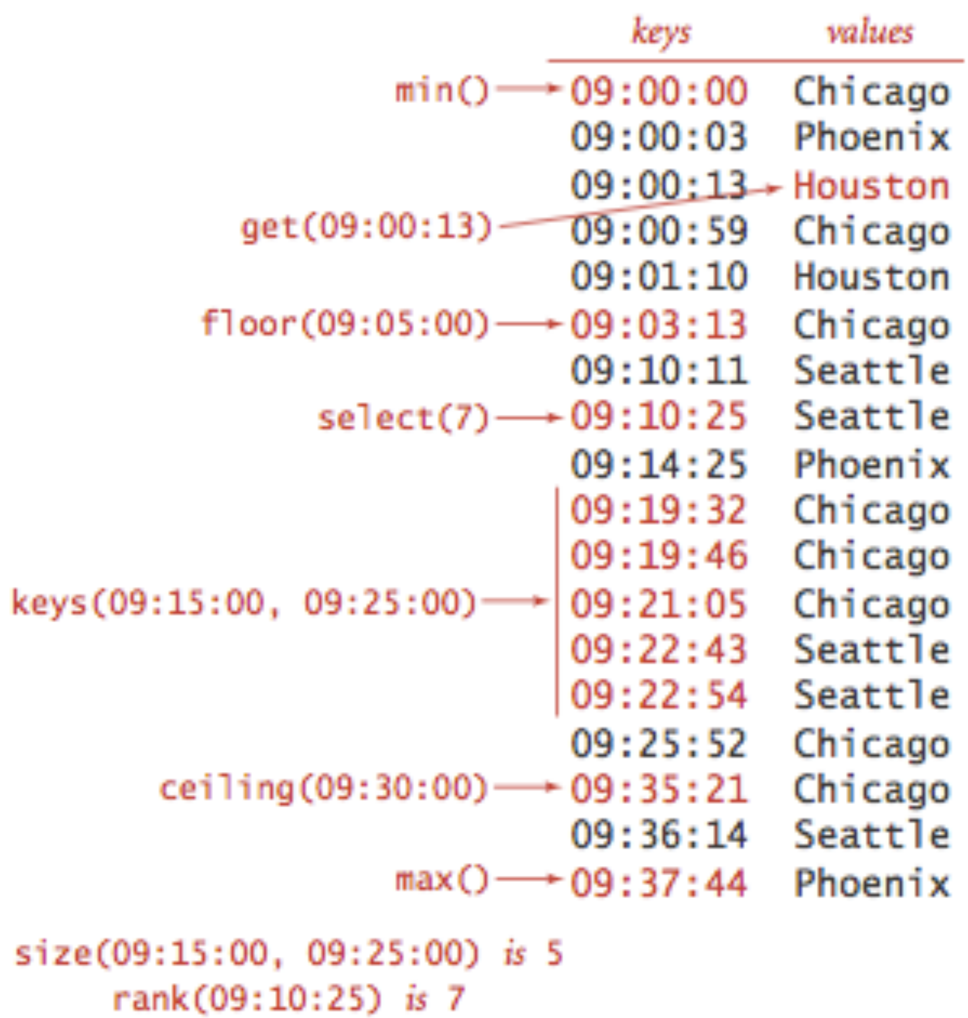
- Also known as: symbol tables, maps, indices, associative arrays.
- Key-value pair abstractions that support two operations:
    - Insert a key-value pair.
    - Given a key, search for the corresponding value.
- Supported either with built-in or external libraries by the majority of programming languages.

Basic symbol table API

Application	Key	Value
Phonebook	Name	Phone
Web search	Keyword	List of page
Book index	Term	List of page
Compiler	Variable	Type & Value

- `public class ST <Key extends Comparable<Key>, Value>`
  - `Key needs to implement the Comparable interface, but it is a generic (use extends)`
- `ST()`: create an empty symbol table. By convention, values are not `null`.
- `void put(Key key, Value val)`: insert key-value pair.
  - Overwrites old value with new value if key already exists.
- `Value get(Key key)`: return value associated with key.
  - Returns `null` if key not present. Can't distinguish between null values and non-existent pairs
- `boolean contains(Key key)`: is there a value associated with key?
- `Iterable keys()`: all the keys in the symbol table.
- `void delete(Key key)`: delete key and associated value.
- `boolean isEmpty()`: is the symbol table empty?
- `int size()`: number of key-value pairs.

Ordered symbol tables



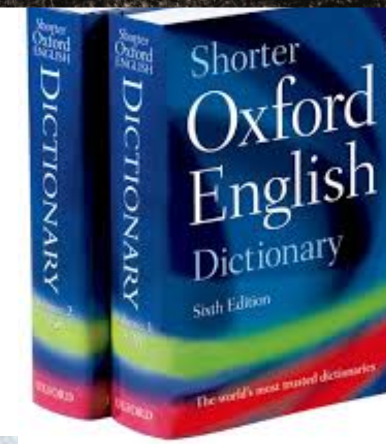
Ordered symbol table API

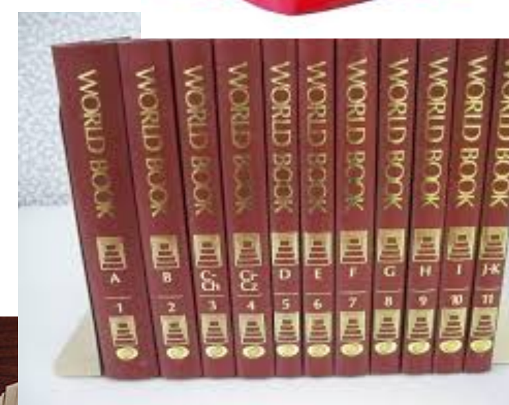
- `Key min()`: smallest key.
- `Key max()`: largest key.
- `Key floor(Key key)`: largest key less than or equal to given key.
- `Key ceiling(Key key)`: smallest key greater than or equal to given key.
- `int rank(Key key)`: number of keys less that given key.
- `Key select(int k)`: key with rank `k`.
- `Iterable keys()`: all keys in symbol table in sorted order.
- `Iterable keys(int lo, int hi)`: keys in `[lo, …, hi]` in sorted order.

DICTIONARIES

Printed symbol tables are all around us

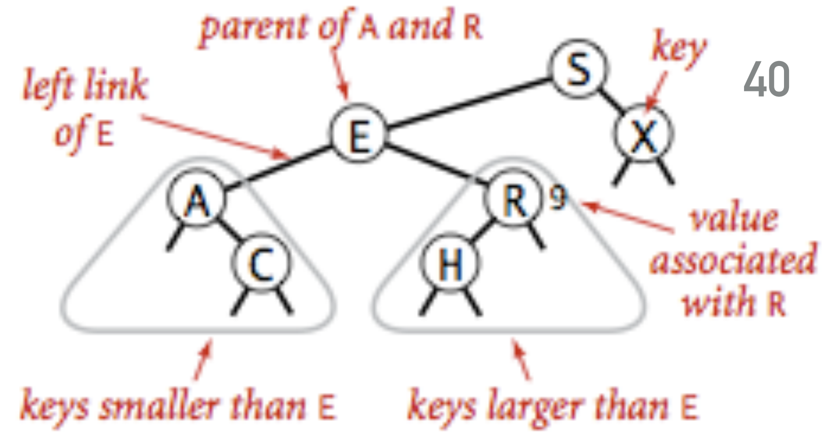
- Dictionary: key = word, value = definition.
- Encyclopedia: key = term, value = article.
- Phonebook: key = name, value = phone number.
- Math table: key = math functions and input, value = function output.
- Unsupported operations:
  - Add a new key and associated value.
  - Remove a given key and associated value.
  - Change value associated with a given key.





Lecture 23: Binary Search Trees

- Dictionaries
  - Unordered linked lists (Node with key and value)
    - Insertion and search are linear
  - Sorted array for keys and parallel array for values
    - Search is logarithmic, but insertion is linear
- Binary search Trees



Definitions

- Binary Search Tree: A binary tree in symmetric order.
- Symmetric order: Each node has a key, and every node's key is:
  - Larger than all keys in its left subtree.
  - Smaller than all keys in its right subtree.
- Our textbook uses BSTs to implement dictionaries, therefore each node holds a key-value pair. Other implementations hold only a key.

Differences between heaps and BSTs

	Heap	BST
Used to implement	Priority queues	Dictionaries
Supported operations	Insert, delete max	insert, search, delete, ordered operations
What is inserted	Keys	Key-value pairs
Underlying data structure	(Resizing) array	Linked nodes
Tree shape	Complete binary tree	Depends on data
Ordering of keys	Heap-ordered	Symmetrically-ordered
Duplicate keys allowed?	Yes	No*

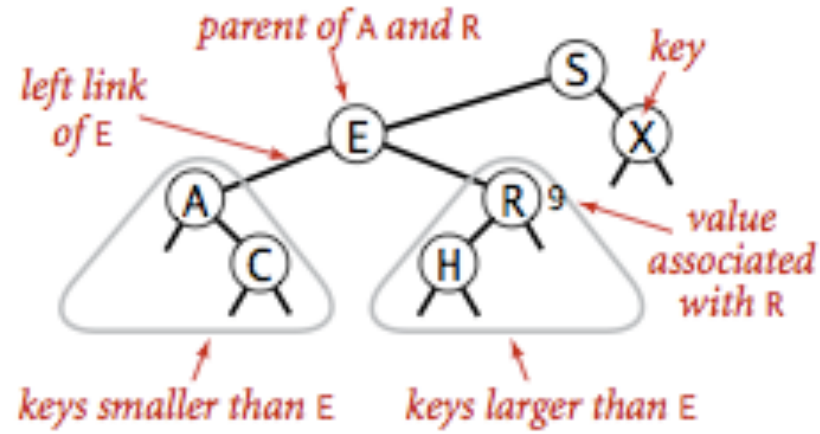
*: when BSTs used to implement dictionaries.

BST representation of dictionaries

- We will use an inner class Node that is composed by:
  - A Key that is comparable and a `Value`
  - A reference to the root nodes of the left (smaller keys) and right (larger keys) subtrees.
  - Potentially, the total number of nodes in the subtree that has root at this node.
- A BST has a reference to a Node `root`.

BST and Node implementation

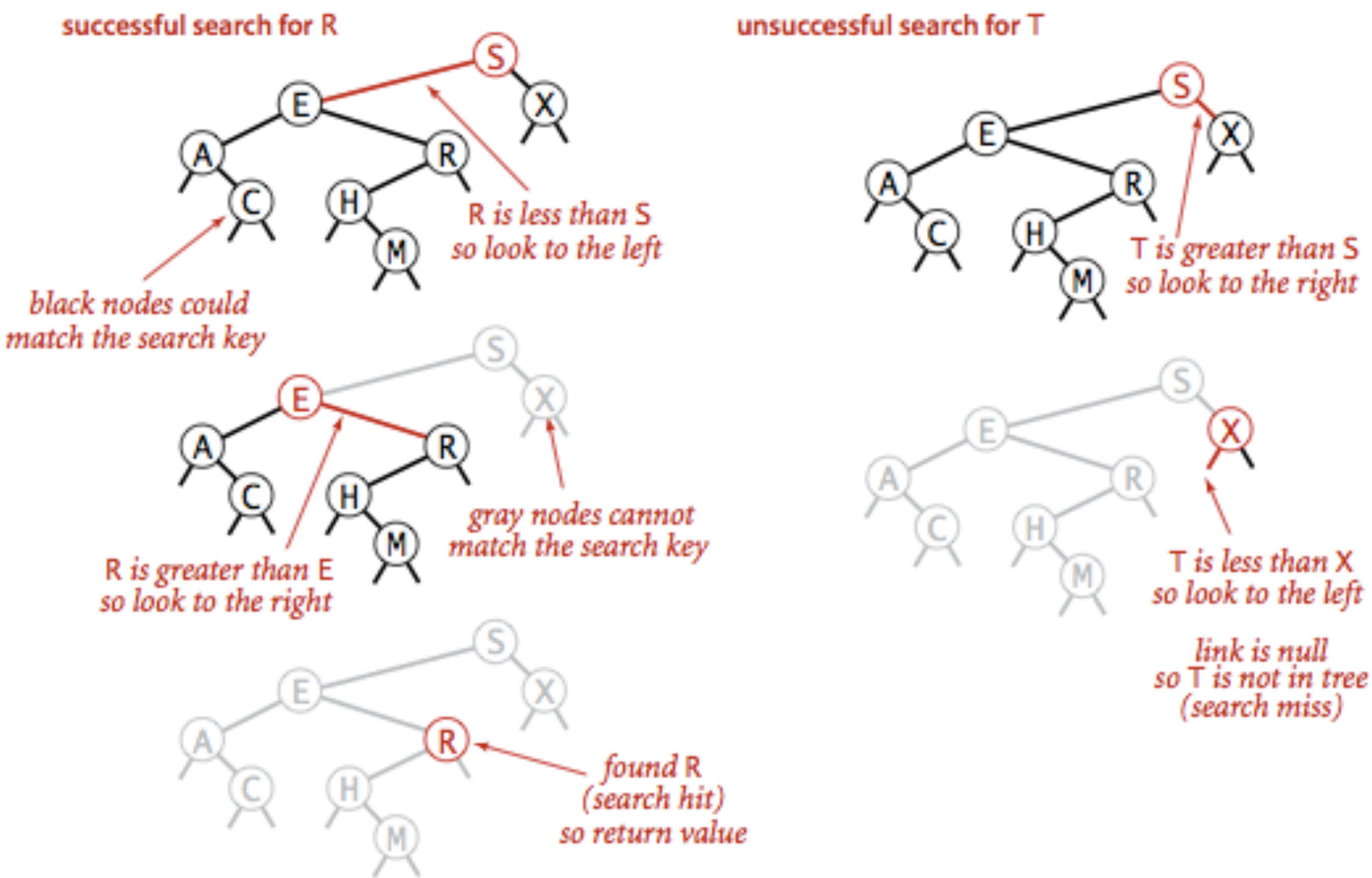
```
public class BST<Key extends Comparable<Key>, Value> {
   private Node root;                    // root of BST

   private class Node {
        private Key key;                 // sorted by key
        private Value val;               // associated value
        private Node left, right;        // roots of left and right subtrees
        private int size;                // #nodes in subtree rooted at this

        public Node(Key key, Value val, int size) {
            this.key = key;
            this.val = val;
            this.size = size;
        }
    }
```



Search for a key

- If less than key in node go to left subtree.
- If greater than key in node go to right subtree.
- If given key and key at examined node are equal, search hit.
- Return value corresponding to given key, or `null` if no such key.
  - In other implementations, you return the last node you reached.
- Number of compares is equal to the depth of the node + 1.

Search example



Successful (left) and unsuccessful (right) search in a BST

Search - iterative implementation

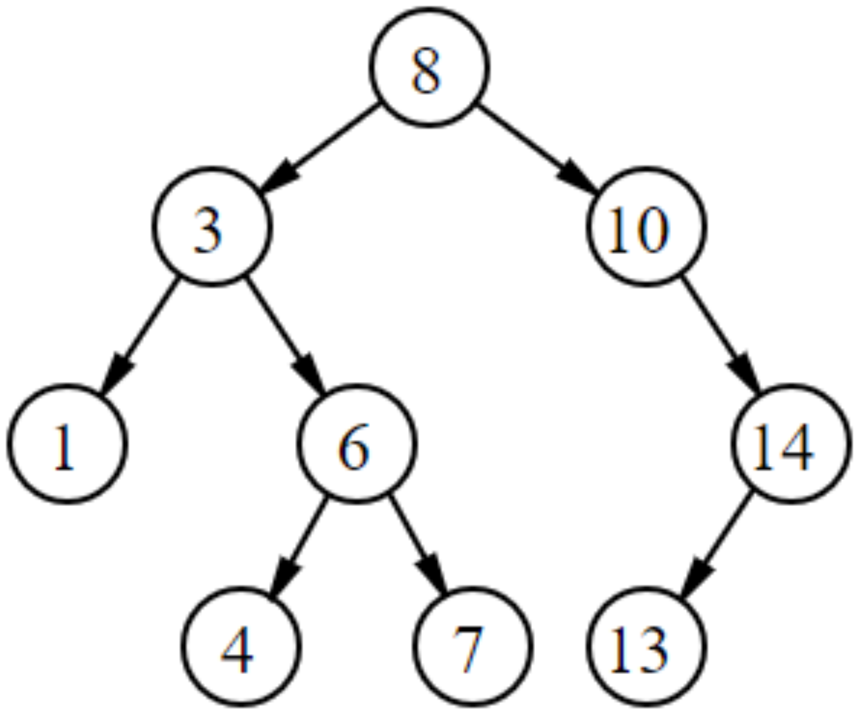
```
public Value get(Key key) {
      Node x = root;
      while (x != null) {
            int cmp = key.compareTo(x.key);
            if (cmp < 0)
                    x = x.left;
            else if (cmp > 0)
                    x = x.right;
            else if (cmp == 0)
                    return x.val;
       }
       return null;
}
```

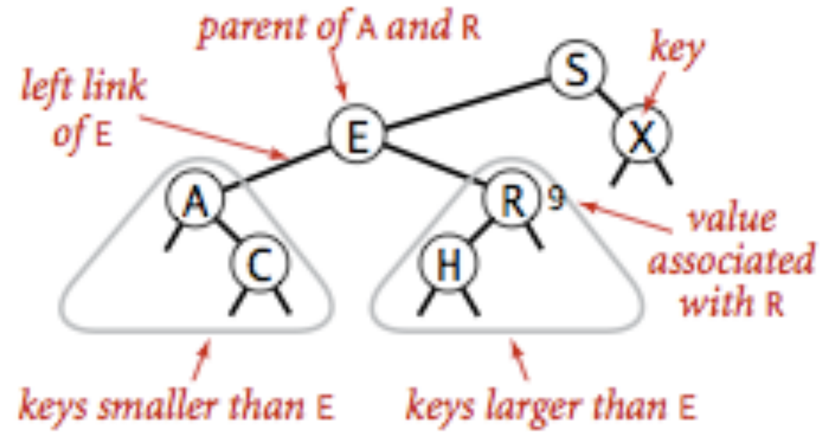
Search - recursive implementation

- ```
  public Value get(Key key) {
       return get(root, key);
  }
  ```
- ```
  private Value get(Node x, Key key) {
       if (x == null)
            return null;
       int cmp = key.compareTo(x.key);
       if (cmp < 0)
           return get(x.left, key);
       else if (cmp > 0)
           return get(x.right, key);
       else
           return x.val;
  }
  ```

Practice Time

- Search for the keys 4 and 9 in the following BST:

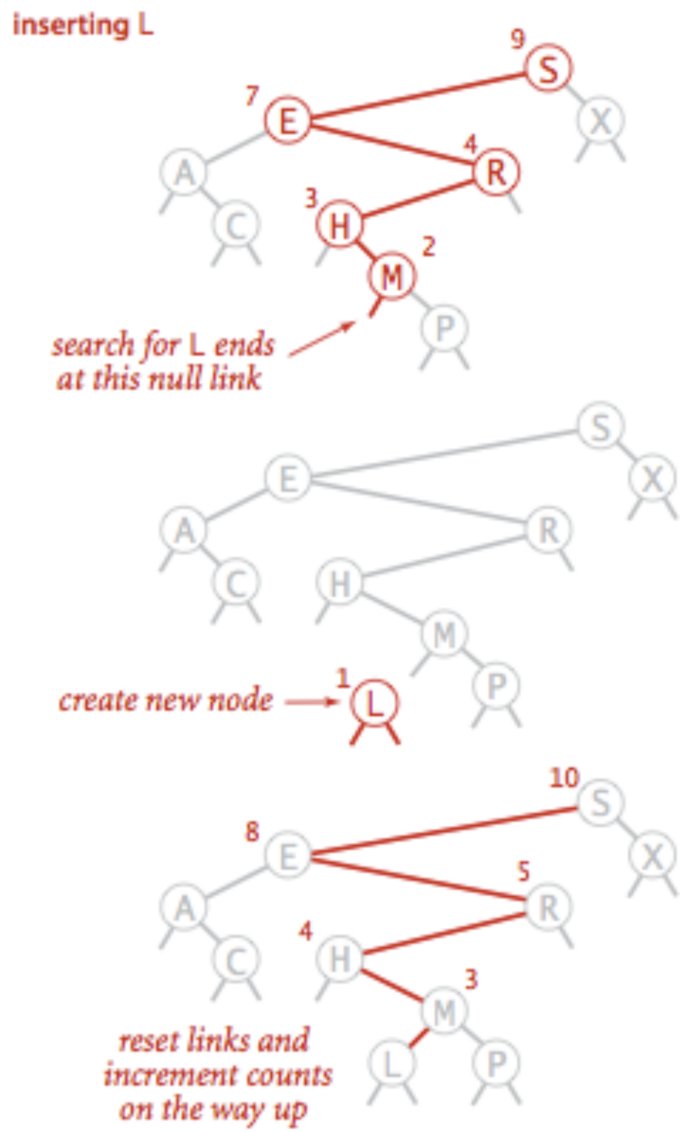




Insert

- If less than key in node go left.
- If greater than key in node go right.
- If null, insert.
- If already exists, update value.
- Number of compares is equal to the depth of the node + 1.

Insert example



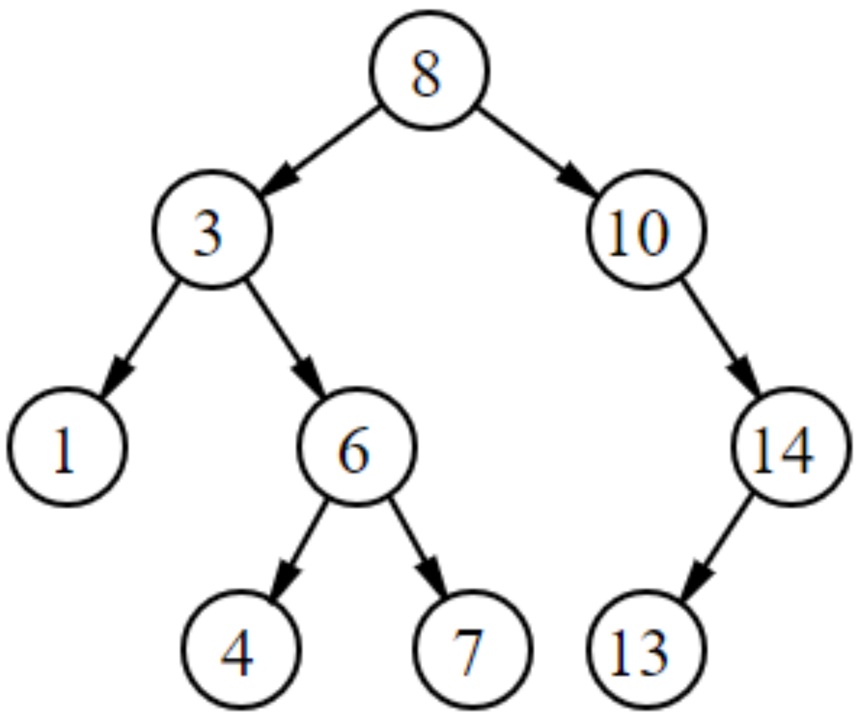
Insertion into a BST

Insert

```
public void put(Key key, Value val) {
    root = put(root, key, val);
}
private Node put(Node x, Key key, Value val) {
    if (x == null)
        return new Node(key, val, 1);
    int cmp = key.compareTo(x.key);
    if (cmp < 0)
        x.left = put(x.left, key, val);
    else if (cmp > 0)
        x.right = put(x.right, key, val);
    else
        x.val = val;
    x.size = 1 + size(x.left) + size(x.right);
    return x;
}
```

Practice Time

- Add the key-value pairs (4,3) and (9,2) in the following BST:



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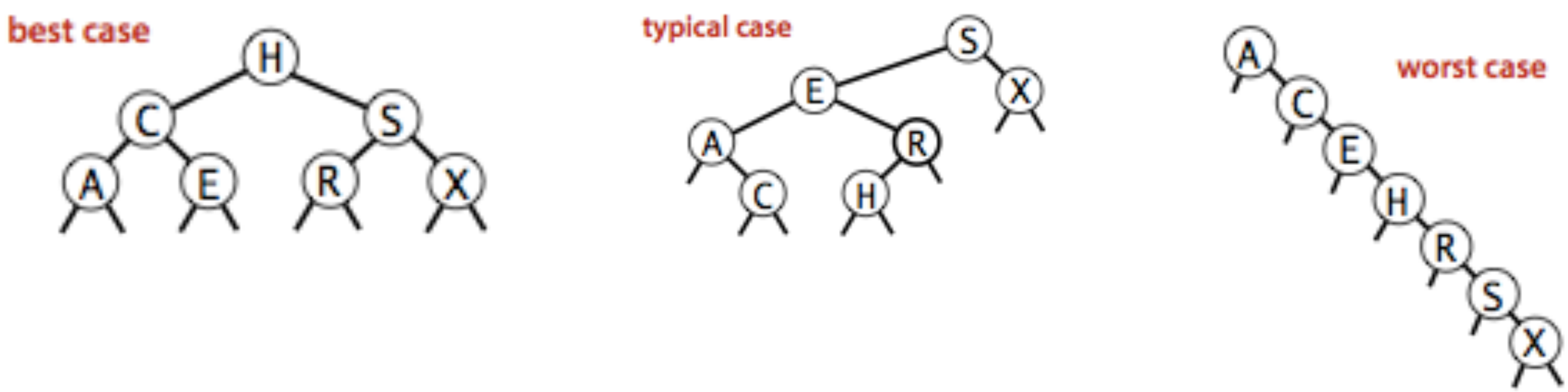
3.2 BINARY SEARCH TREE DEMO

Tree shape

- The same set of keys can result to different BSTs based on their order of insertion.
- Number of compares for search/insert is equal to depth of node +1.

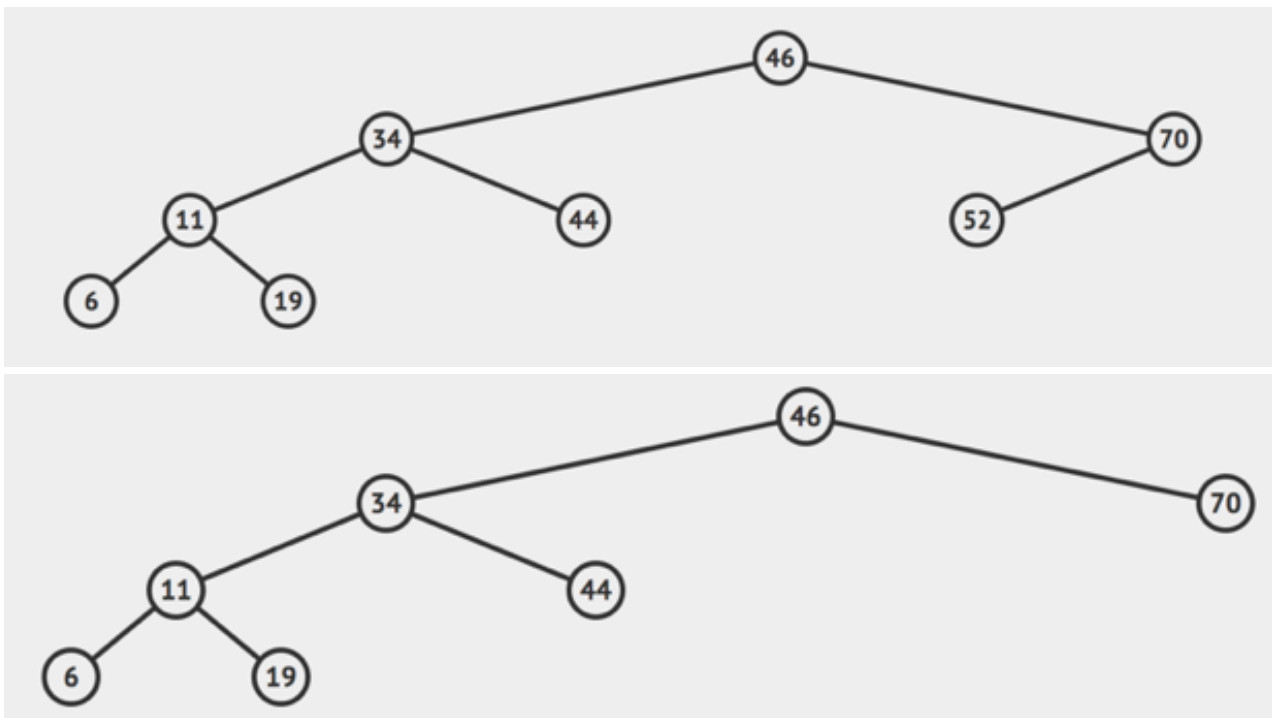


BSTs mathematical analysis

- If n distinct keys are inserted into a BST in random order, the expected number of compares of search/insert is $O(\log n)$.
  - If n distinct keys are inserted into a BST in random order, the expected height of tree is $O(\log n)$. [Reed, 2003].
- Worst case height is n but highly unlikely.
  - Keys would have to come (reversely) sorted!
- All ordered operations in a dictionary implemented with a BST depend on the height of the BST.

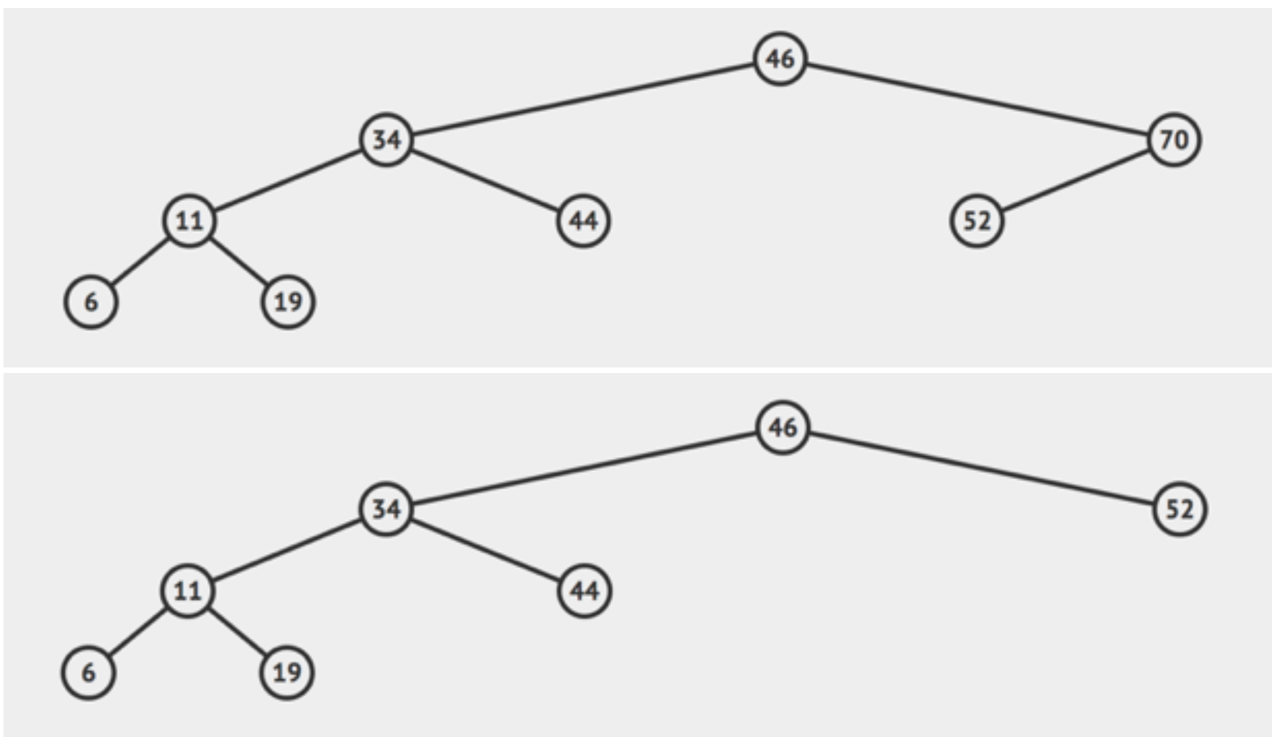
Hibbard deletion: Delete node which is a leaf

- Simply delete node.
- Example: delete 52 locates a node which is a leaf and removes it.



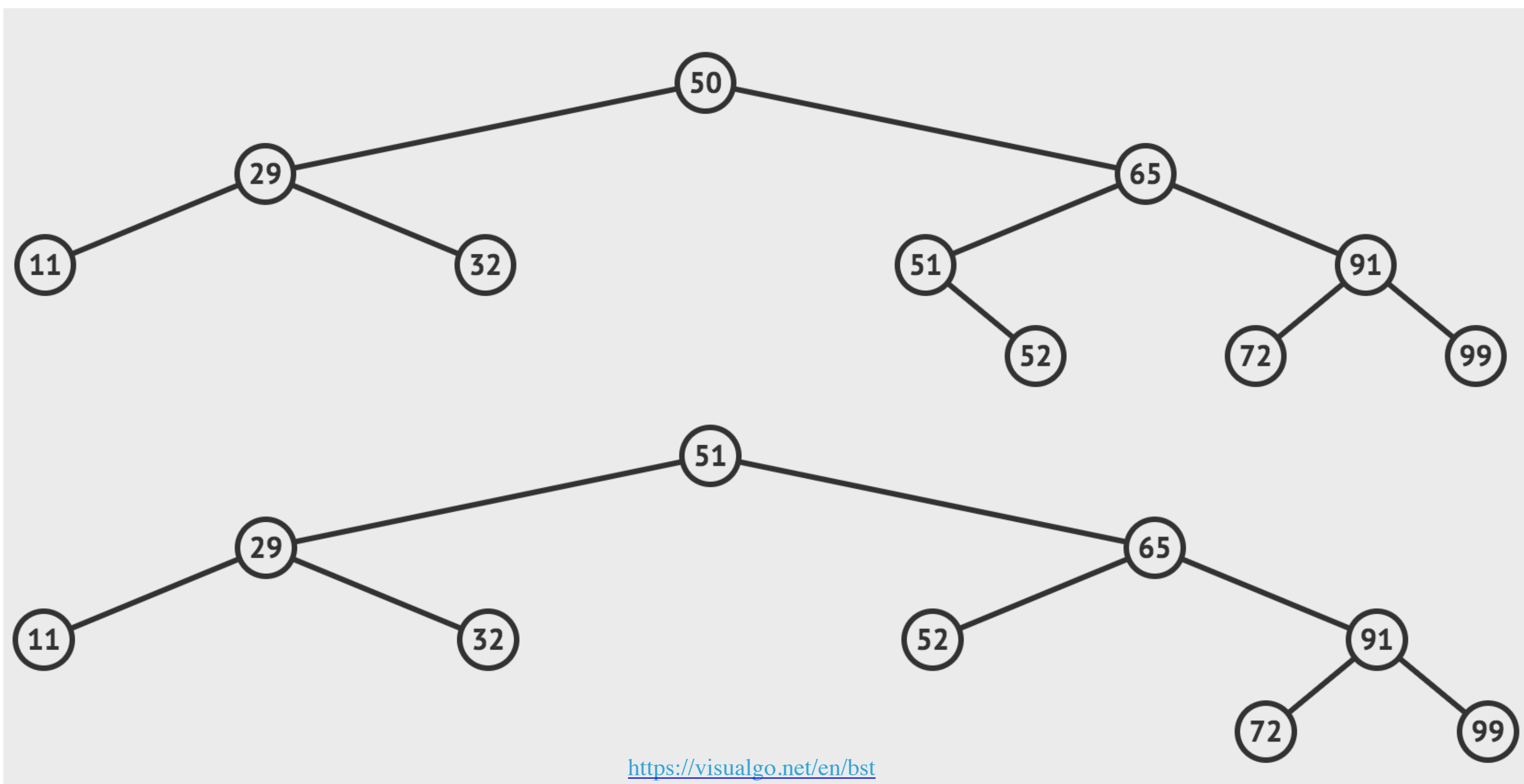
Hibbard deletion: Delete node with one child

- Delete node and replace it with its child.
- Example: delete 70 locates a node which has one child and replaces it with the child.



Hibbard deletion: Delete node with two children

- Delete node and replace it with successor (node with smallest of the larger keys). Move successor's child (if any) where successor was.
- Example: delete 50 locates a node which has two children. Successor is 51.



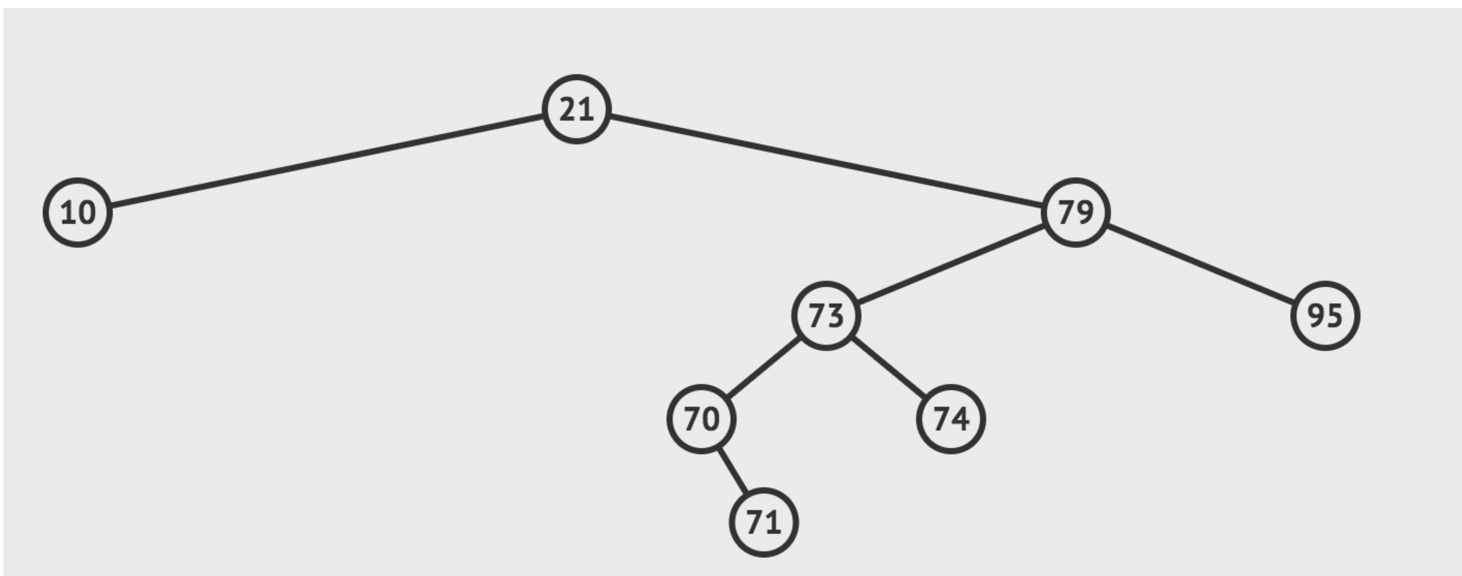
```
public void delete(Key key) {
    root = delete(root, key);
}

 private Node delete(Node x, Key key) {
     if (x == null) return null;

     int cmp = key.compareTo(x.key);
     if (cmp < 0)
         x.left  = delete(x.left,  key);
     else if (cmp > 0)
         x.right = delete(x.right, key);
     else {
         if (x.right == null)
             return x.left;
         if (x.left  == null)
             return x.right;
         Node t = x; //replace with successor
         x = min(t.right);
         x.right = deleteMin(t.right);
         x.left = t.left;
     }
     x.size = size(x.left) + size(x.right) + 1;
     return x;
 }
```

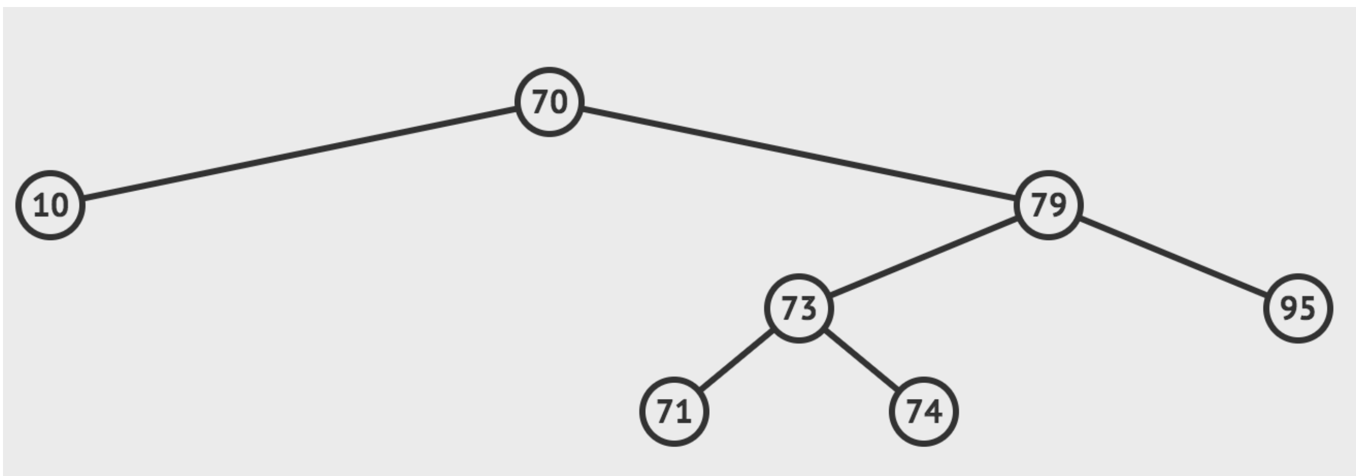
Practice Time

- Delete the node 21 following Hibbard's deletion



Answer

- Delete the node 21 following Hibbard's deletion



Hibbard's deletion

- Unsatisfactory solution. If we were to perform many insertions and deletions the BST ends up being not symmetric and skewed to the left.
  - Extremely complicated analysis, but average cost of deletion ends up being $\sqrt{n}$. Let's simplify things by saying it stays $O(\log n)$.
  - No one has proven that alternating between the predecessor and successor will fix this.
- Hibbard devised the algorithm in 1962. Still no algorithm for efficient deletion in Binary Search Trees!
- Overall, BSTs can have $O(n)$ worst-case for search, insert, and delete. We want to do better (see future lectures).

Lecture 23: Binary Search Trees

- Dictionaries
- Binary Search Trees

Readings:

- Textbook: Chapters 3.1 (Pages 362–386) and 3.2 (Pages 396–414)
- Website:
  - https://algs4.cs.princeton.edu/31elementary/
  - https://algs4.cs.princeton.edu/32bst/
- Visualization:
  - https://visualgo.net/en/bst

Practice Problems:

- 3.1.1-3.1.6, 3.2.1-3.2.13