SORTING

# CS062

## DATA STRUCTURES AND ADVANCED PROGRAMMING

## 16: Quicksort, Binary Trees and Heaps

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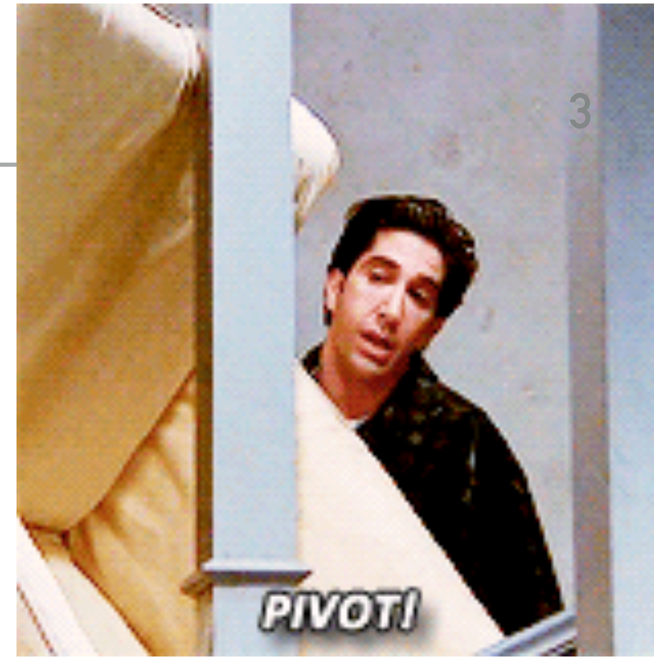
he/him/his

## Lecture 16: Quicksort, Binary Trees and Heaps

- Quicksort




# Algorithm sketch:

- Shuffle the array.
- Partition so that, for some pivot `j`:
  - Entry `a[j]` is in place.
  - There is no larger entry to the left of `j`.
  - No smaller entry to the right of `j`.
- Sort each subarray recursively.

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| input | Q | U | I | C | K | S | O | R | T | E | X | A | M | P | L | E |
| shuffle | K | R | A | T | E | L | E | P | U | I | M | Q | C | X | O | S |
| partition | E | C | A | I | E | K | L | P | U | T | M | Q | R | X | O | S |
| sort left | A | C | E | E | I | K | L | P | U | T | M | Q | R | X | O | S |
| sort right | A | C | E | E | I | K | L | M | O | P | Q | R | S | T | U | X |
| result | A | C | E | E | I | K | L | M | O | P | Q | R | S | T | U | X |

*partitioning element*

*not greater*  *not less*

**Quicksort overview**

# Quicksort Trace

```java
private static void sort(Comparable[] a, int lo, int hi) {
    if (hi <= lo) return;
    int j = partition(a, lo, hi);
    sort(a, lo, j-1);
    sort(a, j+1, hi);
}
```

| | lo | j | hi | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| initial values | | | | Q | U | I | C | K | S | O | R | T | E | X | A | M | P | L | E |
| random shuffle | | | | K | R | A | T | E | L | E | P | U | I | M | Q | C | X | O | S |
| | 0 | 5 | 15 | E | C | A | I | E | K | L | P | U | T | M | Q | R | X | O | S |
| | 0 | 3 | 4 | E | C | A | E | I | K | L | P | U | T | M | Q | R | X | O | S |
| | 0 | 2 | 2 | A | C | E | E | I | K | L | P | U | T | M | Q | R | X | O | S |
| | 0 | 0 | 1 | A | C | E | E | I | K | L | P | U | T | M | Q | R | X | O | S |
| | 1 | | 1 | A | C | E | E | I | K | L | P | U | T | M | Q | R | X | O | S |
| | 4 | | 4 | A | C | E | E | I | K | L | P | U | T | M | Q | R | X | O | S |
| | 6 | 6 | 15 | A | C | E | E | I | K | L | P | U | T | M | Q | R | X | O | S |
| | 7 | 9 | 15 | A | C | E | E | I | K | L | M | O | P | T | Q | R | X | U | S |
| | 7 | 7 | 8 | A | C | E | E | I | K | L | M | O | P | T | Q | R | X | U | S |
| | 8 | | 8 | A | C | E | E | I | K | L | M | O | P | T | Q | R | X | U | S |
| | 10 | 13 | 15 | A | C | E | E | I | K | L | M | O | P | S | Q | R | T | U | X |
| | 10 | 12 | 12 | A | C | E | E | I | K | L | M | O | P | R | Q | S | T | U | X |
| | 10 | 11 | 11 | A | C | E | E | I | K | L | M | O | P | Q | R | S | T | U | X |
| | 10 | | 10 | A | C | E | E | I | K | L | M | O | P | Q | R | S | T | U | X |
| | 14 | 14 | 15 | A | C | E | E | I | K | L | M | O | P | Q | R | S | T | U | X |
| | 15 | | 15 | A | C | E | E | I | K | L | M | O | P | Q | R | S | T | U | X |
| result | | | | A | C | E | E | I | K | L | M | O | P | Q | R | S | T | U | X |

*no partition for subarrays of size 1*

# Great algorithms are better than good ones

- Your laptop executes $10^8$ comparisons per second
- A supercomputer executes $10^{12}$ comparisons per second

| | Insertion sort | | | Mergesort | | | Quicksort | | |
|---|---|---|---|---|---|---|---|---|---|
| **Computer** | **Thousand inputs** | **Million inputs** | **Billion inputs** | **Thousand inputs** | **Million inputs** | **Billion inputs** | **Thousand inputs** | **Million inputs** | **Billion inputs** |
| **Home** | Instant | 2 hours | 300 years | instant | 1 sec | 15 min | Instant | 0.5 sec | 10 min |
| **Supercomputer** | Instant | 1 second | 1 week | instant | instant | instant | instant | instant | Instant |

# Quicksort analysis: best case

- Quicksort divides everything exactly in half.
- Similar to merge sort.
- Number of compares is $\sim n \log n$.

# Quicksort analysis: worst case

- Data are already sorted or when we always pick the smallest or largest key as pivot.
- Number of compares is ~$n^2$ - quadratic!
- Extremely unlikely (less likely than the probably that your computer is struck by lightning) if we shuffle and our shuffling is not broken.

# Things to remember about quick sort

- $O(n \log n)$ average, $O(n^2)$ worst, in practice faster than mergesort.

- 39% more compares than merge sort but in practice it is faster because it does not move data much.
  - Quicksort compares and increments index pointer
  - Mergesort moves items into and out of aux array

- Random shuffle = probabilistic guarantee against worst case

- In-place sorting.
- **Not** stable.

## Quicksort practical improvements

- Use insertion sort for small subarrays.
- Best choice of pivot is the median of a small sample.

- For years, Java used quicksort for collections of primitives and mergesort for collections of objects due to stability.
  - Has moved to dual-pivot quick sort (Yaroslavskiy, Bentley, and Bloch, 2009) and timsort (Peters, 1993), respectively.

# Sorting: the story so far

| Which Sort | In place | Stable | Best | Average | Worst | Remarks |
|---|---|---|---|---|---|---|
| Selection | X | | $O(n^2)$ | $O(n^2)$ | $O(n^2)$ | $n$ exchanges |
| Insertion | X | X | $O(n)$ | $O(n^2)$ | $O(n^2)$ | Use for small arrays or partially ordered |
| Merge | | X | $O(n \log n)$ | $O(n \log n)$ | $O(n \log n)$ | Guaranteed performance; stable |
| Quick | X | | $O(n \log n)$ | $O(n \log n)$ | $O(n^2)$ | $n \log n$ probabilistic guarantee; fastest in practice |

## Sorting based on comparisons

- All sorting algorithms we have seen so far and we will see in this class are compare-based.
- No compare-based sorting algorithm can sort $n$ elements in less than $O(n \log n)$ time in the worst case.
  - Proof and proper notation in CS140.

# Readings:

- Textbook:
  - Chapter 2.3 (Pages 288-296)
- Website:
  - Quicksort: https://algs4.cs.princeton.edu/23quicksort/
  - Code: https://algs4.cs.princeton.edu/23quicksort/Quick.java.html

# Practice Problems:

- 2.3.1-2.3.4

Basic data structures

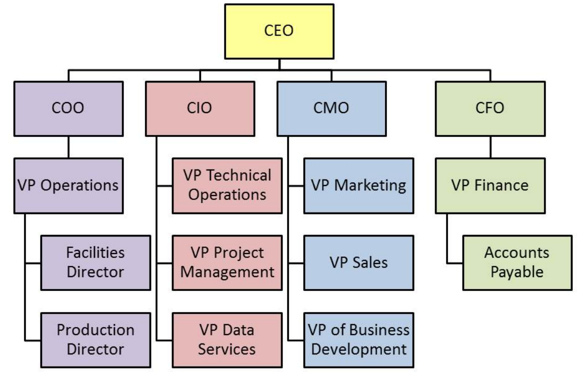
- Arrays,
- Resizing arrays or arraylists,
- Linked Lists,
- Queues, and
- Stacks.
- Runtime and memory analysis for each one.

# Sorting

- Selection sort,
- Insertion sort,
- Mergesort, and
- Quicksort.
- Runtime (comparisons and exchanges), stability, in-place for each one.
- Comparators: How to sort a data structure with objects of any class.
- Iterators: How to traverse a data structure.

## Lecture 16: Binary Trees and Heaps

- Binary Trees
- Tree traversals
- Binary Heaps



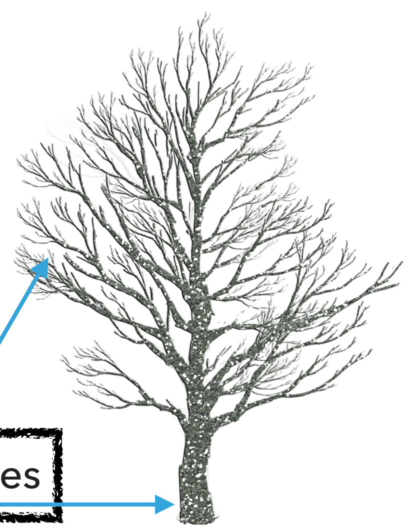

# Trees in Computer Science

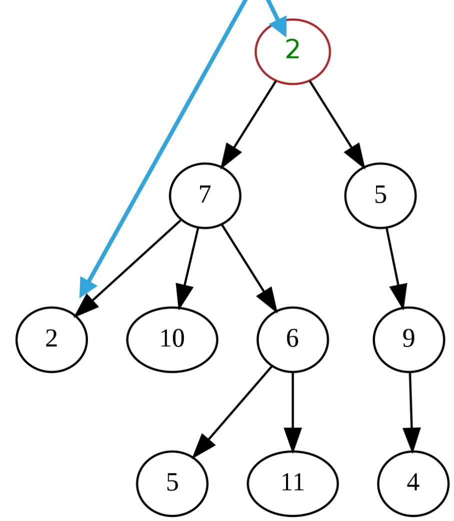
- Abstract data types that store elements **hierarchically** rather than linearly.
- Examples of hierarchical structures:
  - Organization charts for
    - Companies (CEO at the top followed by CFO, CMO, COO, CTO, etc).
    - Universities (Board of Trustees at the top, followed by President, then by VPs, etc).
  - Sitemaps (home page links to About, Products, etc. They link to other pages).
  - Computer file systems (user at top followed by Documents, Downloads, Music, etc. Each folder can hold more folders.).

## Trees in Computer Science

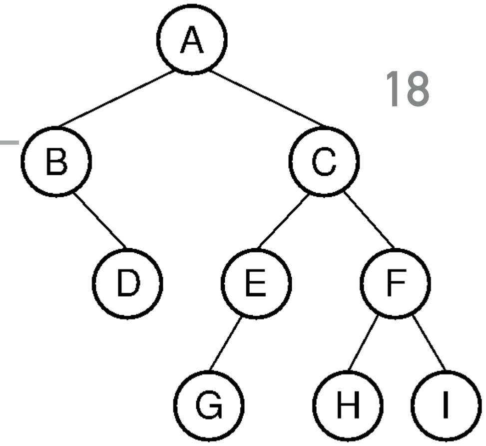
- Hierarchical: Each element in a tree has a **single** parent (immediate ancestor) and zero or more children (immediate descendants).
- What if you have multiple parents?
  - graphs

CS tree root and leaves



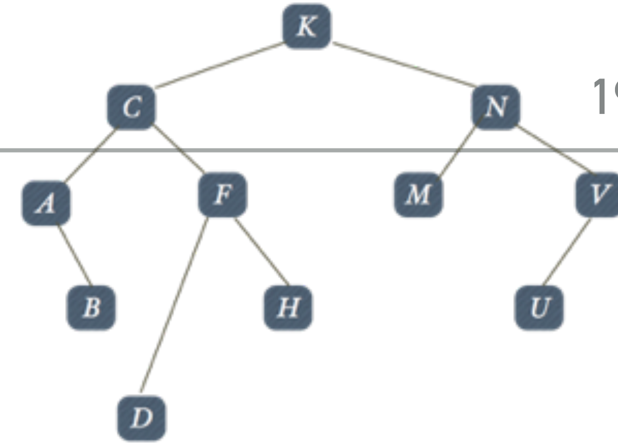



Real tree root and leaves



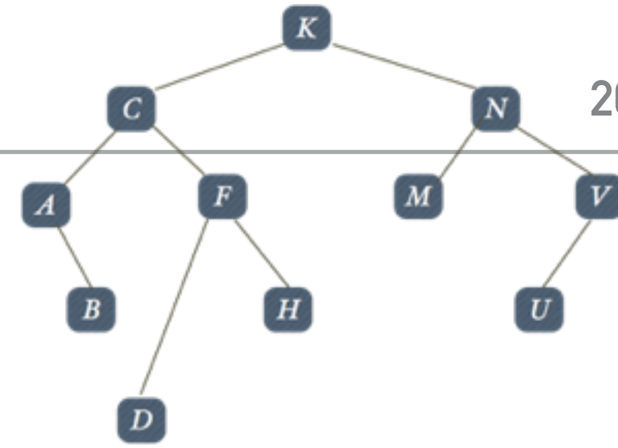

## Definition of a tree

- A tree $T$ is a set of nodes that store elements based on a parent-child relationship:
  - If $T$ is non-empty, it has a node called the root of $T$, that has no parent.
    - Here, the root is A.
  - Each node $v$, other than the root, has a unique parent node $u$. Every node with parent $u$ is a child of $u$.
    - E.g., E's parent is C and F has two children, H and I.



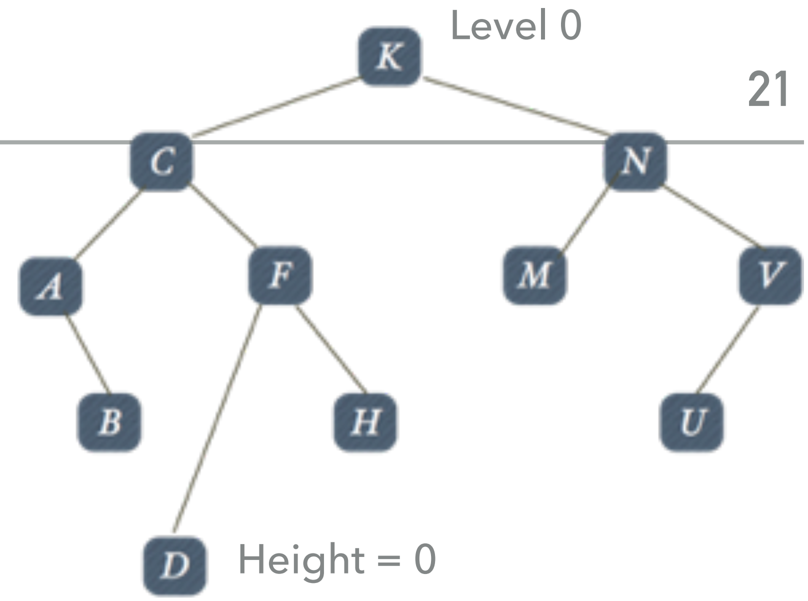

## Tree Terminology

- Edge: a pair of nodes s.t. one is the parent of the other, e.g., (K,C).
- Parent node is directly above child node, e.g., K is parent of C and N.
- Sibling nodes have same parent, e.g., A and F.
- K is ancestor of B.
- B is descendant of K.
- Node plus all descendants gives subtree. Which nodes are in the subtree at N?
- Nodes without descendants are called leaves or external. The rest are called internal. Which ones are leaves? Which are internal?
- A set of trees is called a forest.



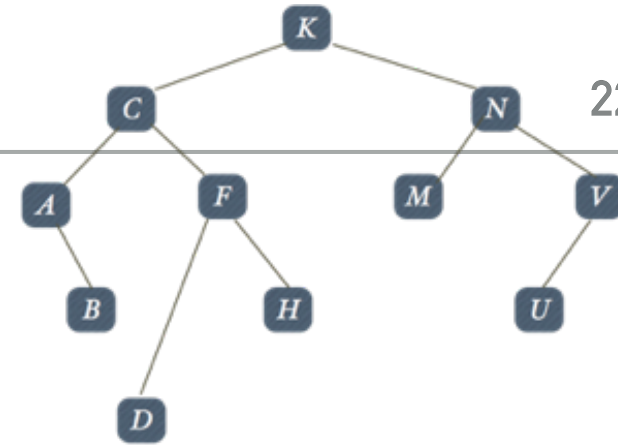

# More Terminology

- Simple path: a series of distinct nodes s.t. there are edges between successive nodes, e.g., K-N-V-U.
- Path length: number of edges in path, e.g., path K-C-A has length 2.
- Height of node: length of longest path from the node to a leaf. What is height of C?
- Height of tree: length of longest path from the root to a leaf. Height of root?
- Degree of node: number of its children. Degree of C?
- Degree of tree (arity): max degree of any of its nodes. Degree of this tree?
- Binary tree: a tree with arity of 2.




## Even More Terminology

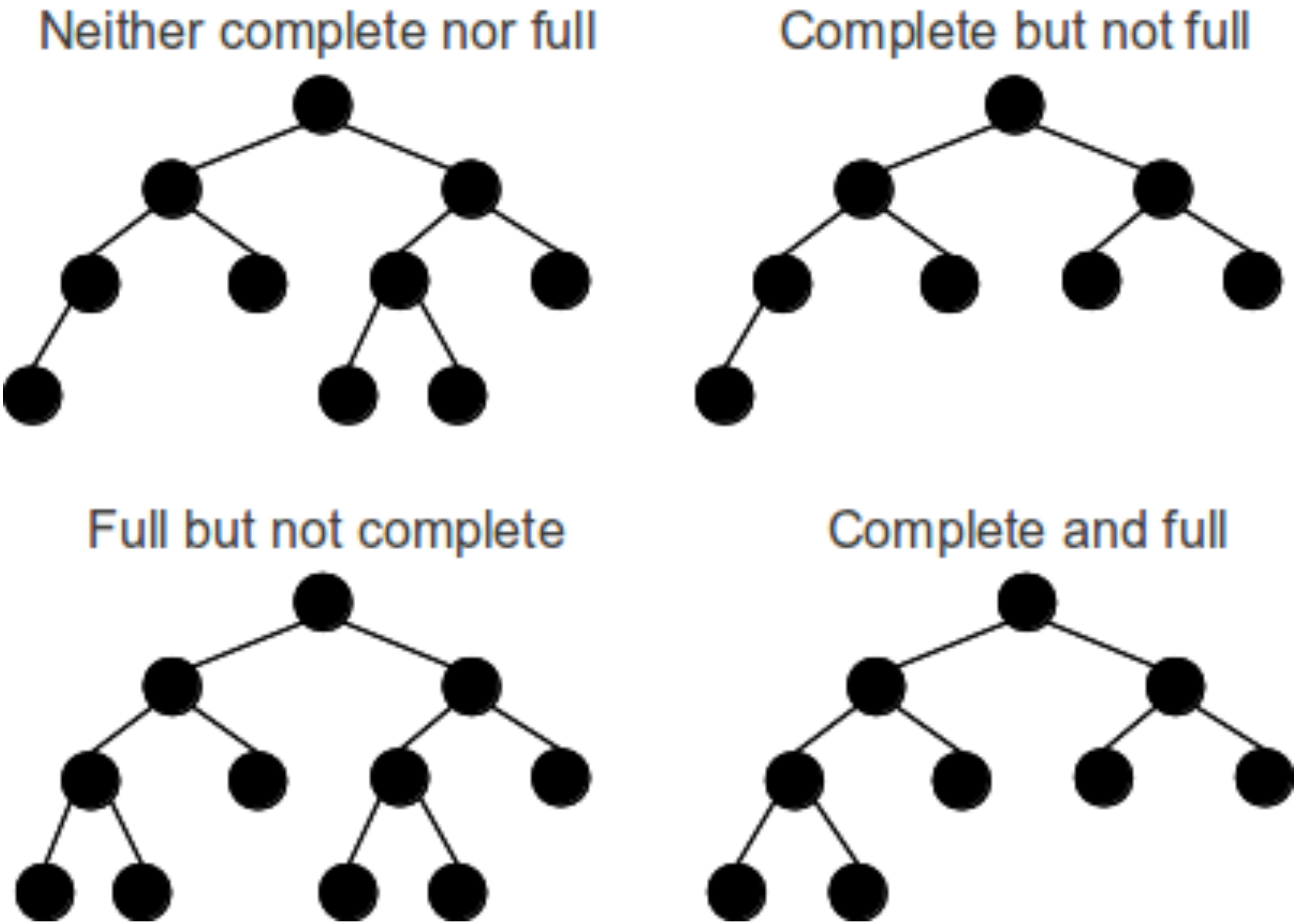
- Level/depth of node defined recursively:
  - Root is at level 0.
  - Level of any other node is equal to level of parent + 1. What is level of M?
  - It is also known as the length of path from root or number of ancestors excluding itself.
- Height of node defined recursively:
  - If leaf, height is 0.
  - Else, height is max height of child + 1. What is height of N?




But wait there's more!

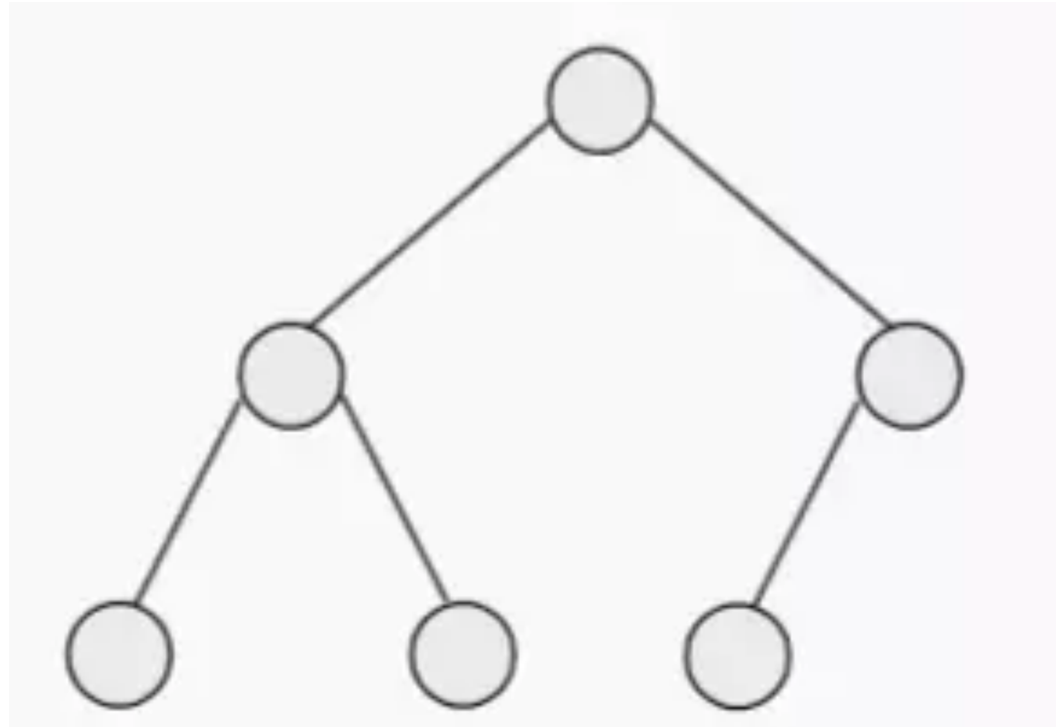
- Full (or proper): a binary tree whose every node has 0 or 2 children. Is this tree full?

- Complete: a binary tree with minimal height. Any holes in tree would appear at last level to right, i.e., all nodes of last level are as left as possible.




http://code.cloudkaksha.org/binary-tree/types-binary-tree

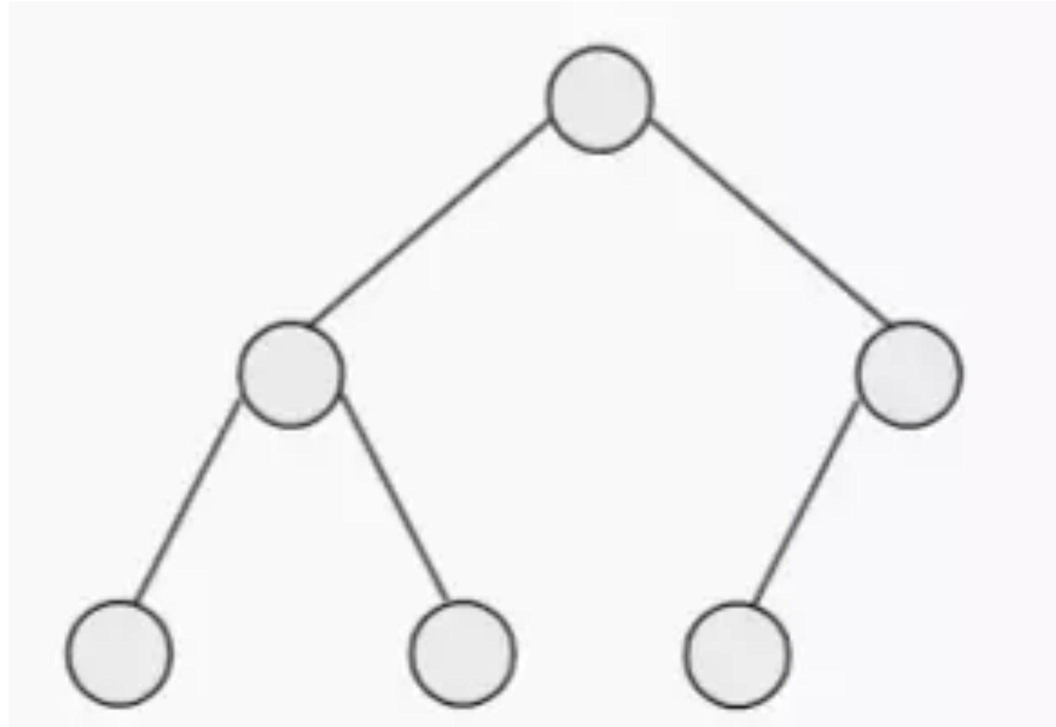
Practice Time: This tree is



- A: Full
- B: Complete
- C: Full and Complete
- D: Neither Full nor Complete

## Answer

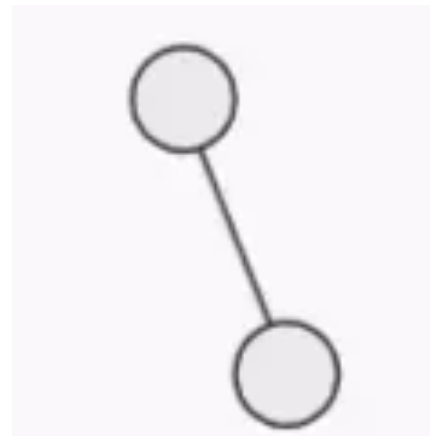
- A: Full
- **B: Complete**
- C: Full and Complete
- D: Neither Full nor Complete



How do we make it full?

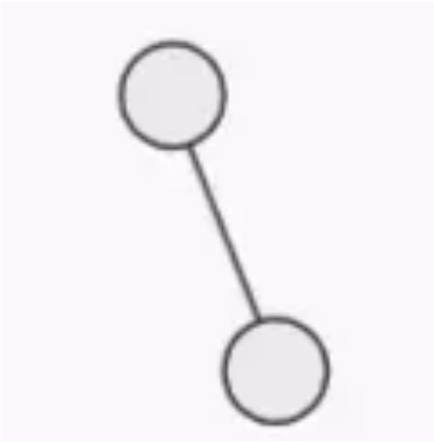
Practice Time: This tree is

- A: Full
- B: Complete
- C: Full and Complete
- D: Neither Full nor Complete



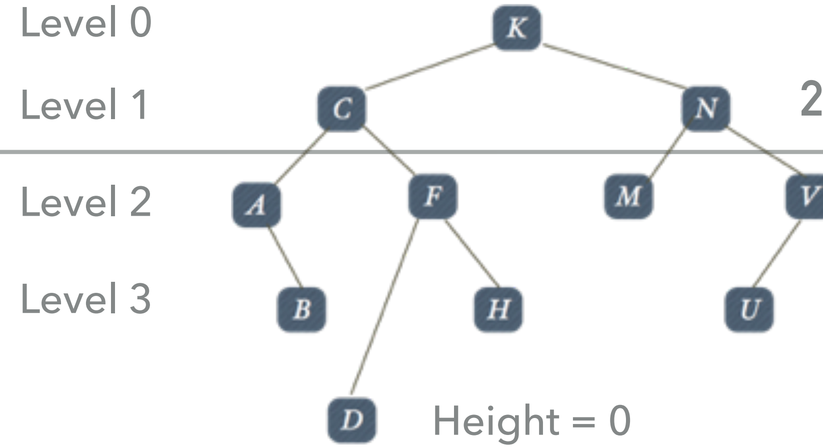
# Answer

- A: Full
- B: Complete
- C: Full and Complete
- **D: Neither Full nor Complete**

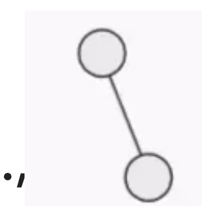


How do we make it full?
complete?

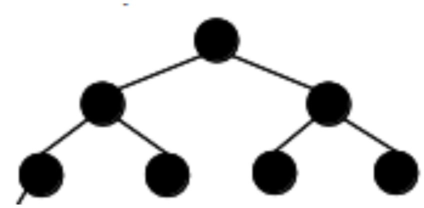
# Counting in binary trees




- Lemma: if $T$ is a binary tree, then at level $k$, $T$ has $\leq 2^k$ nodes.
  - E.g., at level 2, at most 2^2 = 4 nodes (A, F, M, V)
- Theorem: If $T$ has height $h$, then # of nodes $n$ in $T$ satisfy: $h + 1 \leq n \leq 2^{h+1} - 1$.
- Equivalently, if $T$ has $n$ nodes, then $log(n + 1) - 1 \leq h \leq n - 1$.
  - Worst case (Max height): When $h = n - 1$ or $O(n)$, the tree looks like a left or right-leaning "stick".



  - Best case (Min height): When a tree is as compact as possible (e.g., complete) it has $O(\log n)$ height.



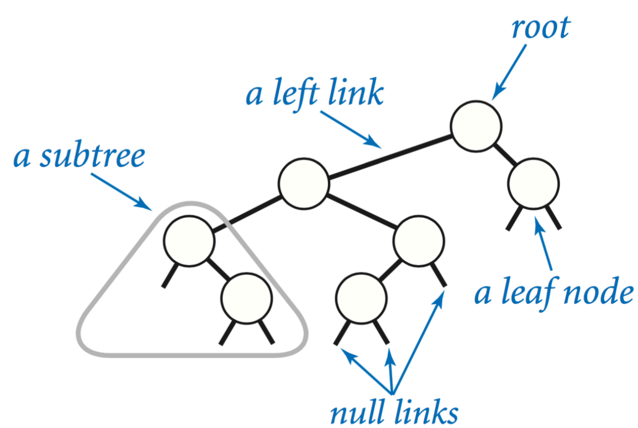
# Basic idea behind a simple implementation

```
public class BinaryTree<Item> {
    private Node root;

    /**
     * A node subclass which contains various recursive methods
     *
     * @param <Item>  The type of the contents of nodes
     */
    private class Node {
        private Item item;

        private Node left;
        private Node right;

        /**
         * Node constructor with subtrees
         *
         * @param left     the left node child
         * @param right    the right node child
         * @param item     the item contained in the node
         */
        public Node(Node left, Node right, Item item) {
            this.left = left;
            this.right = right;
            this.item = item;
        }
```

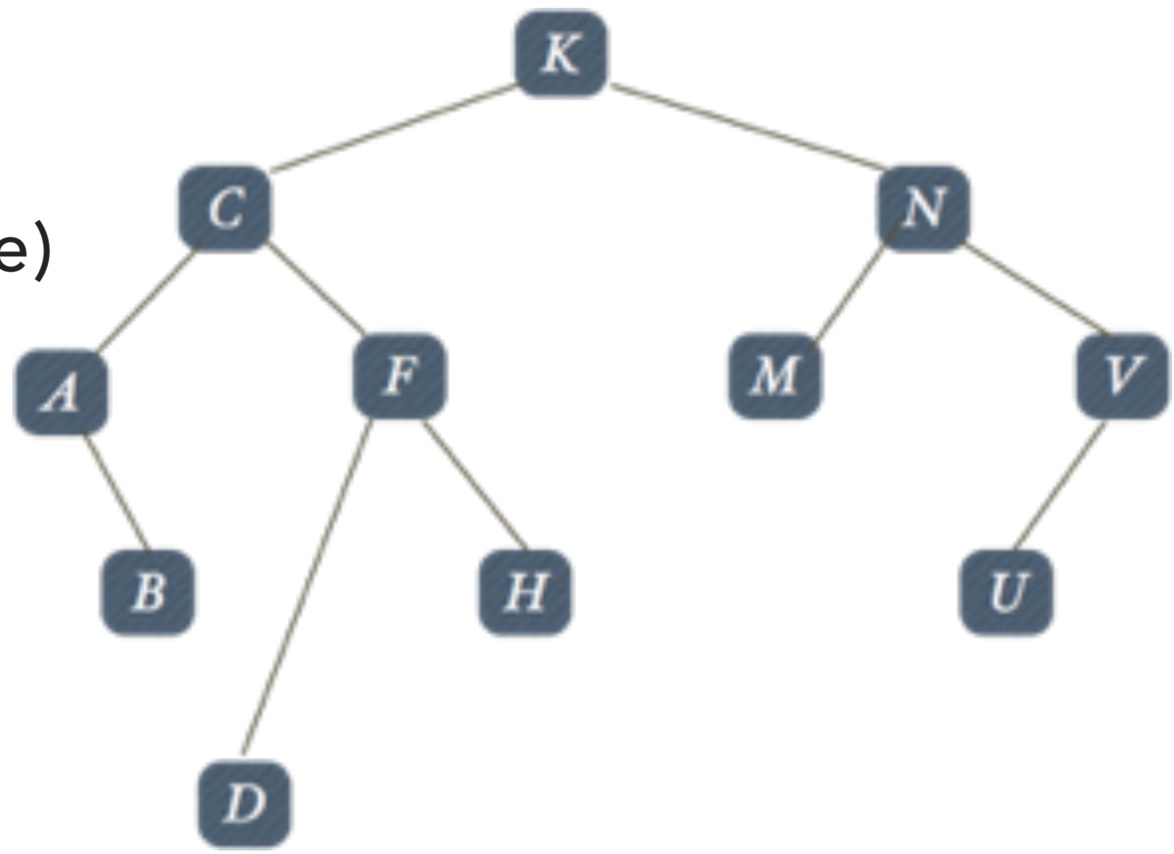


## Lecture 16: Binary Trees and Heaps

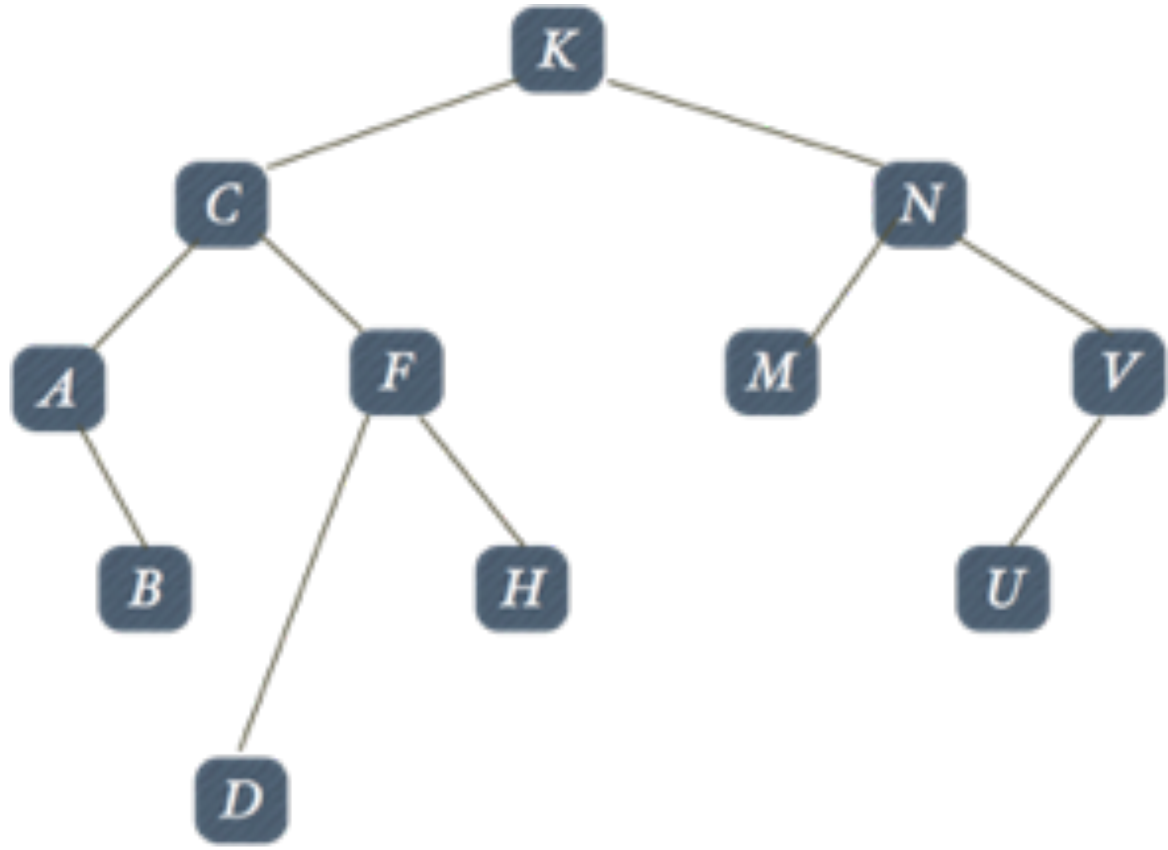
- Binary Trees
- Tree traversals
  - Pre-order, in-order, and post-order
  - Prefix indicates order of marking the root of the subtree as visited
  - Before, between, and after left and right subtrees
- Binary Heaps

## Pre-order traversal

- Preorder(Tree)
  - Mark root as visited
  - Preorder(Left Subtree)
  - Preorder(Right Subtree)
- K C A B F D H N M V U



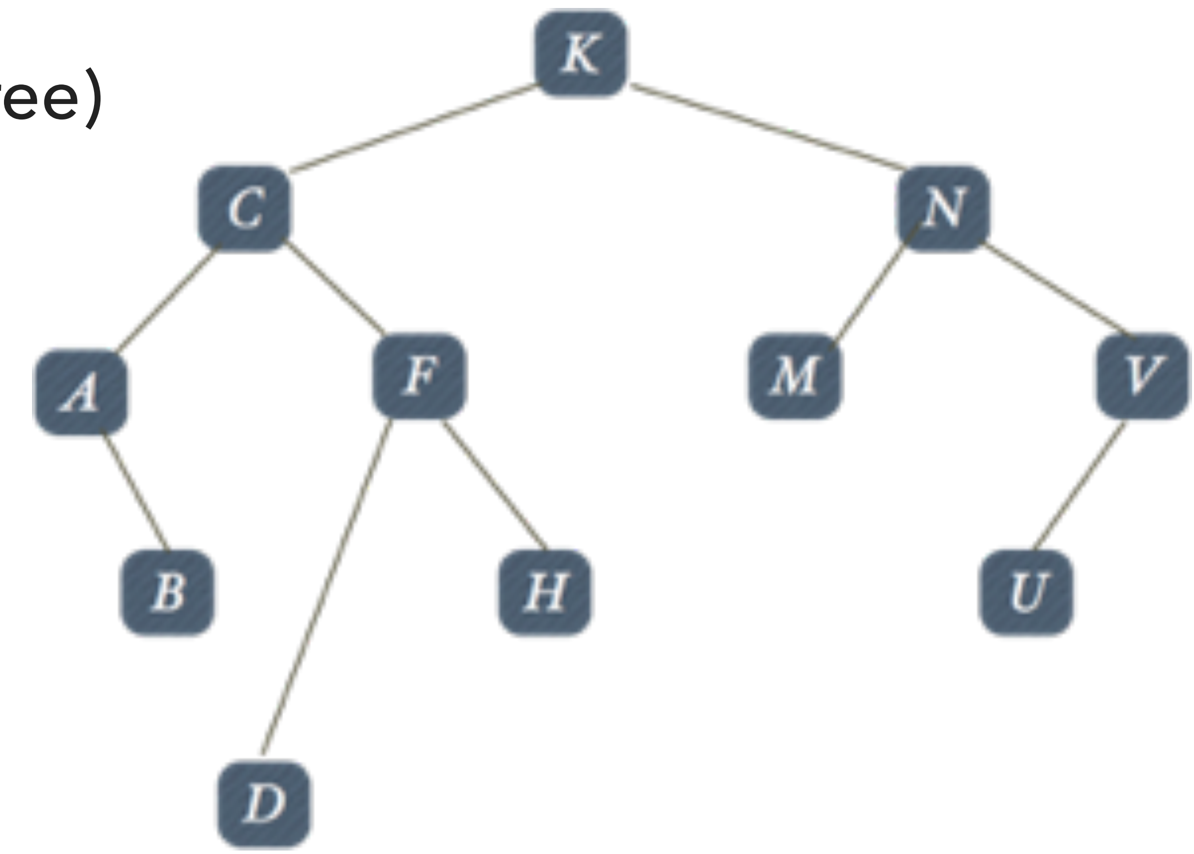
In-order traversal




- Inorder(Tree)
    - Inorder(Left Subtree)
    - Mark root as visited
    - Inorder(Right Subtree)
- A B C D F H K M N U V
- In-order traversals of binary search tree visits the nodes in sorted order

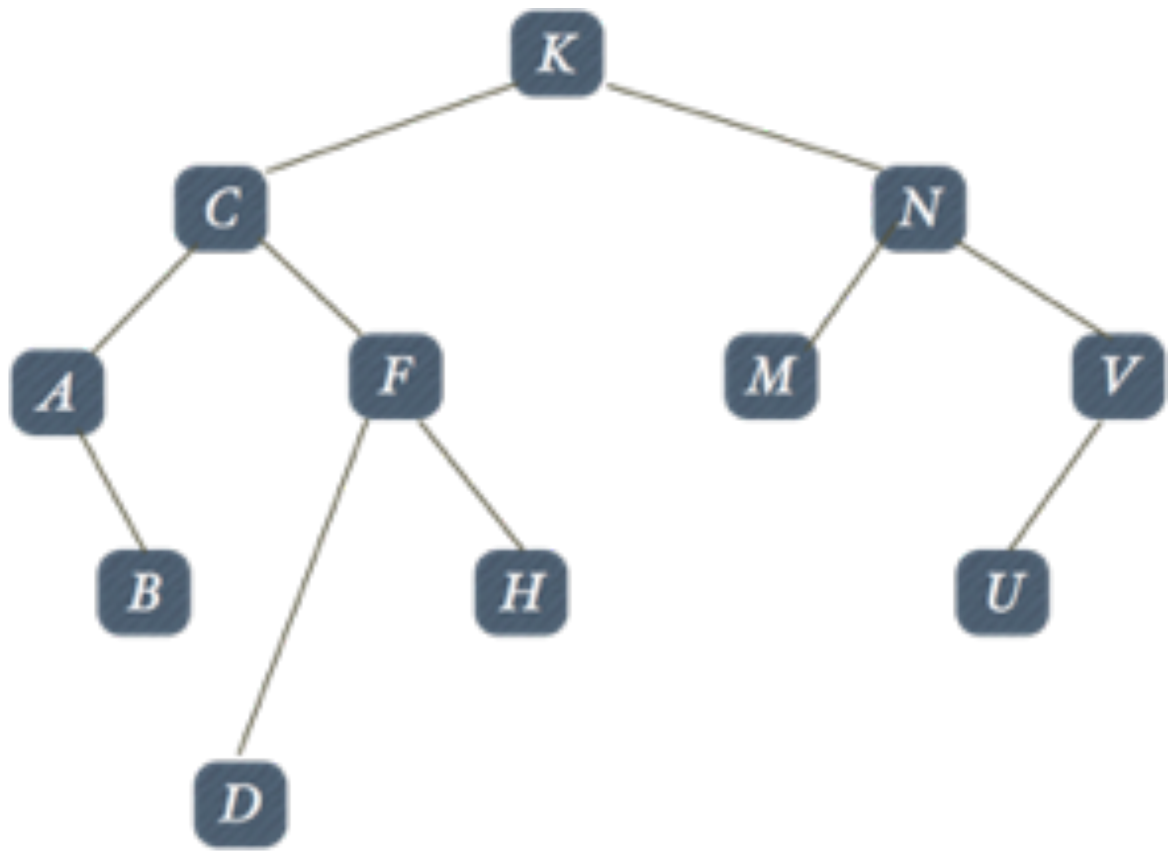
## Post-order traversal

- Postorder(Tree)
  - Postorder(Left Subtree)
  - Postorder(Right Subtree)
  - Mark root as visited
- B A D H F C M U V N K



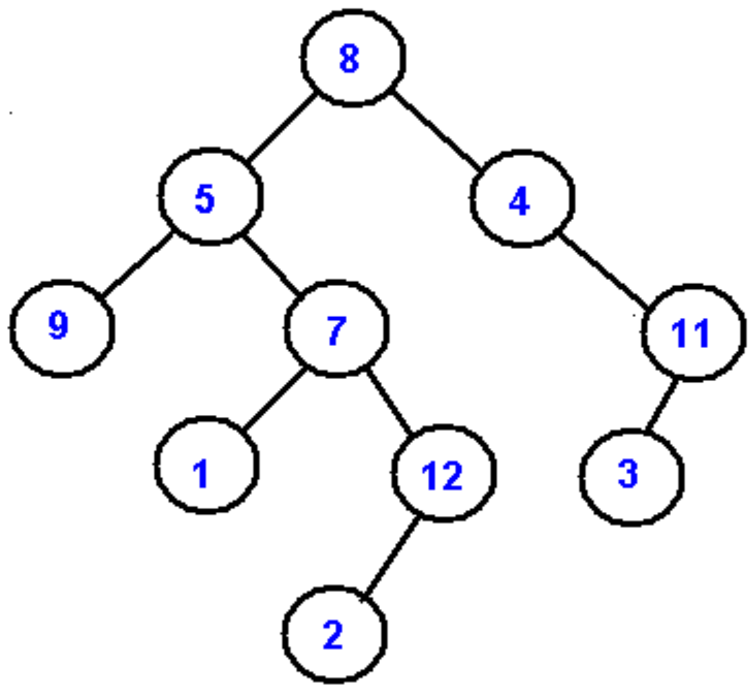
# Level-order traversal

- From left to right, mark nodes of level $i$ as visited before nodes in level $i + 1$. Start at level 0.
- K C N A F M V B D H U

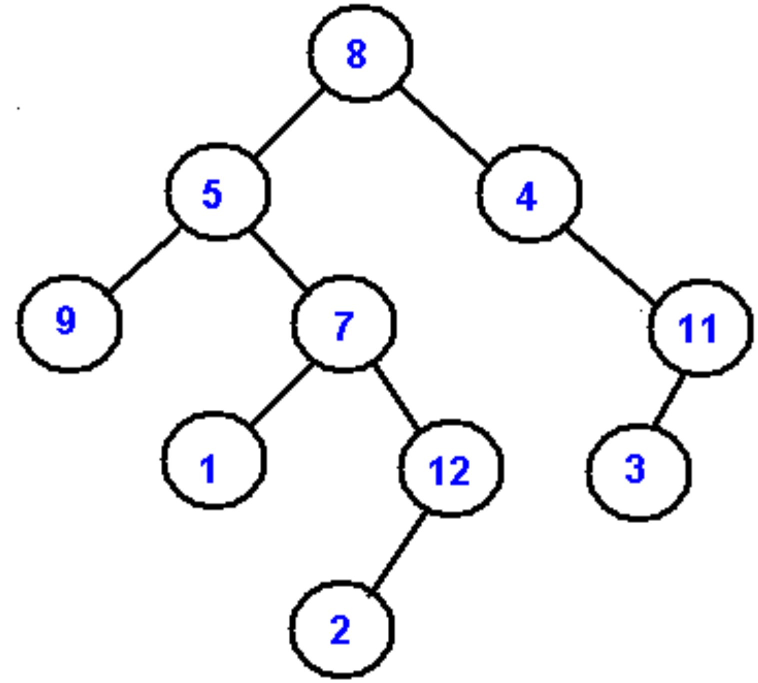


## Practice Time

- List the nodes in pre-order, in-order, post-order, and level order:



Answer




- Pre-order: 8, 5, 9, 7, 1, 12, 2, 4, 11, 3
- In-order: 9, 5, 1, 7, 2, 12, 8, 4, 3, 11
- Post-order: 9, 1, 2, 12, 7, 5, 3, 11, 4, 8
- Level-order: 8, 5, 4, 9, 7, 11, 1, 12, 3, 2

## Lecture 16: Binary Trees and Heaps

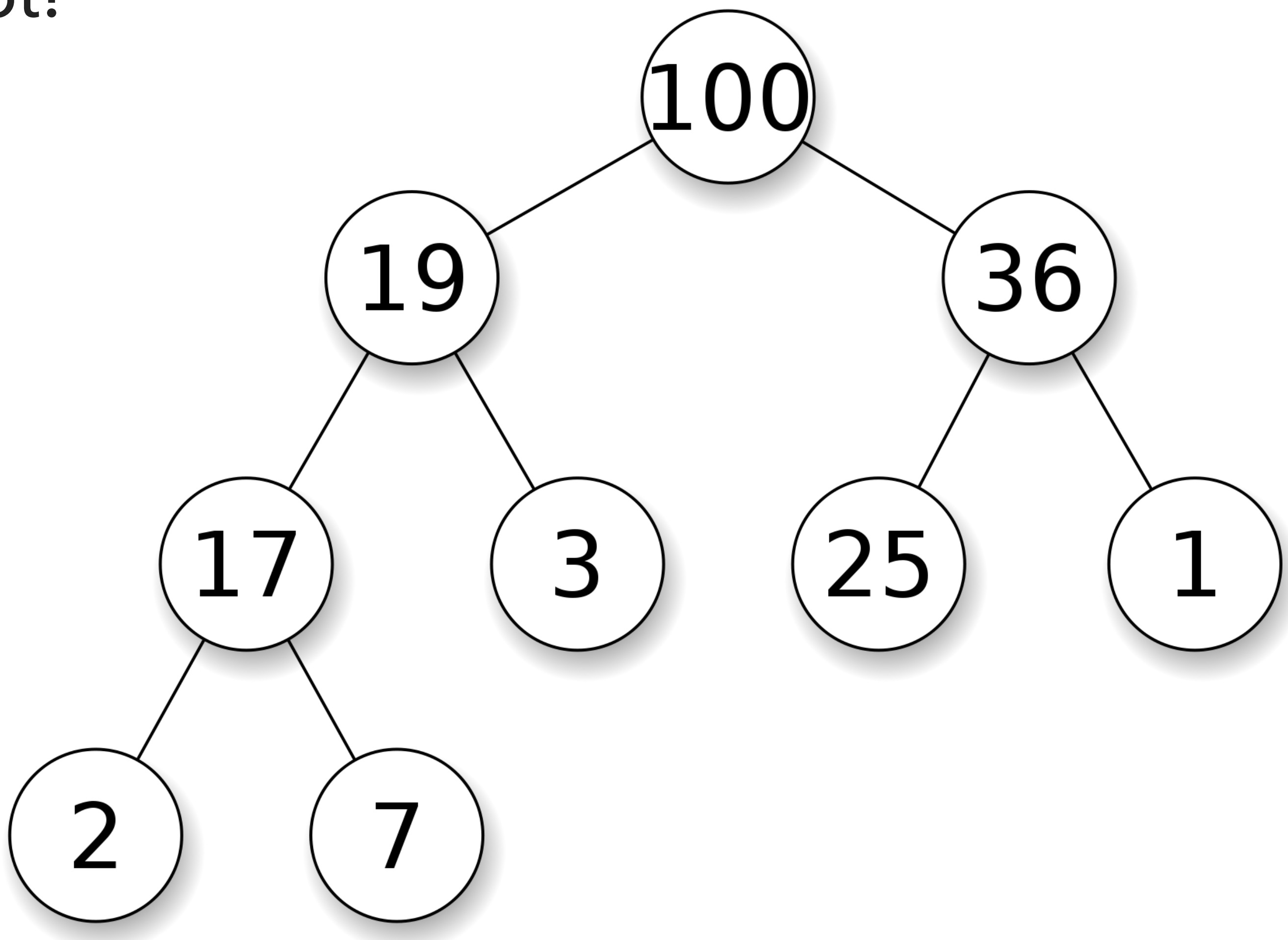
- Binary Trees
- Tree traversals
- Binary Heaps

## Heap-ordered binary trees

- A binary tree is heap-ordered if the key in each node is larger than or equal to the keys in that node's two children (if any).
- Equivalently, the key in each node of a heap-ordered binary tree is smaller than or equal to the key in that node's parent (if any).
- No assumption of which child is smaller.
- Moving up from any node, we get a non-decreasing sequence of keys.
- Moving down from any node we get a non-increasing sequence of keys.

## Heap-ordered binary trees

- The largest key in a heap-ordered binary tree is found at the root!



## Binary heap representation

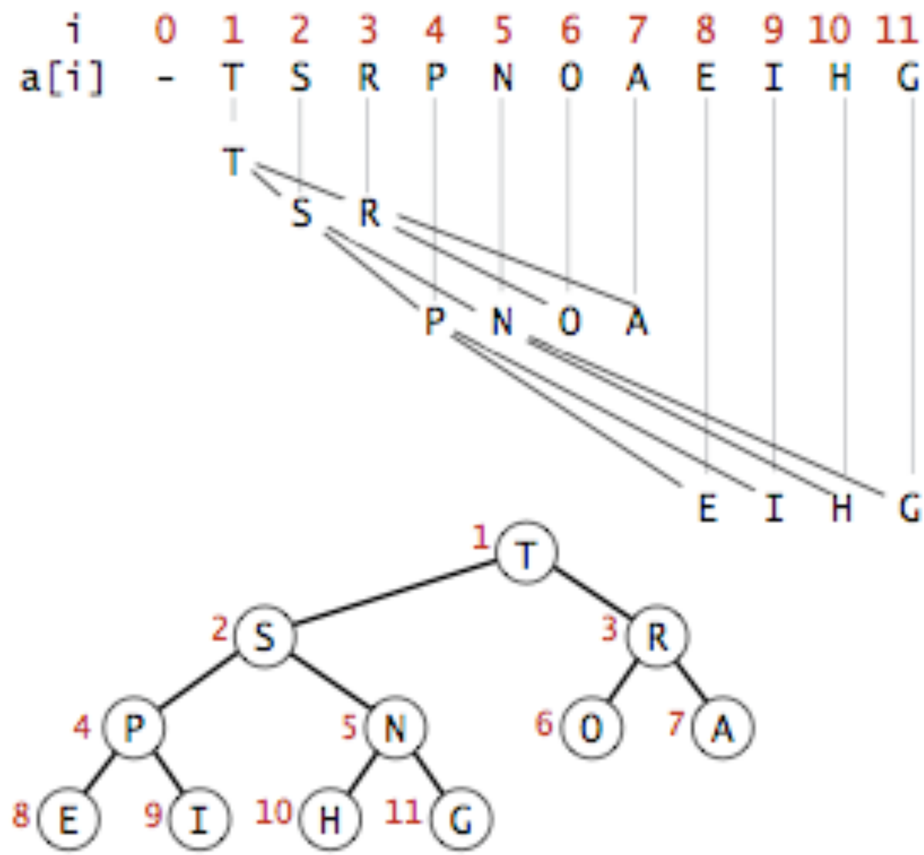
- We could use a linked representation but we would need three links for every node (one for parent, one for left subtree, one for right subtree).
- If we use complete binary trees, we can use an array instead.
  - Compact arrays vs explicit links means memory savings and faster execution!

## Binary heaps

- Binary heap: the array representation of a complete heap-ordered binary tree.
  - Items are stored in an array such that each key is guaranteed to be larger (or equal to) than the keys at two other specific positions (children).
- Max-heap but there are min-heaps, too.

# Array representation of heaps

- Nothing is placed at index 0.
- Root is placed at index 1.
- Rest of nodes are placed in level order.
- No unnecessary indices and no wasted space because it's complete.
- What's the relationship between node index and 2 children?



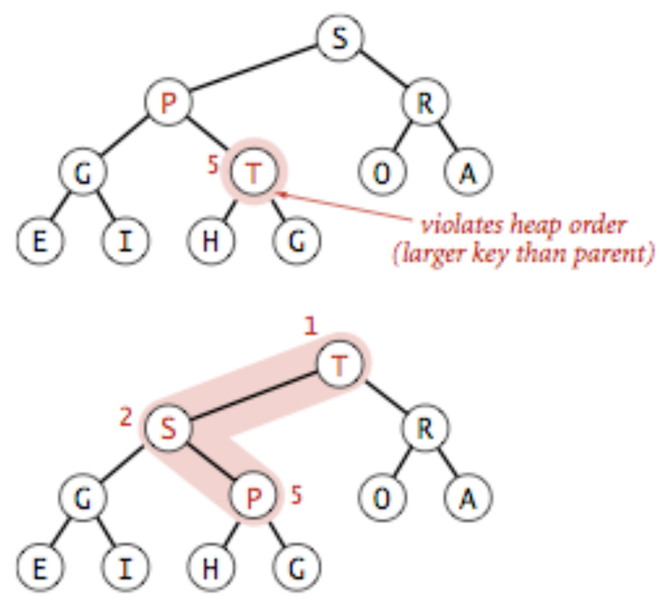

Heap representations

Reuniting immediate family members.

- For every node at index $k$, its parent is at index $\lfloor k/2 \rfloor$.
- Its two children are at indices $2k$ and $2k + 1$.
- We can travel up and down the heap by using this simple arithmetic on array indices.
- Accesses using indices are much faster than using pointers/references

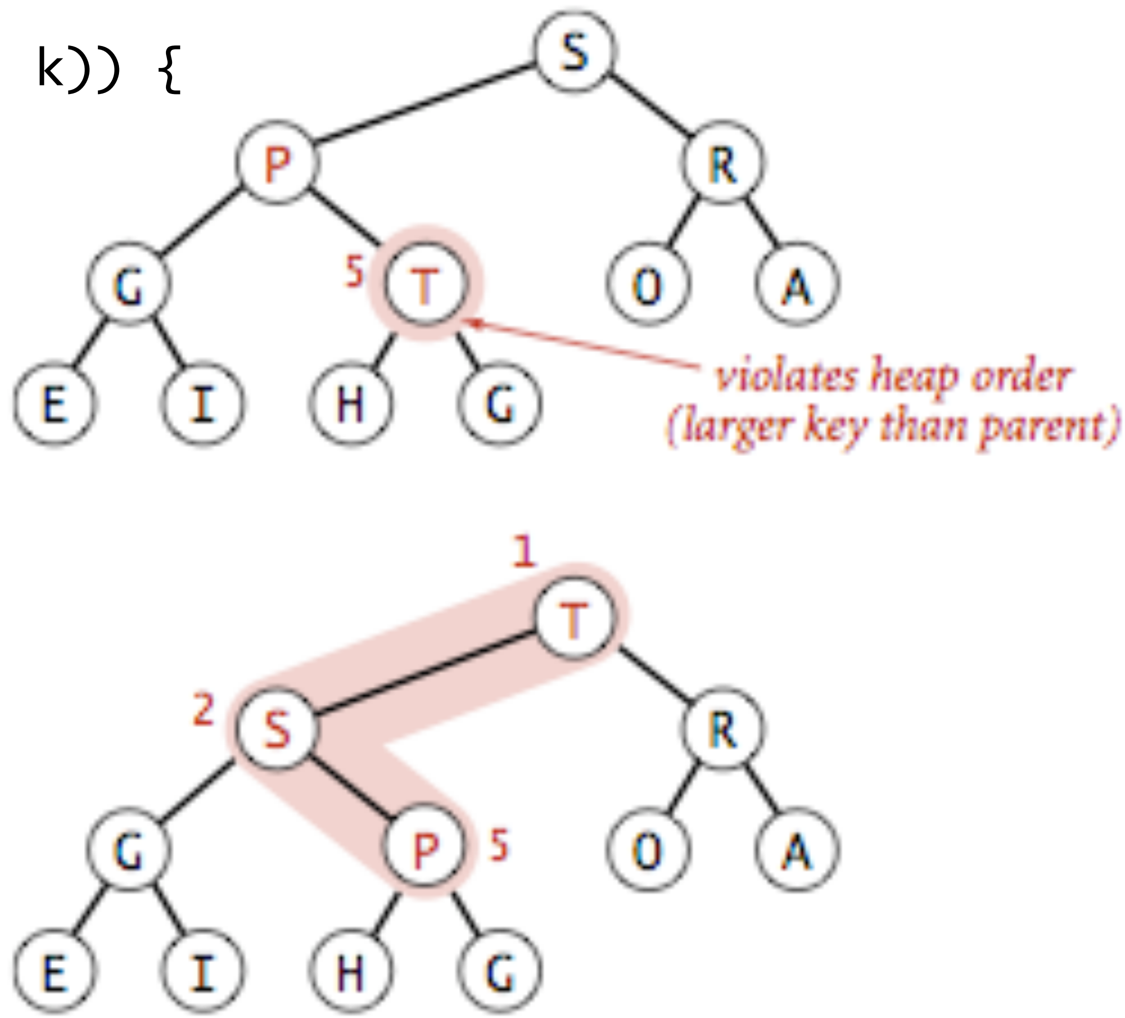
## Swim/promote/percolate up/bottom up reheapify

- Scenario: a key becomes larger than its parent therefore it violates the heap-ordered property.
- To eliminate the violation:
  - Exchange key in child with key in parent.
  - Repeat until heap order restored.



# Swim/promote/percolate up

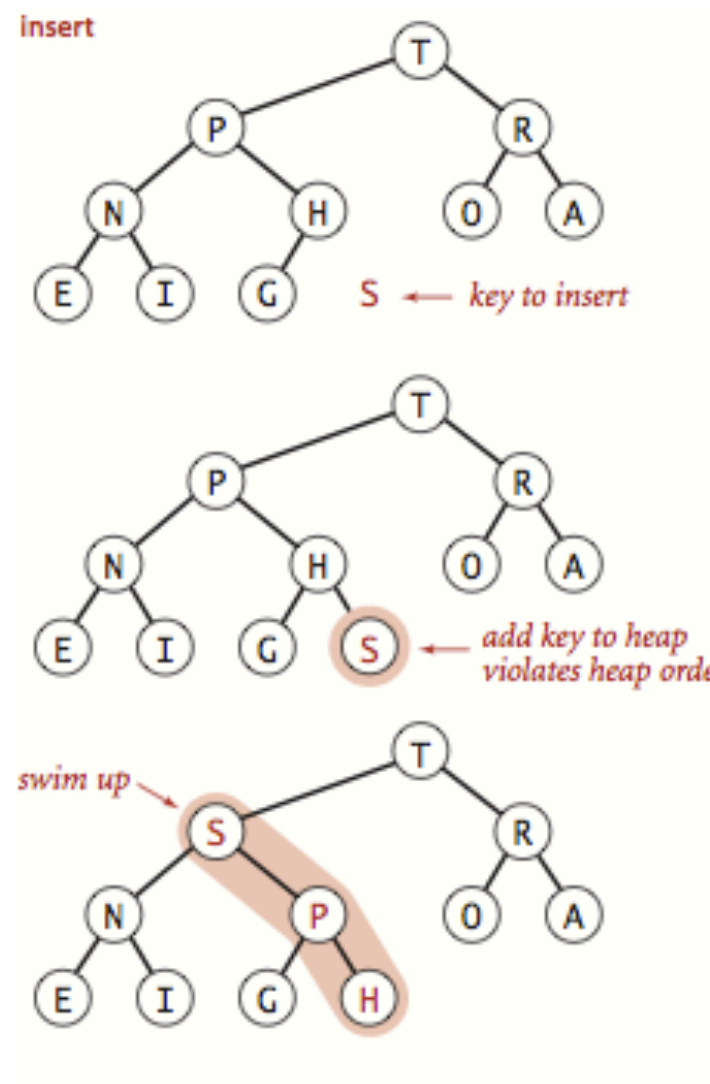
```
private void swim(int k) {
   while (k > 1 && less(k/2, k)) {
      exch(k, k/2);
      k = k/2;
   }
}
```



# Binary heap: insertion

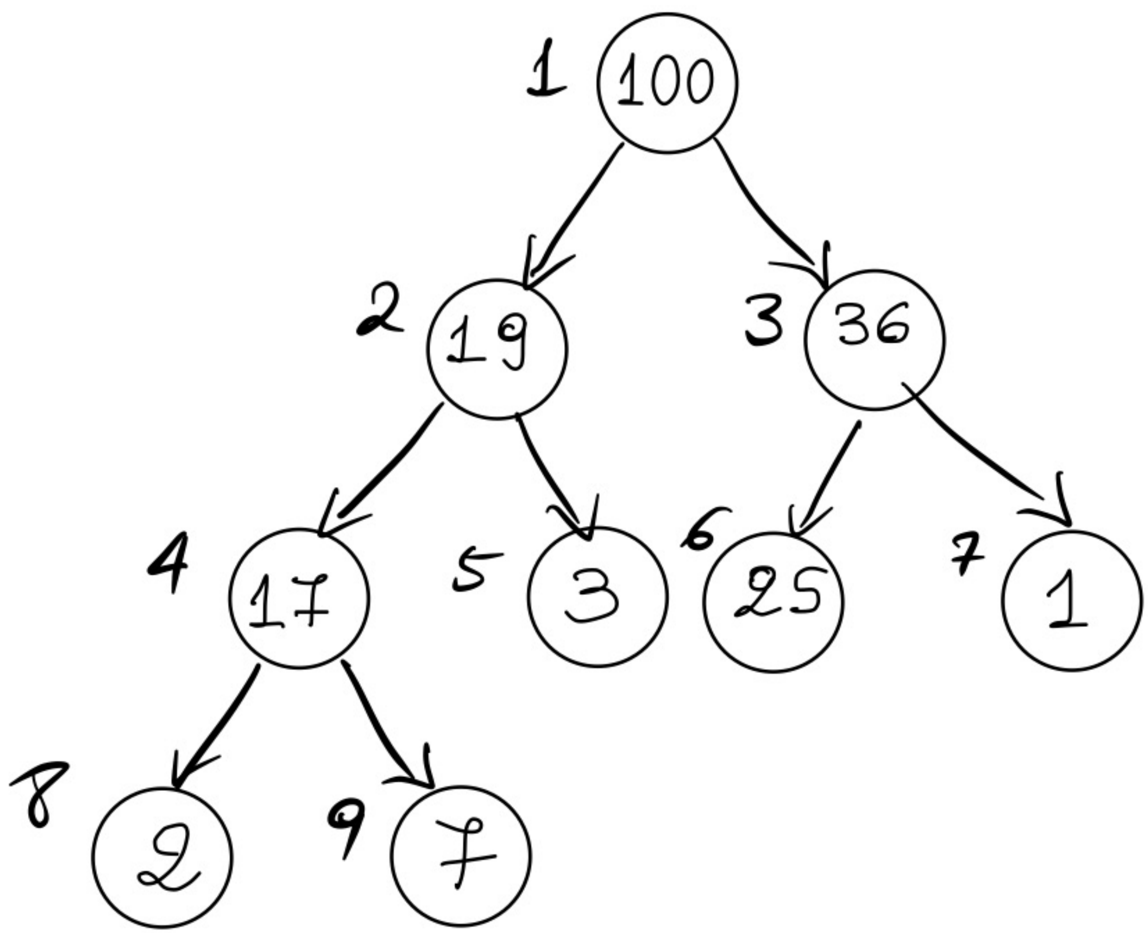
- Insert: Add node **at end in bottom level**, then **swim it up**.
- Cost: At most $\log n + 1$ compares.

```
public void insert(Key x) {
    pq[++n] = x;
    swim(n);
}
```

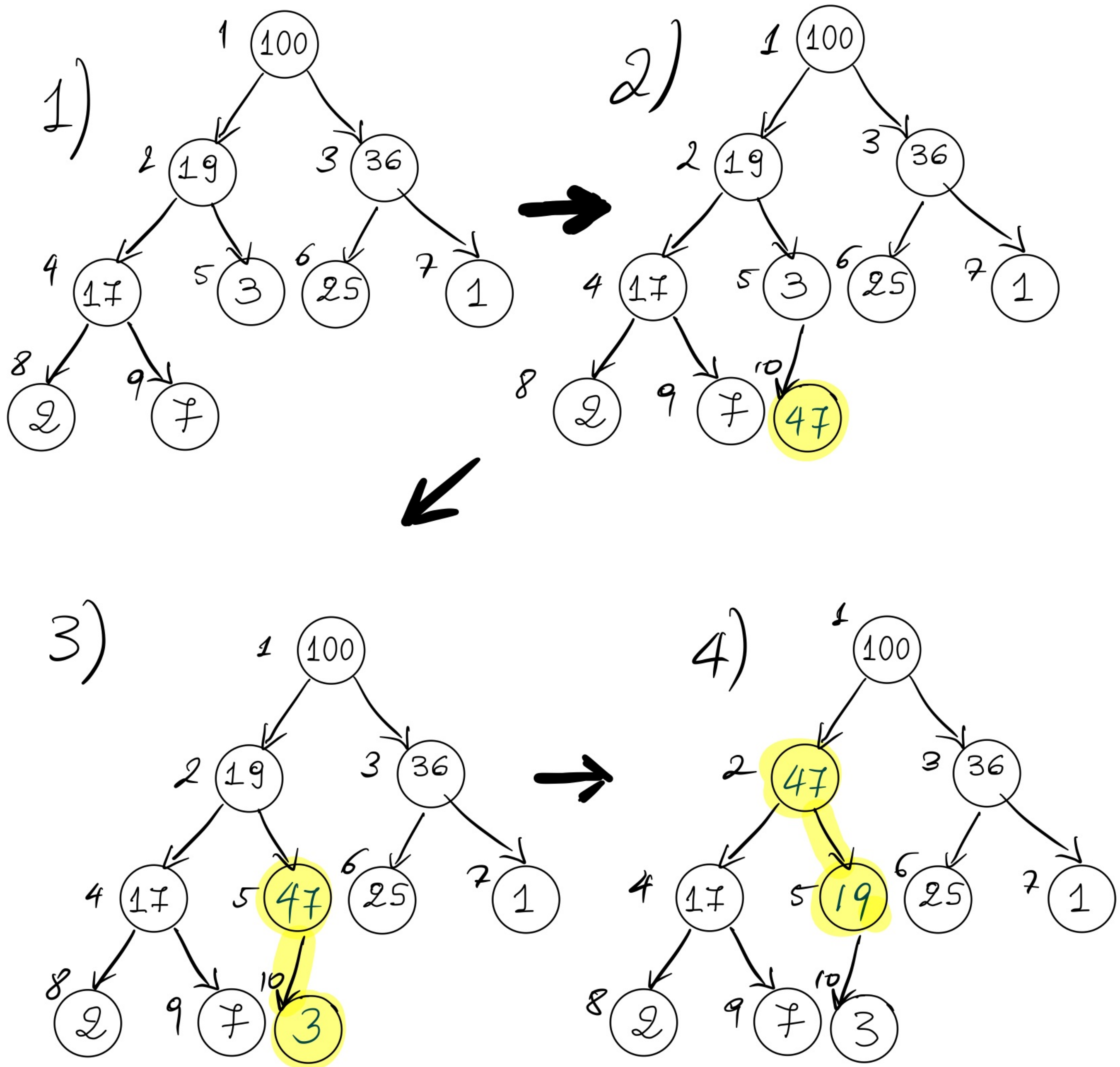


## Practice Time

- Insert 47 in this binary heap.

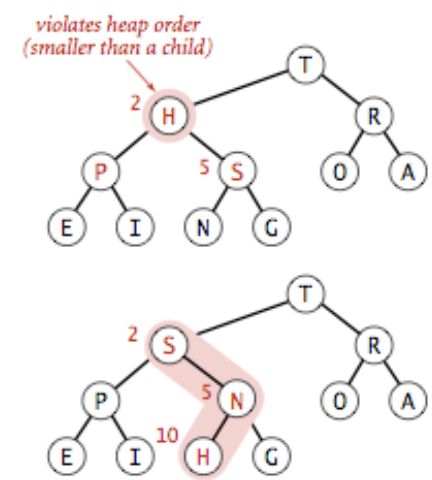


# Answer



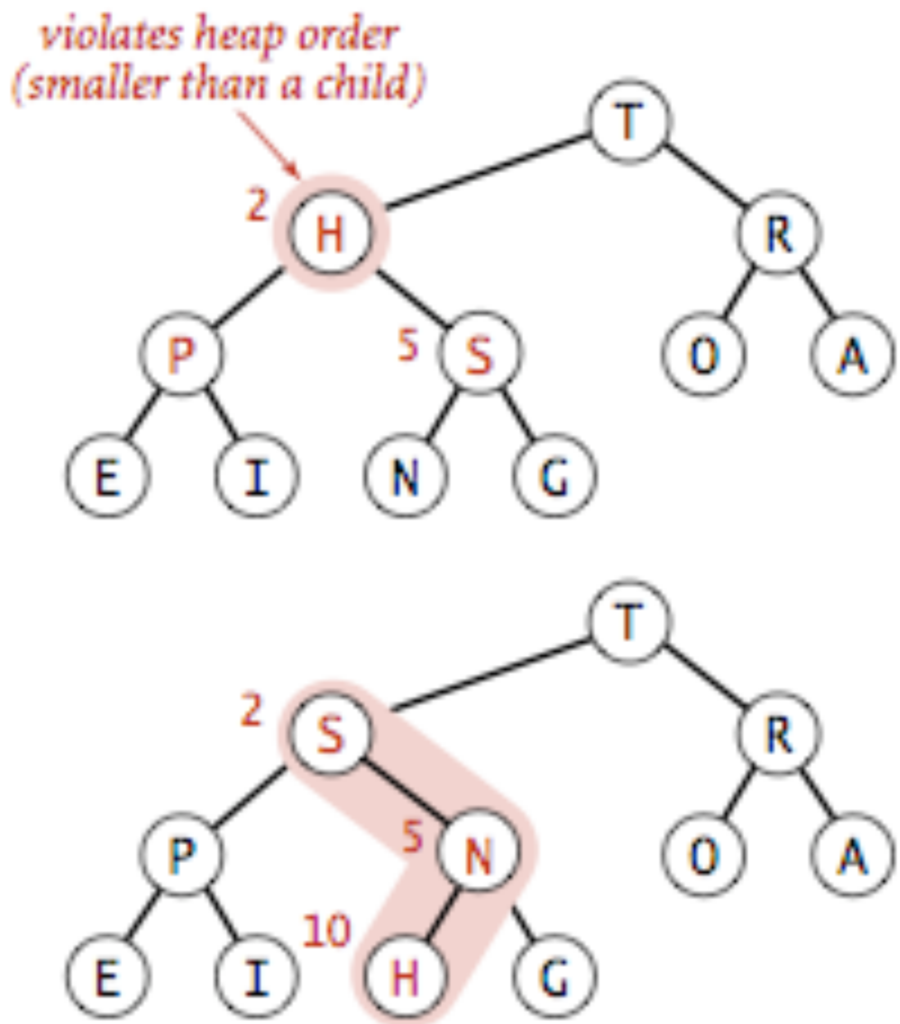
# Sink/demote/top down heapify

- Scenario: a key becomes smaller than one (or both) of its children's keys.
- To eliminate the violation:
  - Exchange key in parent with key in **larger** child.
  - Repeat until heap order is restored.



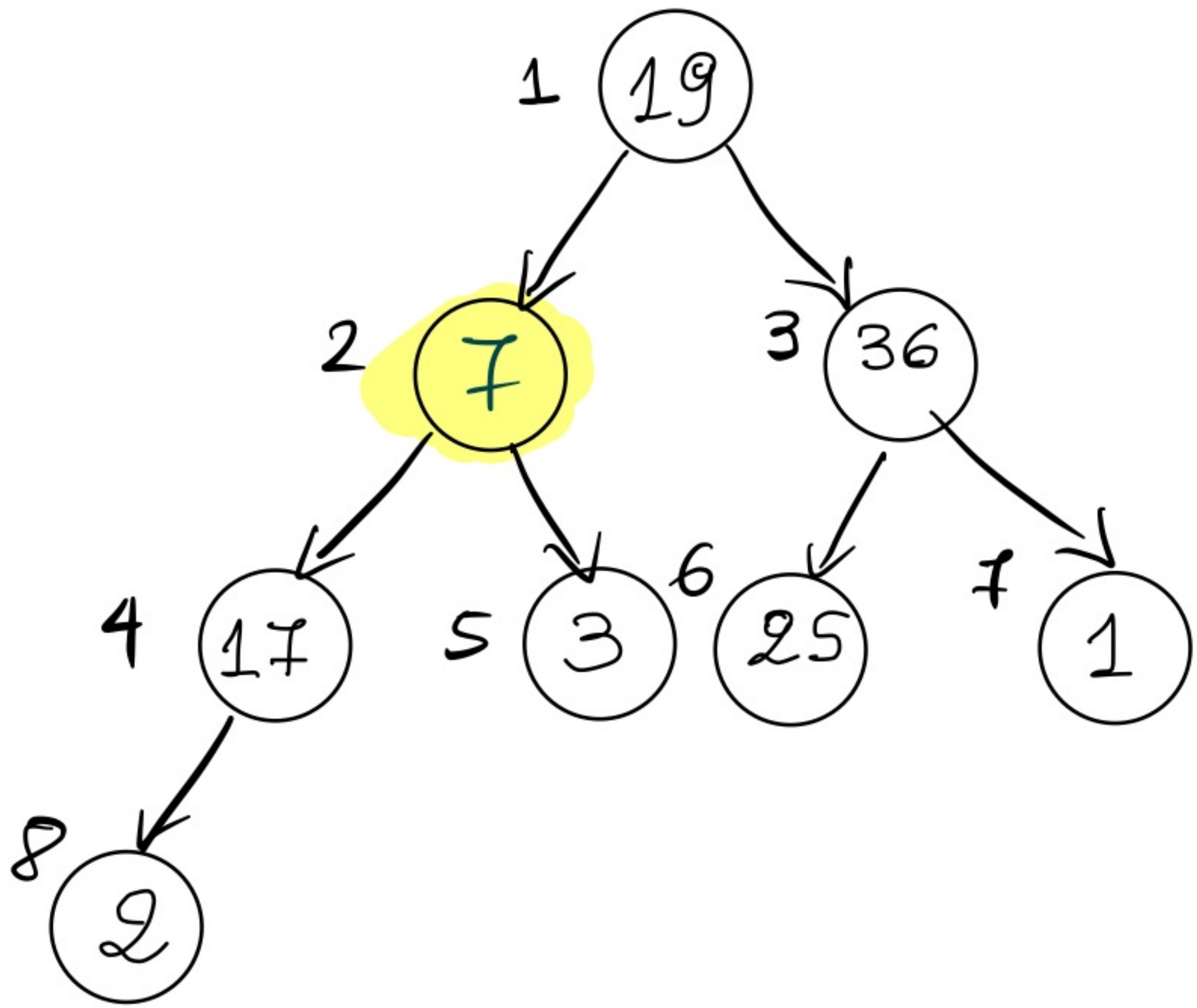
# Sink/demote/top down heapify

```
private void sink(int k) {
    while (2*k <= n) {
        int j = 2*k;
        if (j < n && less(j, j+1))
            j++;
        if (!less(k, j))
            break;
        exch(k, j);
        k = j;
    }
}
```

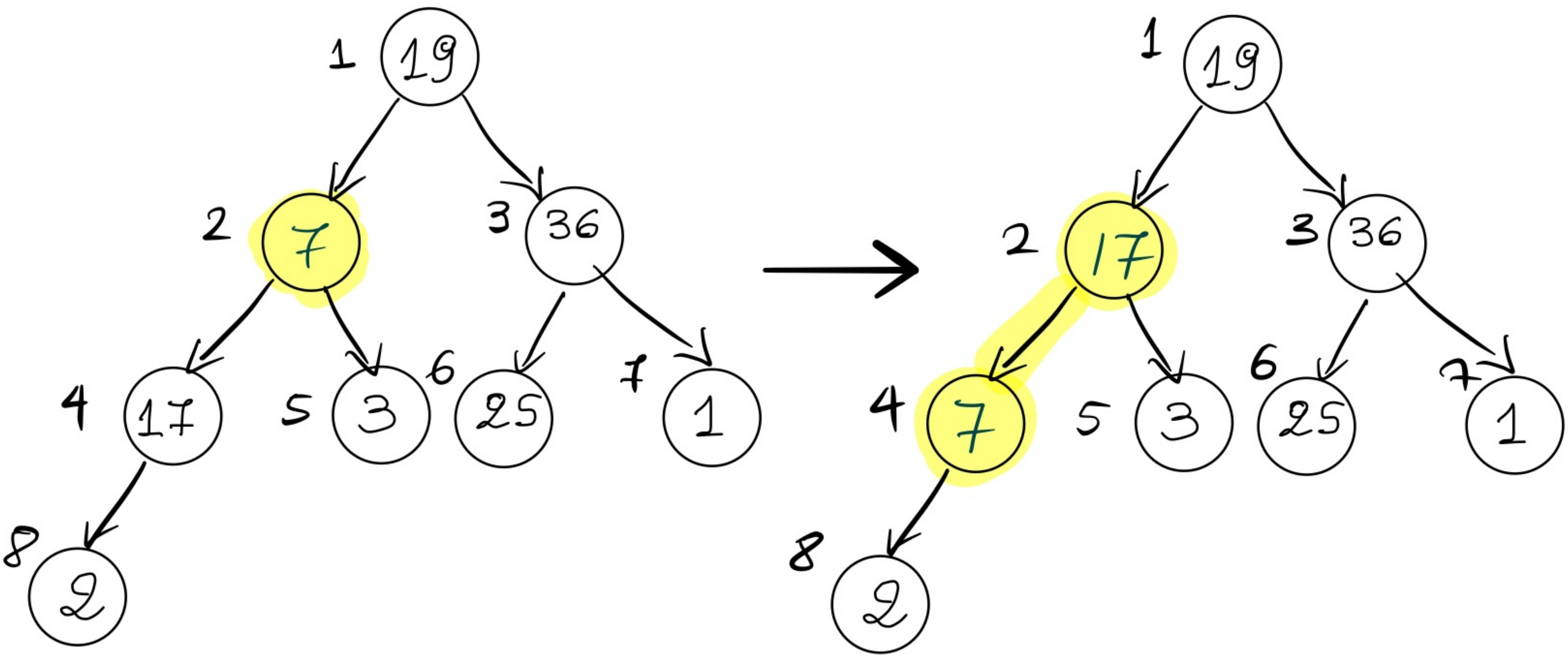


## Practice Time

- Sink 7 to its appropriate place in this binary heap.



# Answer

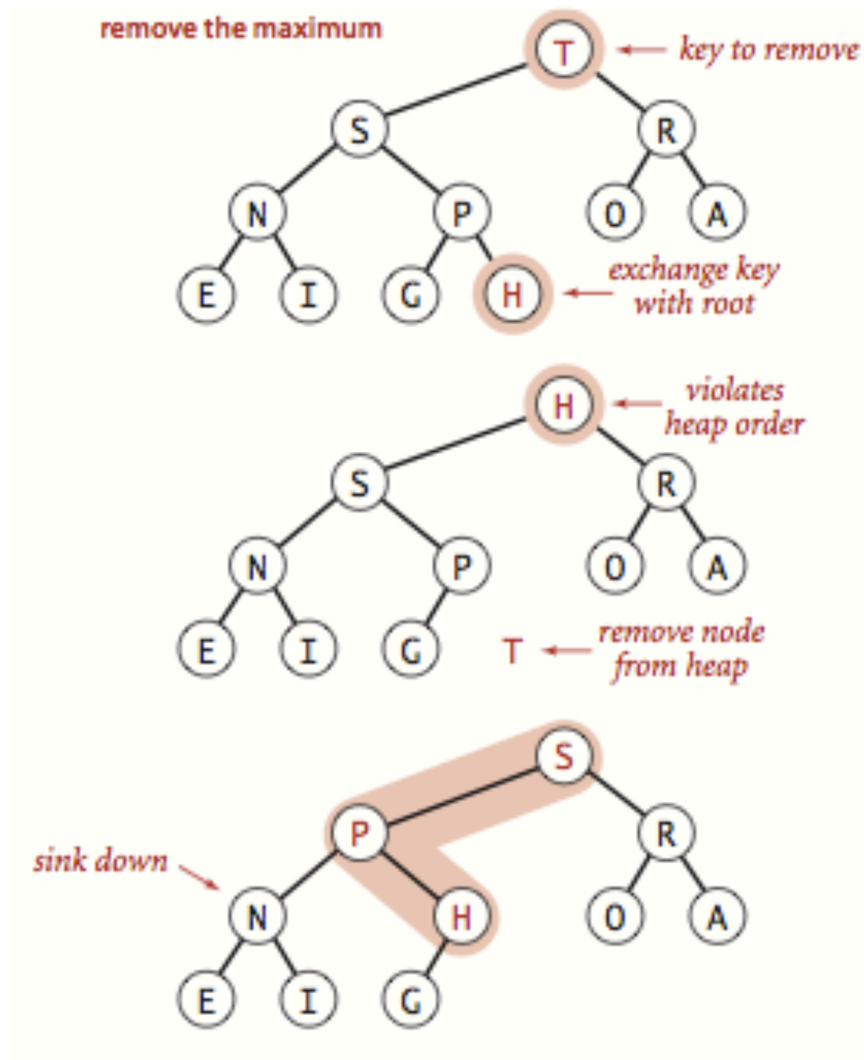


# Binary heap: return (and delete) the maximum

- Delete max: Exchange root with node at end. Return it and delete it. Sink the new root down.
- Cost: At most $2 \log n$ compares.

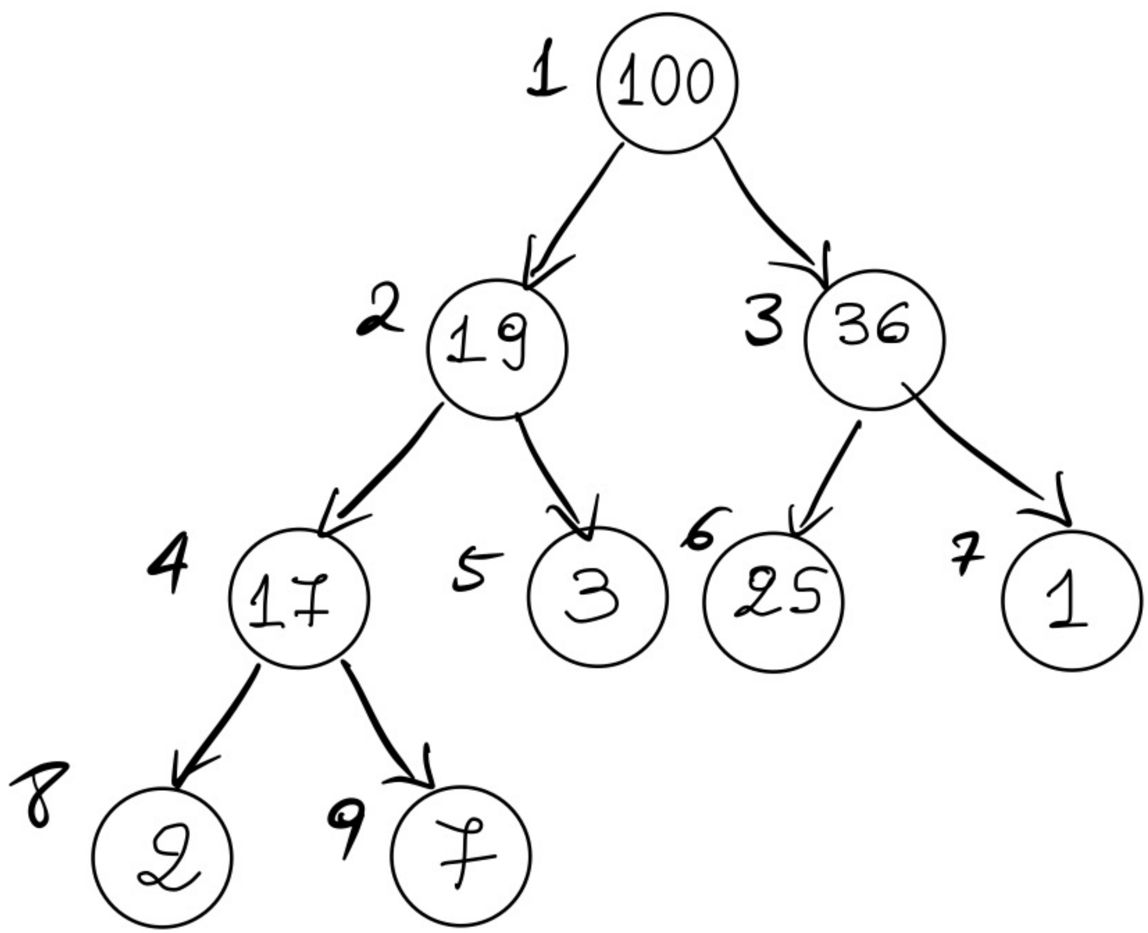
```
public Key delMax() {
    Key max = pq[1];
    exch(1, n--);
    sink(1);
    pq[n+1] = null;
    return max;
}
```

# Binary heap: delete and return maximum

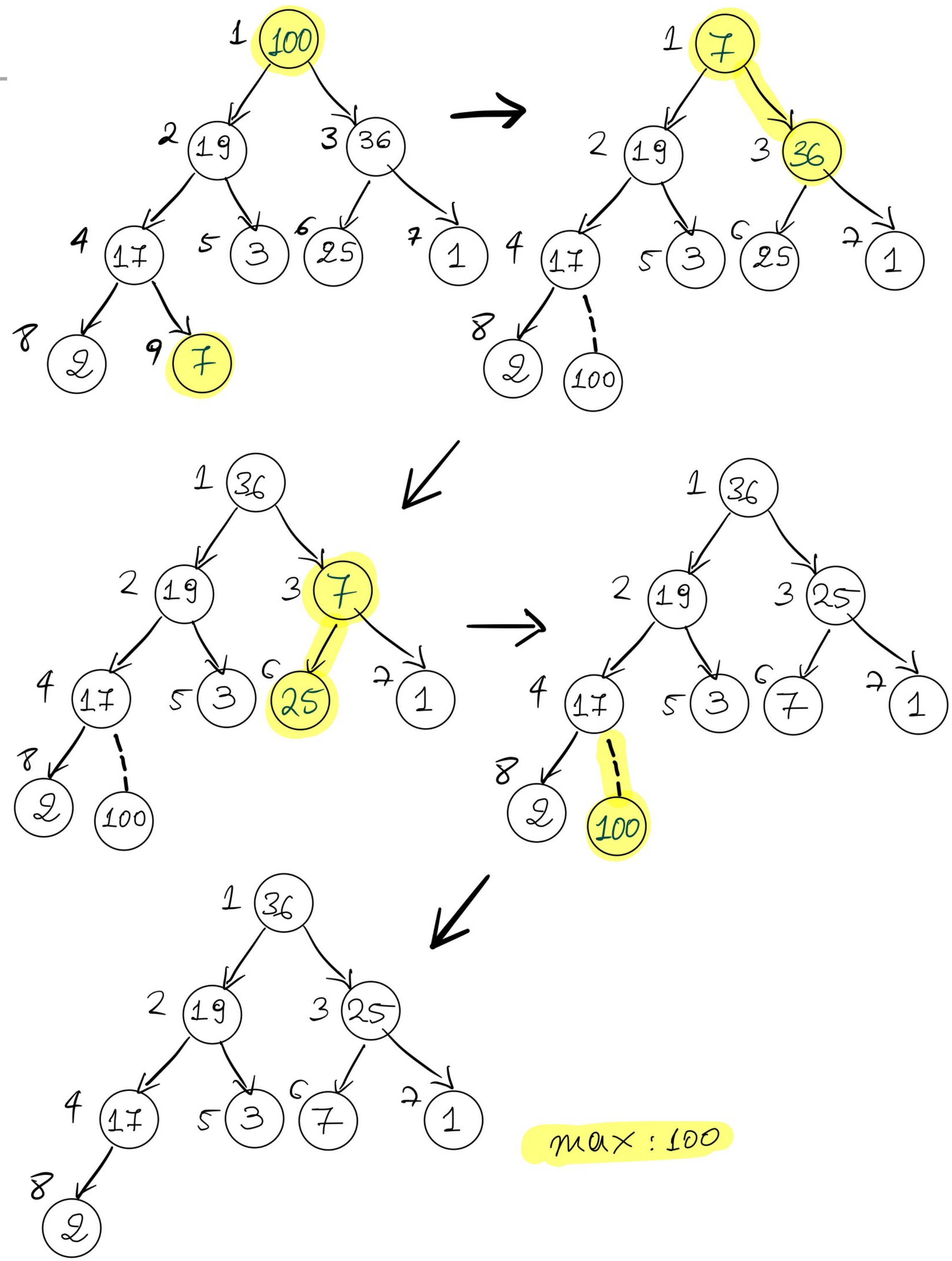


## Practice Time

- Delete max (and return it!)



# Answer



Things to remember about runtime complexity of heaps

- Insertion is $O(\log n)$.
- Delete max is $O(\log n)$.
- Space efficiency is $O(n)$.


http://algs4.cs.princeton.edu

# 2.4 BINARY HEAP DEMO

## Lecture 16: Binary Trees and Heaps

- Binary Trees
- Tree traversals
- Binary Heaps

# Readings:

- Textbook:
  - Chapter 2.4 (Pages 308-327)
- Website:
  - Priority Queues: https://algs4.cs.princeton.edu/24pq/
- Visualization:
  - Insert and ExtractMax: https://visualgo.net/en/heap

# Practice Problems:

- Practice with traversals of trees and insertions and deletions in binary heaps