GRAPHS: SHORTEST PATHS

David Kauchak
CS 62 – Spring 2021

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Admin

Last quiz!

Last assignment due next Friday (5/7)

Next week:

- Tuesday: balanced trees
- Wednesday: course feedback forms, ethics discussion, work session
- Thursday: recap/review

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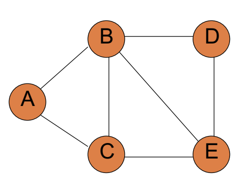
Shortest paths

What is the shortest path from a to d?



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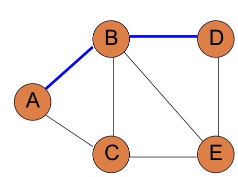
Shortest paths

How can we find this?



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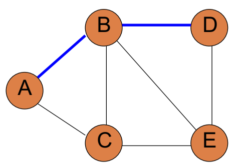
Shortest paths

BFS visits vertices in increasing distance!



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BFS with distances

Look at ShortestPaths.bfsDistances in GraphExamples

https://github.com/pomonacs622021sp/LectureCode/tree/master/GraphExamples

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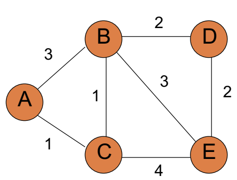
Shortest paths

What is the shortest path from a to d?



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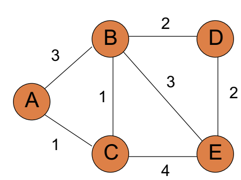
Shortest paths

We can still use BFS



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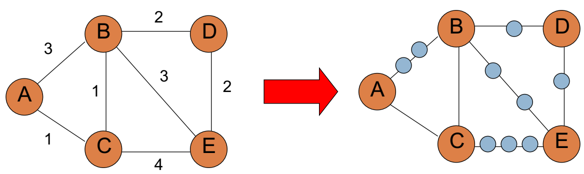
Shortest paths

We can still use BFS



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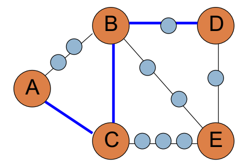
Shortest paths

We can still use BFS



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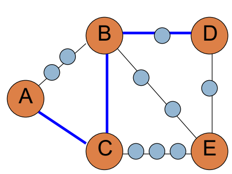
Shortest paths

What is the problem?



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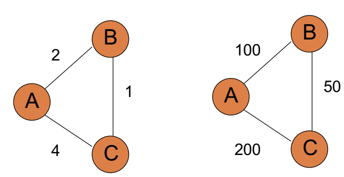
Shortest paths

Running time is dependent on the weights!



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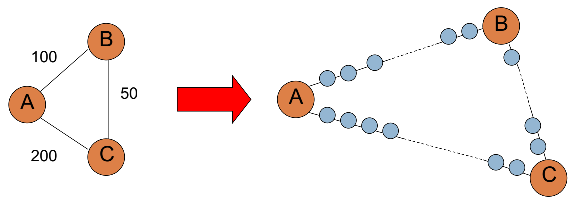
Shortest paths



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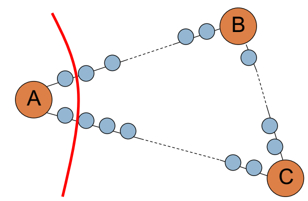
Shortest paths



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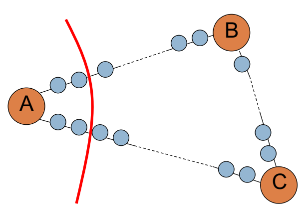
Shortest paths



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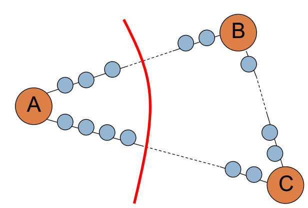
Shortest paths

Nothing will change as we expand the frontier until we've gone out 100 levels



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Key idea

Explore the vertices in order of increasing distance from the starting vertex

Keep track of the distances to each vertex

If we find a better path, update that distance

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Dijkstra's high-level

Explore the vertices in order of increasing distance from the starting vertex

Use a priority queue to keep track of the shortest path found so far to a vertex

```
Initialize: distance to start = 0 and all others infinity

repeat
    get vertex v with shortest distance

    for each vertex, adj, adjacent to v (edge exists v → adj)
        if path v → adj is shortest then best path for adj so far
            update the distance for adj
            update the priority queue
```

18

```
Initialize: distance to start = 0 and all others infinity

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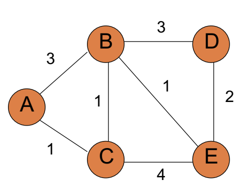

19

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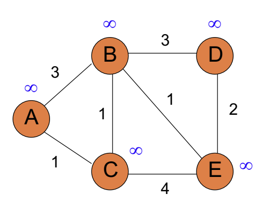

20

Initialize: distance to start = 0 and all others infinity

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    for each vertex, adj, adjacent to v (edge exists v ⟶ adj)
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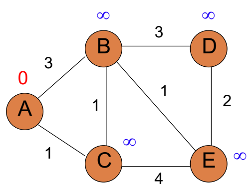
            update the priority queue

PQ

A 0
B ∞
C ∞
D ∞
E ∞



21

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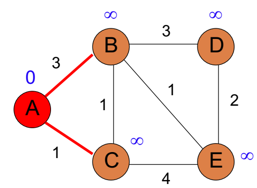
            update the priority queue

PQ

B ∞
C ∞
D ∞
E ∞



22

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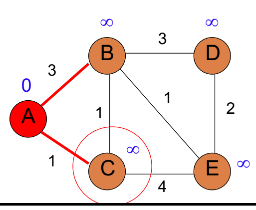
            update the priority queue

PQ

B ∞
C ∞
D ∞
E ∞



23

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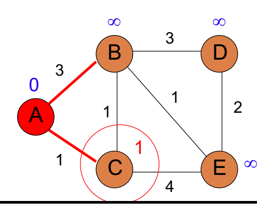
PQ

C 1
B ∞
D ∞
E ∞



24

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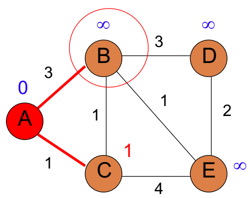
            update the priority queue

PQ

C 1
B ∞
D ∞
E ∞



25

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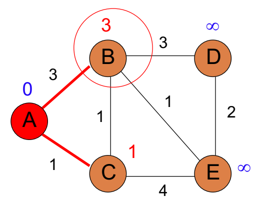
            update the priority queue

PQ

C 1
B 3
D ∞
E ∞



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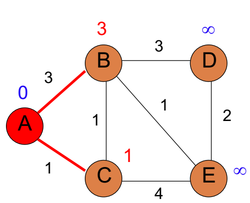
PQ

C 1
B 3
D ∞
E ∞



27

Initialize: distance to start = 0 and all others infinity

repeat
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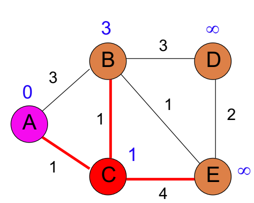
            update the priority queue

PQ

B 3
D ∞
E ∞



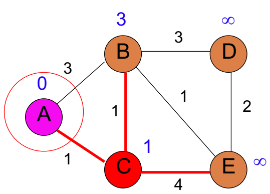
28

Initialize: distance to start = 0 and all others infinity

repeat
- get vertex v with shortest distance
- for each vertex, adj, adjacent to v (edge exists v ⟶ adj)
  - if path v ⟶ adj is shortest then best path for adj so far
    - update the distance for adj
    - update the priority queue

PQ

B 3
D ∞
E ∞



29

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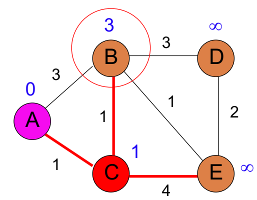
PQ

B 3
D ∞
E ∞



30

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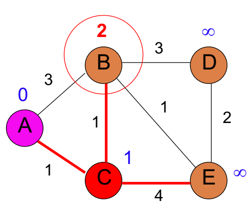
- for each vertex, adj, adjacent to v (edge exists v ⟶ adj)
  - if path v ⟶ adj is shortest then best path for adj so far
    - update the distance for adj
    - update the priority queue

PQ

B 2
D ∞
E ∞



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Initialize: distance to start = 0 and all others infinity

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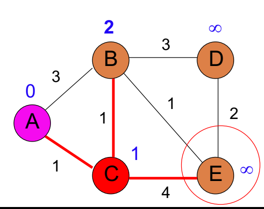
    - update the distance for adj
    - update the priority queue

PQ

B 2
D ∞
E ∞



32

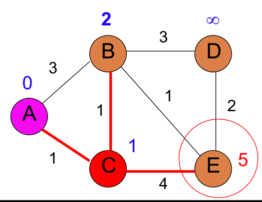
Initialize: distance to start = 0 and all others infinity

repeat
  get vertex v with shortest distance

  for each vertex, adj, adjacent to v (edge exists v ⟶ adj)
    if path v ⟶ adj is shortest then best path for adj so far
      update the distance for adj
      update the priority queue

PQ

B 2
E 5
D ∞



33

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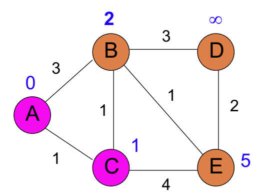
PQ

B 2
E 5
D ∞



Frontier?

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Initialize: distance to start = 0 and all others infinity

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  get vertex v with shortest distance

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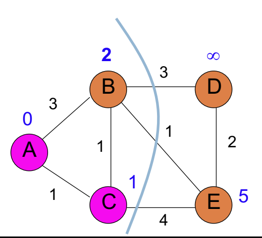
      update the priority queue

PQ

B 2
E 5
D ∞



All nodes reachable from starting node within a given distance

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Initialize: distance to start = 0 and all others infinity

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    if path v ⟶ adj is shortest then best path for adj so far
      update the distance for adj
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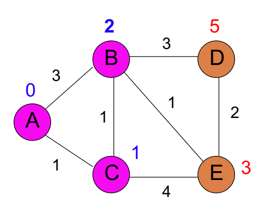
PQ

E 3
D 5



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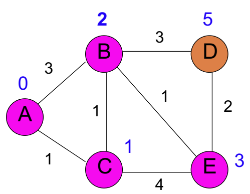
Initialize: distance to start = 0 and all others infinity

repeat
    get vertex v with shortest distance

    for each vertex, adj, adjacent to v (edge exists v ⟶ adj)
        if path v ⟶ adj is shortest then best path for adj so far
            update the distance for adj
            update the priority queue

PQ

D 5



37

Initialize: distance to start = 0 and all others infinity

repeat
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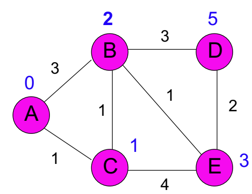
PQ



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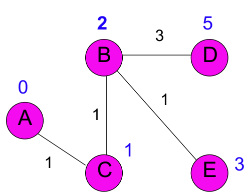
PQ



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Dijkstra's algorithm

```
public static void dijkstra(WeightedGraph g, int start) {
    IndexMinPQ<Double> pq = new IndexMinPQ<Double>(g.numberOfVertices());
    int[] edgeTo = new int[g.numberOfVertices()];
    double[] distTo = new double[g.numberOfVertices()];

    for( int v = 0; v < g.numberOfVertices(); v++ ) {
        distTo[v] = Double.POSITIVE_INFINITY;
        pq.insert(v, Double.POSITIVE_INFINITY);
    }

    distTo[start] = 0.0;
    pq.decreaseKey(start, 0.0);

    // relax vertices in order of distance from s
    while( !pq.isEmpty() ) {
        int v = pq.delMin();

        for (WeightedEdge e : g.adj(v)) {
            int adj = e.to();

            if( distTo[v] + e.weight() < distTo[adj] ) {
                distTo[adj] = distTo[v] + e.weight();
                edgeTo[adj] = v;
                pq.decreaseKey(adj, distTo[adj]);
            }
        }
    }
}
```

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Dijkstra's algorithm

Dijkstra's

```
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while( !pq.isEmpty() ) {
    int v = pq.delMin();

    for (WeightedEdge e : g.adj(v)) {
        int adj = e.to();

        if( distTo[v] + e.weight() < distTo[adj] ) {
            distTo[adj] = distTo[v] + e.weight();
            edgeTo[adj] = v;
            pq.decreaseKey(adj, distTo[adj]);
        }
    }
}
```

BFS

```
q.addLast(start);
visited[start] = true;
distTo[start] = 0;

while( !q.isEmpty() ) {
    int v = q.removeFirst();

    for( int adj: g.adj(v) ) {
        if( !visited[adj] ) {
            visited[adj] = true;
            edgeTo[adj] = v;
            distTo[adj] = distTo[v] + 1;
            q.addLast(adj);
        }
    }
}
```

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Dijkstra example

Look at ShortestPaths.dijkstra in GraphExamples

https://github.com/pomonacs622021sp/LectureCode/tree/master/GraphExamples

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Why does it work?

When a vertex is removed from the priority queue, distTo[v] is the actual shortest distance from s to v

- The only time a vertex gets removed is when the distance from s to that vertex is smaller than the distance to any remaining vertex
- Therefore, there cannot be any other path that hasn't been visited already that would result in a shorter path

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Why does it work?

When a vertex is removed from the priority queue, distTo[v] is the actual shortest distance from s to v

- The only time a vertex gets removed is when the distance from s to that vertex is smaller than the distance to any remaining vertex
- Therefore, there cannot be any other path that hasn't been visited already that would result in a shorter path

Does this make any assumptions?

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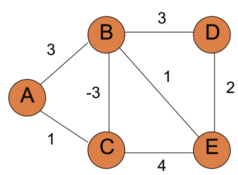
What about this graph?

What's the shortest path from A to C?
What would Dijkstra's do?



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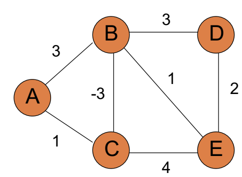
What about this graph?

Dijkstra's only works on graphs with positive edge weights



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Why does it work?

When a vertex is removed from the priority queue, distTo[v] is the actual shortest distance from s to v

- The only time a vertex gets removed is when the distance from s to that vertex is smaller than the distance to any remaining vertex
- Therefore, there cannot be any other path that hasn't been visited already that would result in a shorter path

Assuming no negative edge weights!

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Relaxing an edge

This update is called "relaxing" an edge

```
if( distTo[v] + e.weight() < distTo[adj] ) {
    distTo[adj] = distTo[v] + e.weight();
    edgeTo[adj] = v;
    pq.decreaseKey(adj, distTo[adj]);
}
```

We can apply this to an edge as many times as we want

This idea is used in other shortest paths algorithms (e.g., Bellman-Ford)

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```java
public static void fasterDijkstra(WeightedGraph g, int start) {
    IndexMinPQ<Double> pq = new IndexMinPQ<Double>(g.numberOfVertices());
    int[] edgeTo = new int[g.numberOfVertices()];
    double[] distTo = new double[g.numberOfVertices()];

    for( int v = 0; v < g.numberOfVertices(); v++ ) {
        distTo[v] = Double.POSITIVE_INFINITY;          don't insert everything into pq
    }

    distTo[start] = 0.0;
    pq.insert(start, 0.0);                             only insert starting vertex

    while( !pq.isEmpty() ) {
        int v = pq.delMin();

        for (WeightedEdge e : g.adj(v)) {
            int adj = e.to();

            if( distTo[v] + e.weight() < distTo[adj] ) {
                distTo[adj] = distTo[v] + e.weight();
                edgeTo[adj] = v;

                if( pq.contains(adj) ) {
                    pq.decreaseKey(adj, distTo[adj]);
                } else {
                    pq.insert(adj, distTo[adj]);       insert when we discover a vertex
                }
            }
        }
    }
}
```

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Run-time

```java
public static void dijkstra(WeightedGraph g, int start) {
    IndexMinPQ<Double> pq = new IndexMinPQ<Double>(g.numberOfVertices());
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    for( int v = 0; v < g.numberOfVertices(); v++ ) {
        distTo[v] = Double.POSITIVE_INFINITY;
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            int adj = e.to();

            if( distTo[v] + e.weight() < distTo[adj] ) {
                distTo[adj] = distTo[v] + e.weight();
                edgeTo[adj] = v;
                pq.decreaseKey(adj, distTo[adj]);
            }
        }
    }
}
```

V calls

E calls

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Running time?

Depends on the heap implementation

	V * delMin	E * decreaseKey	Total
Array	O(|V|^2)	O(|E|)	O(|V|^2)
Bin heap	O(|V| log |V|)	O(|E| log |V|)	O((|V|+|E|) log |V|) O(|E| log |V|)

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Running time?

Depends on the heap implementation

	V * delMin	E * decreaseKey	Total
Array	O(|V|^2)	O(|E|)	O(|V|^2)
Bin heap	O(|V| log |V|)	O(|E| log |V|)	O((|V|+|E|) log |V|) O(|E| log |V|)
Fib heap	O(|V| log |V|)	O(|E|)	O(|V| log |V| + |E|)

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Shortest paths

Dijkstra's: single source shortest paths for positive edge weight graphs

What is single source?

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Shortest paths

Dijkstra's: single source shortest paths for positive edge weight graphs

Many other variants:

- graphs with negative edges
- all pairs shortest paths
- …

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