CS062

DATA STRUCTURES AND ADVANCED PROGRAMMING

21: Binary Trees and Heaps


David Kauchak


Alexandra Papoutsaki

Basic data structures

- Arrays,
- Resizing arrays or arraylists,
- Linked Lists,
- Queues, and
- Stacks.
- Runtime and memory analysis for each one.

Sorting

- Selection sort,
- Insertion sort,
- Mergesort, and
- Quicksort.
- Runtime (comparisons and exchanges), stability, in-place for each one.
- Comparators: How to sort a data structure with objects of any class.
- Iterators: How to traverse a data structure.

Lecture 21: Binary Trees and Heaps

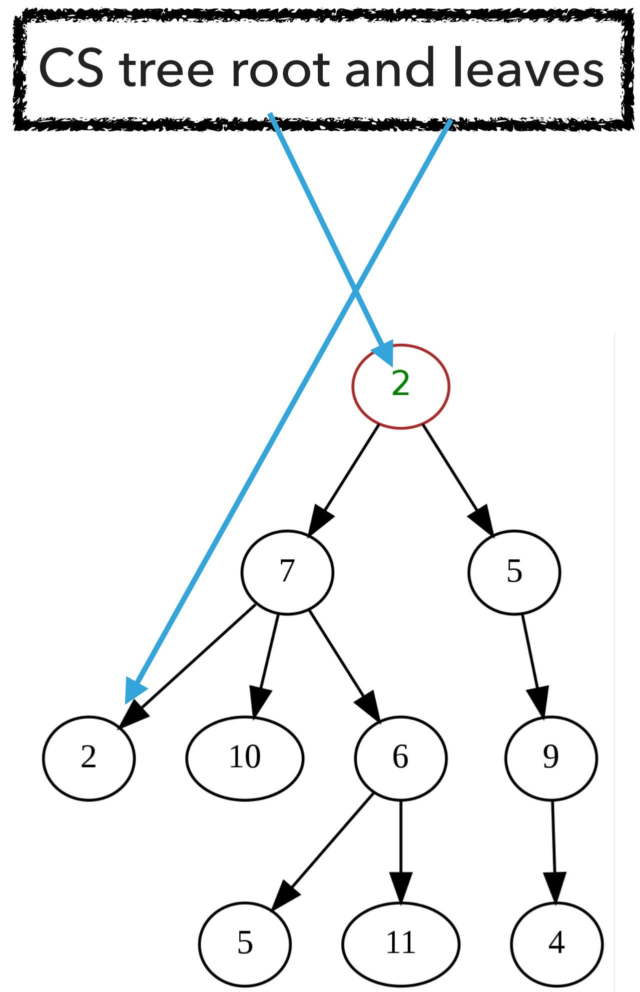
- Binary Trees
- Tree traversals
- Binary Heaps

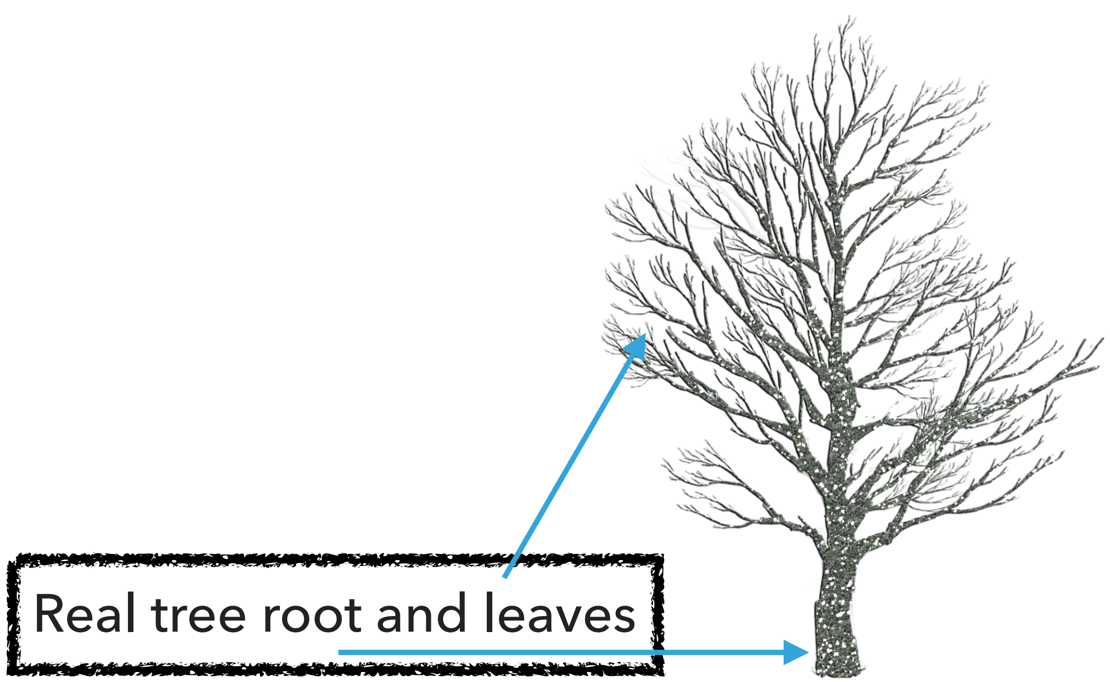
Trees in Computer Science

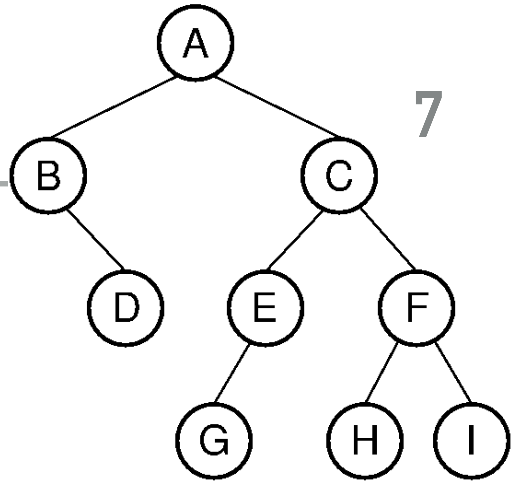
- Abstract data types that store elements hierarchically rather than linearly.
- Examples of hierarchical structures:
  - Organization charts for
    - Companies (CEO at the top followed by CFO, CMO, COO, CTO, etc).
    - Universities (Board of Trustees at the top, followed by President, then by VPs, etc).
  - Sitemaps (home page links to About, Products, etc. They link to other pages).
  - Computer file systems (user at top followed by Documents, Downloads, Music, etc. Each folder can hold more folders.).

Trees in Computer Science

- Hierarchical: Each element in a tree has a **single** parent (immediate ancestor) and zero or more children (immediate descendants).

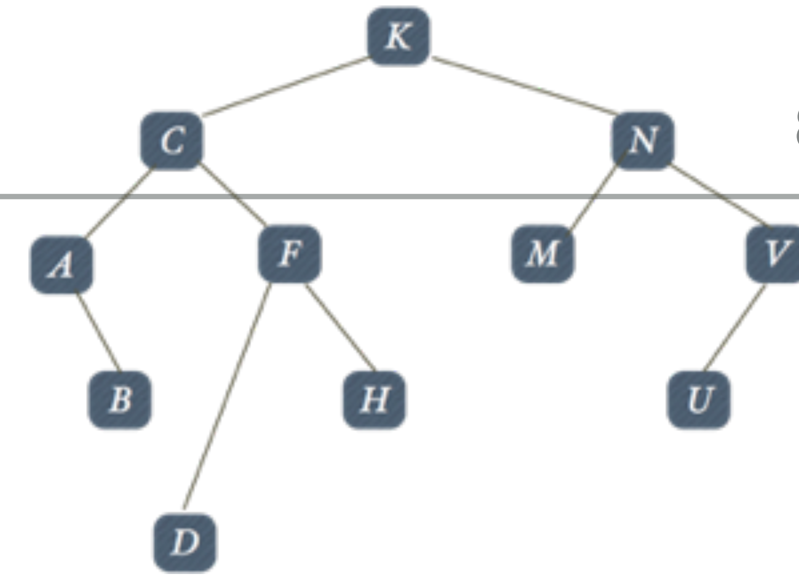






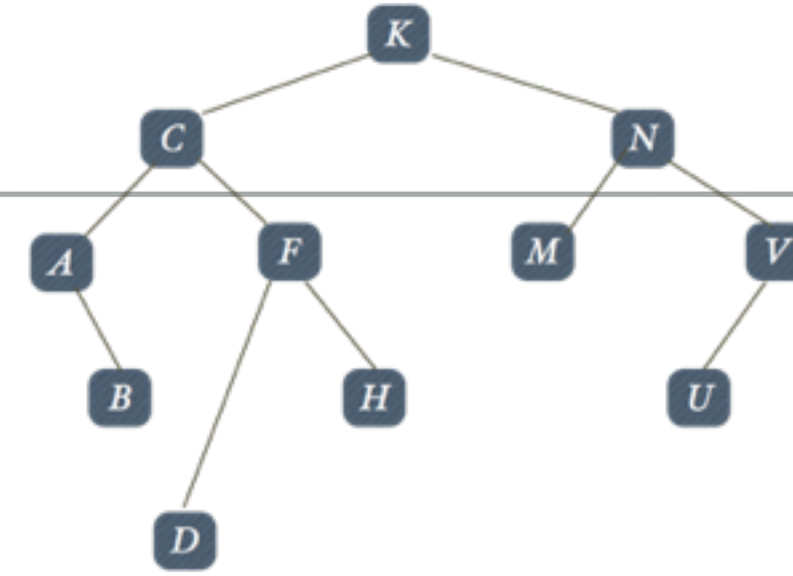
Definition of a tree

- A tree T is a set of nodes that store elements based on a parent-child relationship:
  - If T is non-empty, it has a node called the root of T, that has no parent.
    - Here, the root is A.
  - Each node v, other than the root, has a unique parent node u. Every node with parent u is a child of u.
    - E.g., E's parent is C and F has two children, H and I.



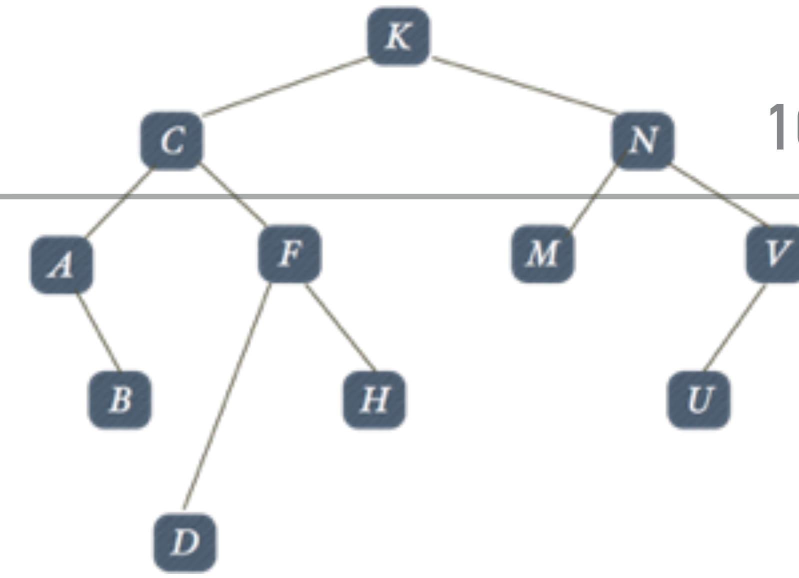
Tree Terminology

- Edge: a pair of nodes s.t. one is the parent of the other, e.g., (K,C).
- Parent node is directly above child node, e.g., K is parent of C and N.
- Sibling nodes have same parent, e.g., A and F.
- K is ancestor of B.
- B is descendant of K.
- Node plus all descendants gives subtree.
- Nodes without descendants are called leaves or external. The rest are called internal.
- A set of trees is called a forest.



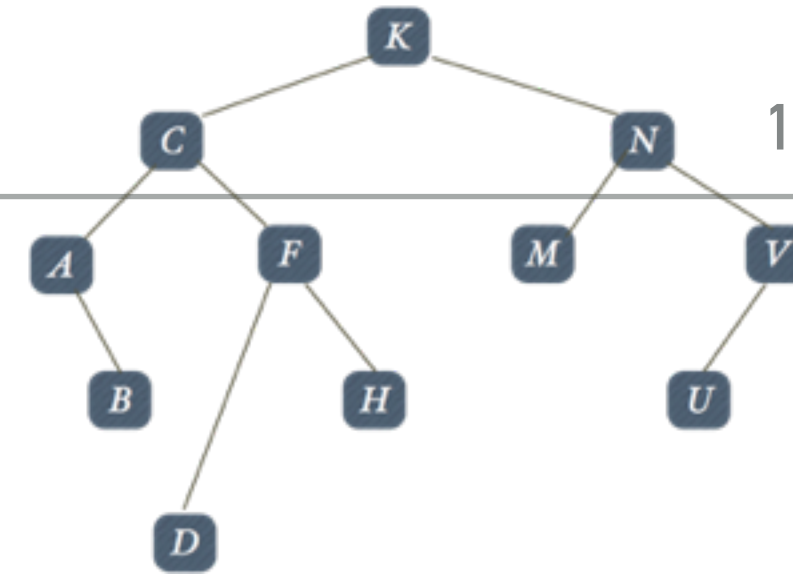
More Terminology

- Simple path: a series of distinct nodes s.t. there are edges between successive nodes, e.g., K-N-V-U.
- Path length: number of edges in path, e.g., path K-C-A has length 2.
- Height of node: length of longest path from the node to a leaf.
- Height of tree: length of longest path from the root to a leaf.
- Degree of node: number of its children.
- Degree of tree (arity): max degree of any of its nodes.
- Binary tree: a tree with arity of 2.



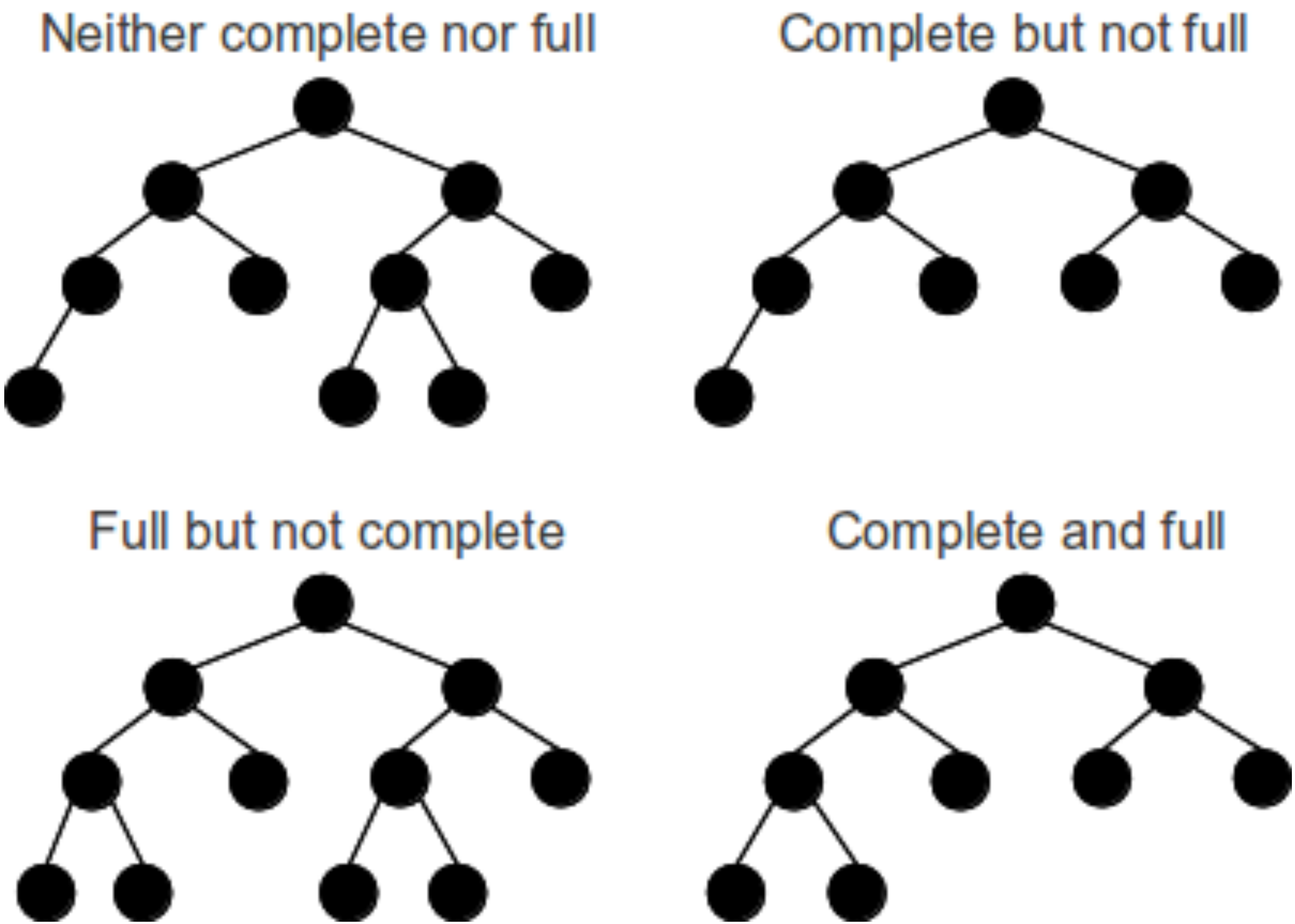
Even More Terminology

- Level/depth of node defined recursively:
  - Root is at level 0.
  - Level of any other node is equal to level of parent + 1.
  - It is also known as the length of path from root or number of ancestors excluding itself.
- Height of node defined recursively:
  - If leaf, height is 0.
  - Else, height is max height of child + 1.



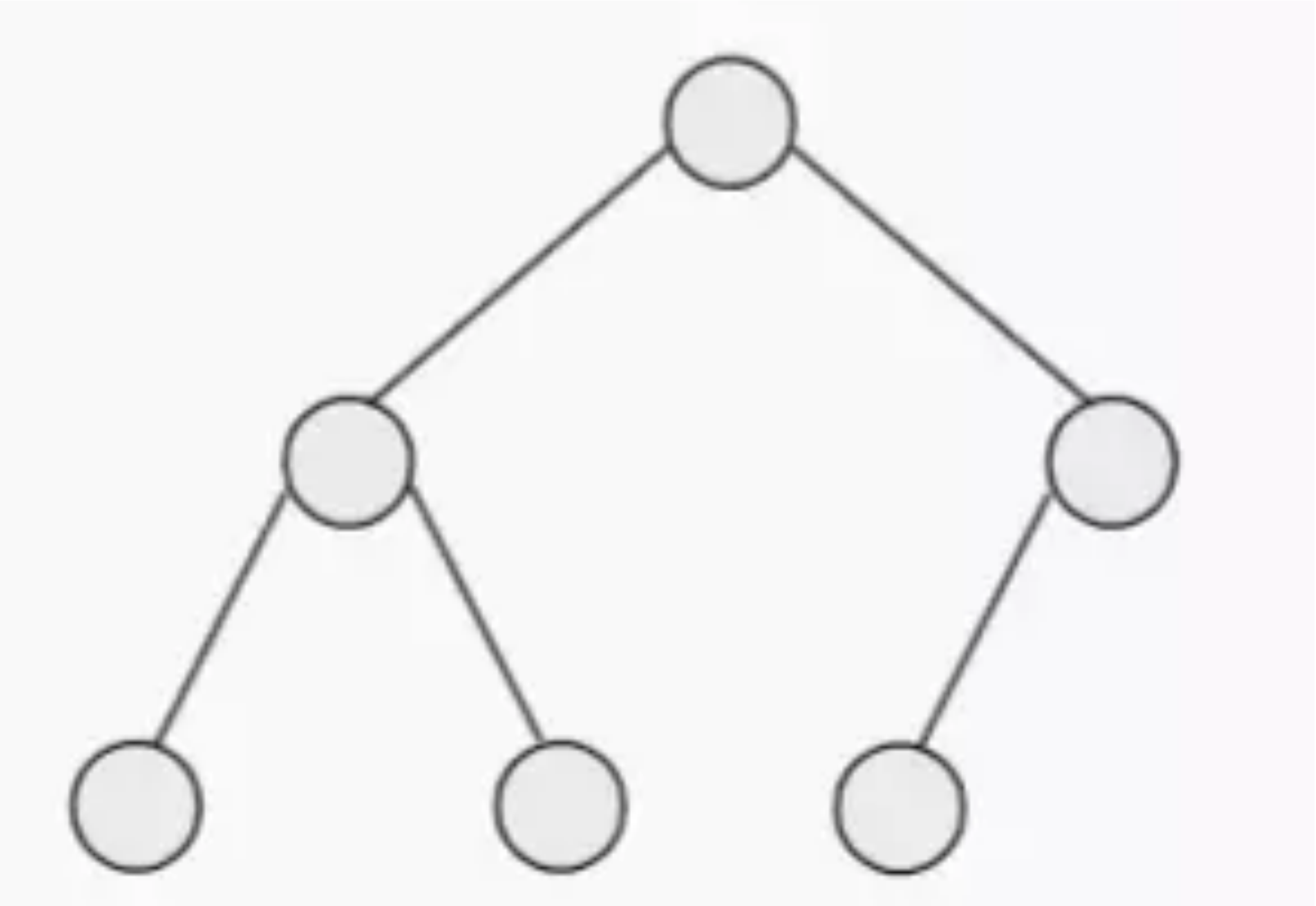
But wait there’s more!

- Full (or proper): a binary tree whose every node has 0 or 2 children.
- Complete: a binary tree with minimal height. Any holes in tree would appear at last level to right, i.e., all nodes of last level are as left as possible.



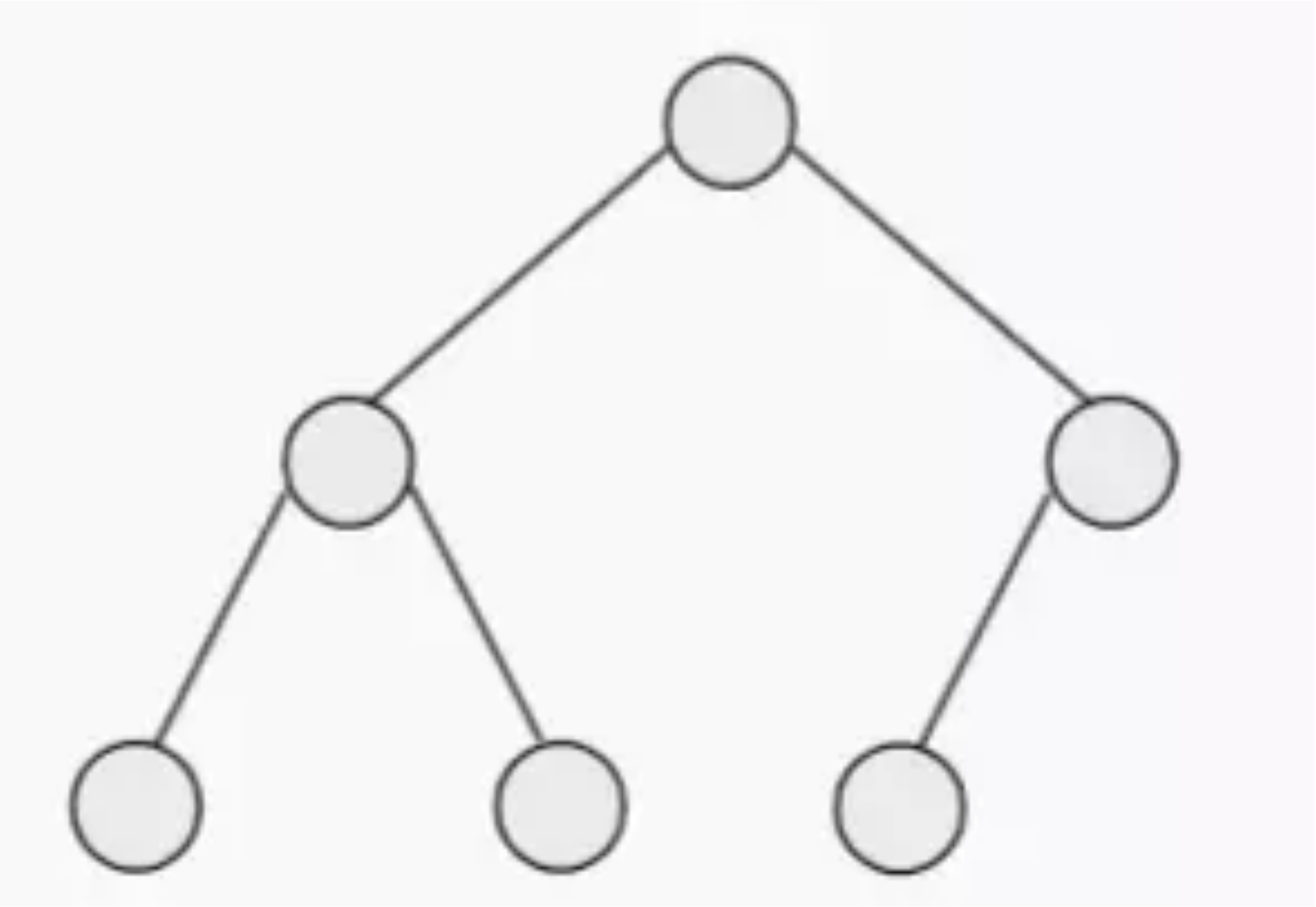
http://code.cloudkaksha.org/binary-tree/types-binary-tree

Practice Time: This tree is



- A: Full
- B: Complete
- C: Full and Complete
- D: Neither Full nor Complete

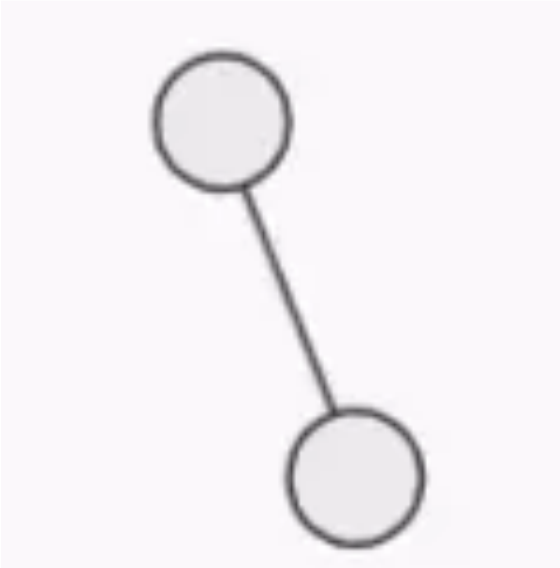
Answer



- A: Full
- **B: Complete**
- C: Full and Complete
- D: Neither Full nor Complete

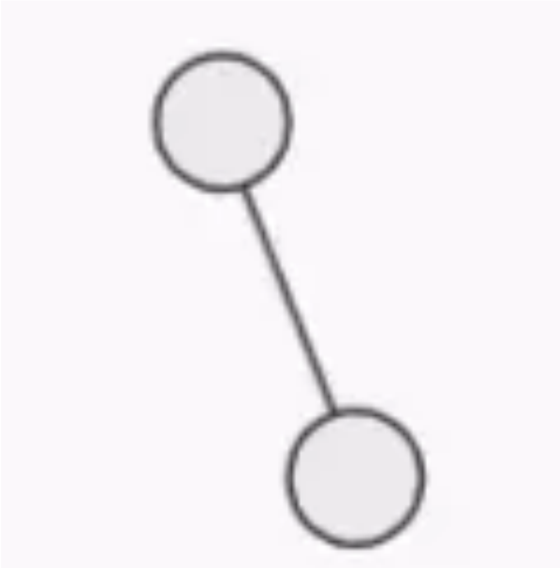
Practice Time: This tree is

- A: Full
- B: Complete
- C: Full and Complete
- D: Neither Full nor Complete

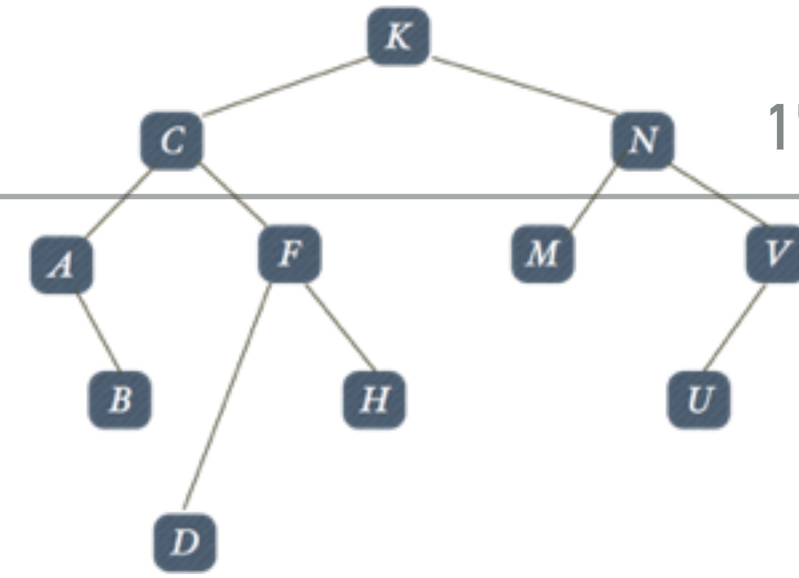


Answer

- A: Full
- B: Complete
- C: Full and Complete
- **D: Neither Full nor Complete**



Counting in binary trees



- Lemma: if T is a binary tree, then at level k, T has $\leq 2^k$ nodes.
    - E.g., at level 2, at most 4 nodes (A, F, M, V)
- Theorem: If T has height h, then # of nodes n in T satisfy: $h + 1 \leq n \leq 2^{h+1} - 1$.
- Equivalently, if T has n nodes, then $log(n+1) - 1 \leq h \leq n - 1$.
    - Worst case: When $h = n - 1$ or $O(n)$, the tree looks like a left or right-leaning "stick".
    - Best case: When a tree is as compact as possible (e.g., complete) it has $O(\log n)$ height.

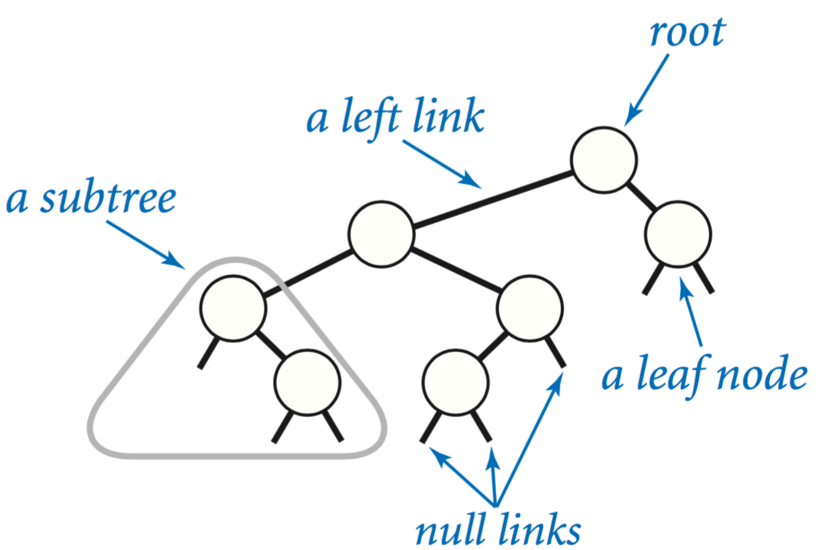
Basic idea behind a simple implementation

```
public class BinaryTree<Item> {
    private Node root;

    /**
     * A node subclass which contains various recursive methods
     *
     * @param <Item>  The type of the contents of nodes
     */
    private class Node {
        private Item item;

        private Node left;
        private Node right;

        /**
         * Node constructor with subtrees
         *
         * @param left   the left node child
         * @param right      the right node child
         * @param item   the item contained in the node
         */
        public Node(Node left, Node right, Item item) {
            this.left = left;
            this.right = right;
            this.item = item;
        }
```

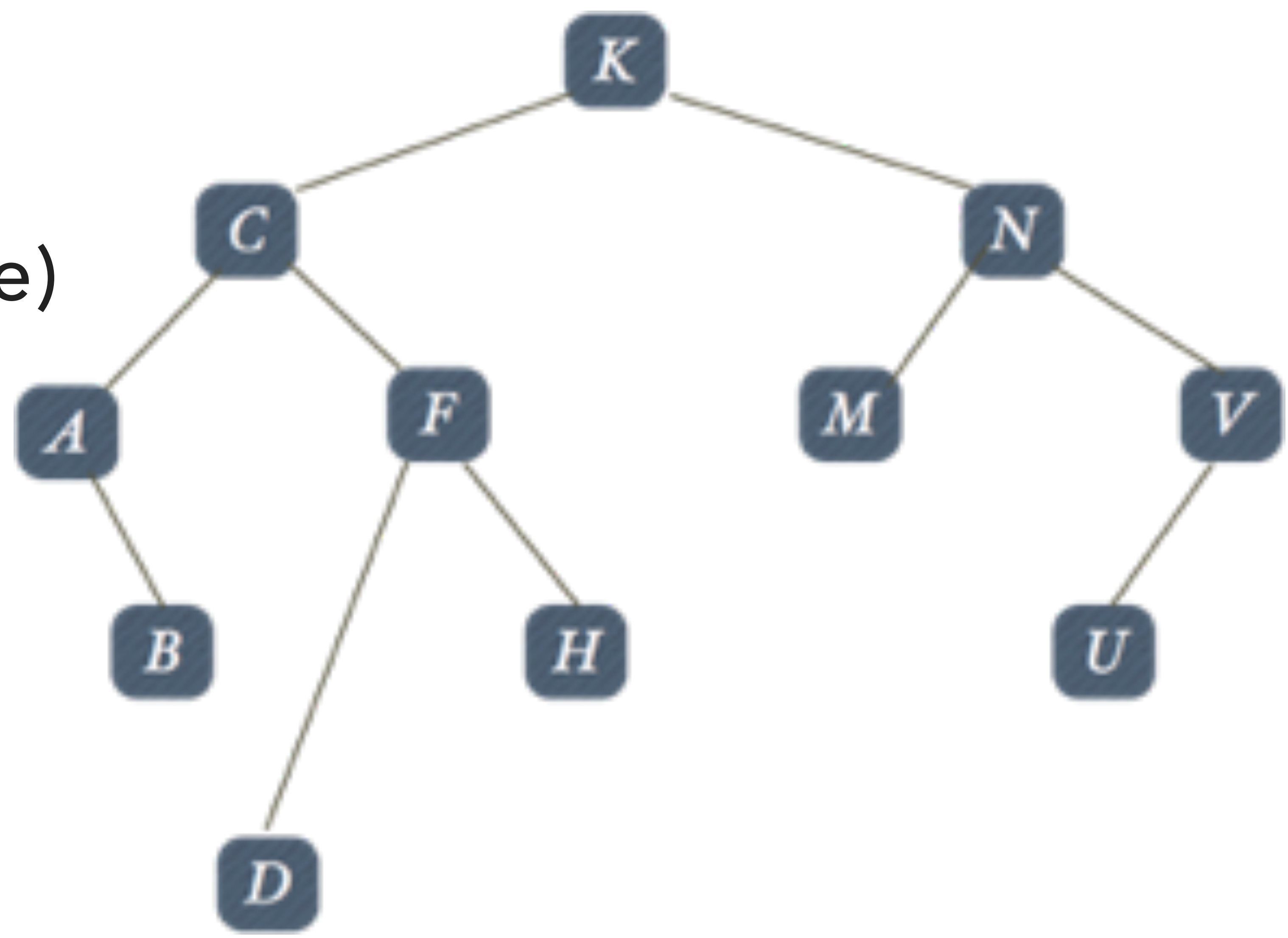


Lecture 21: Binary Trees and Heaps

- Binary Trees
- Tree traversals
- Binary Heaps

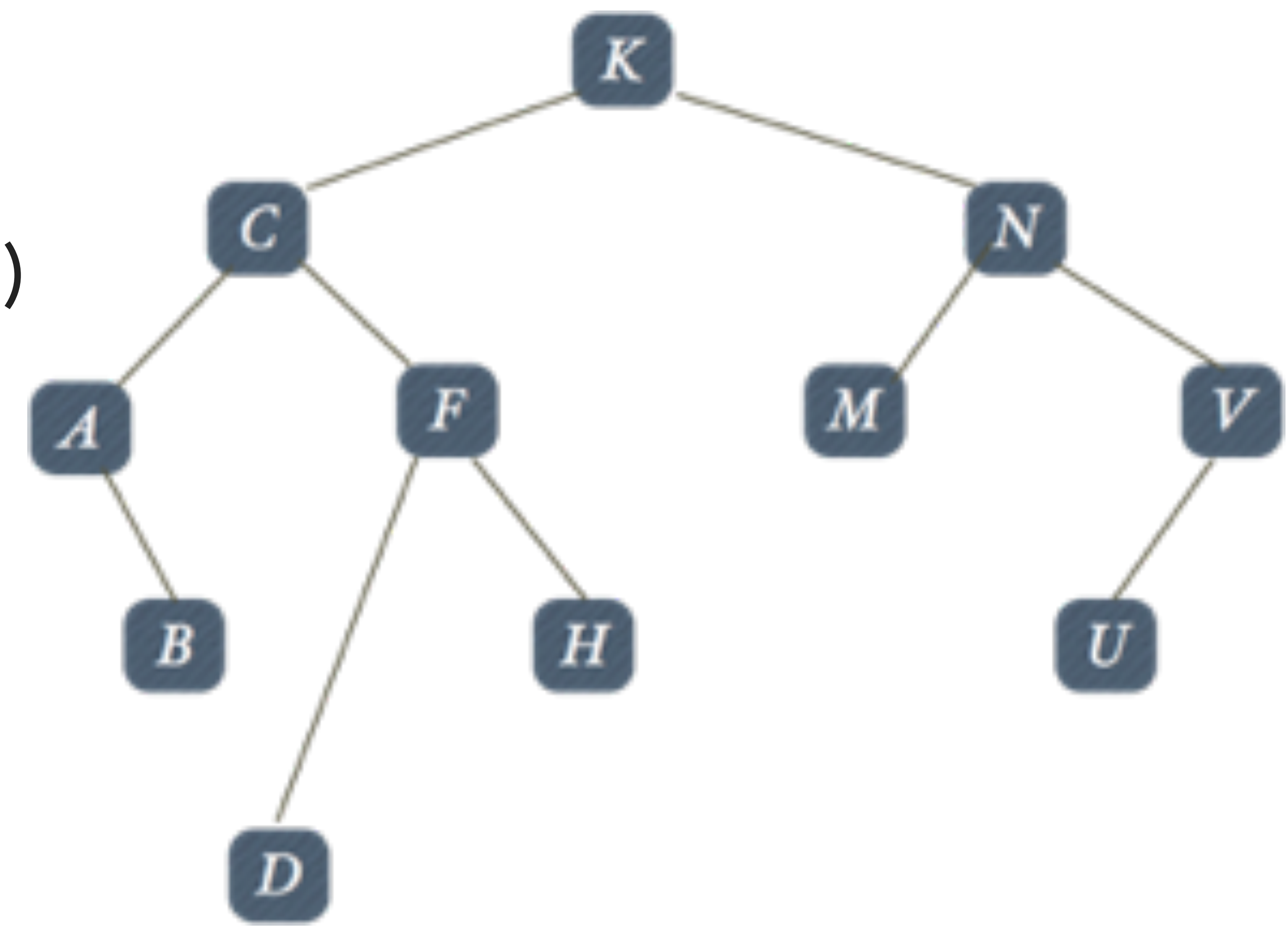
Pre-order traversal

- Preorder(Tree)
  - Mark root as visited
  - Preorder(Left Subtree)
  - Preorder(Right Subtree)
- K C A B F D H N M V U



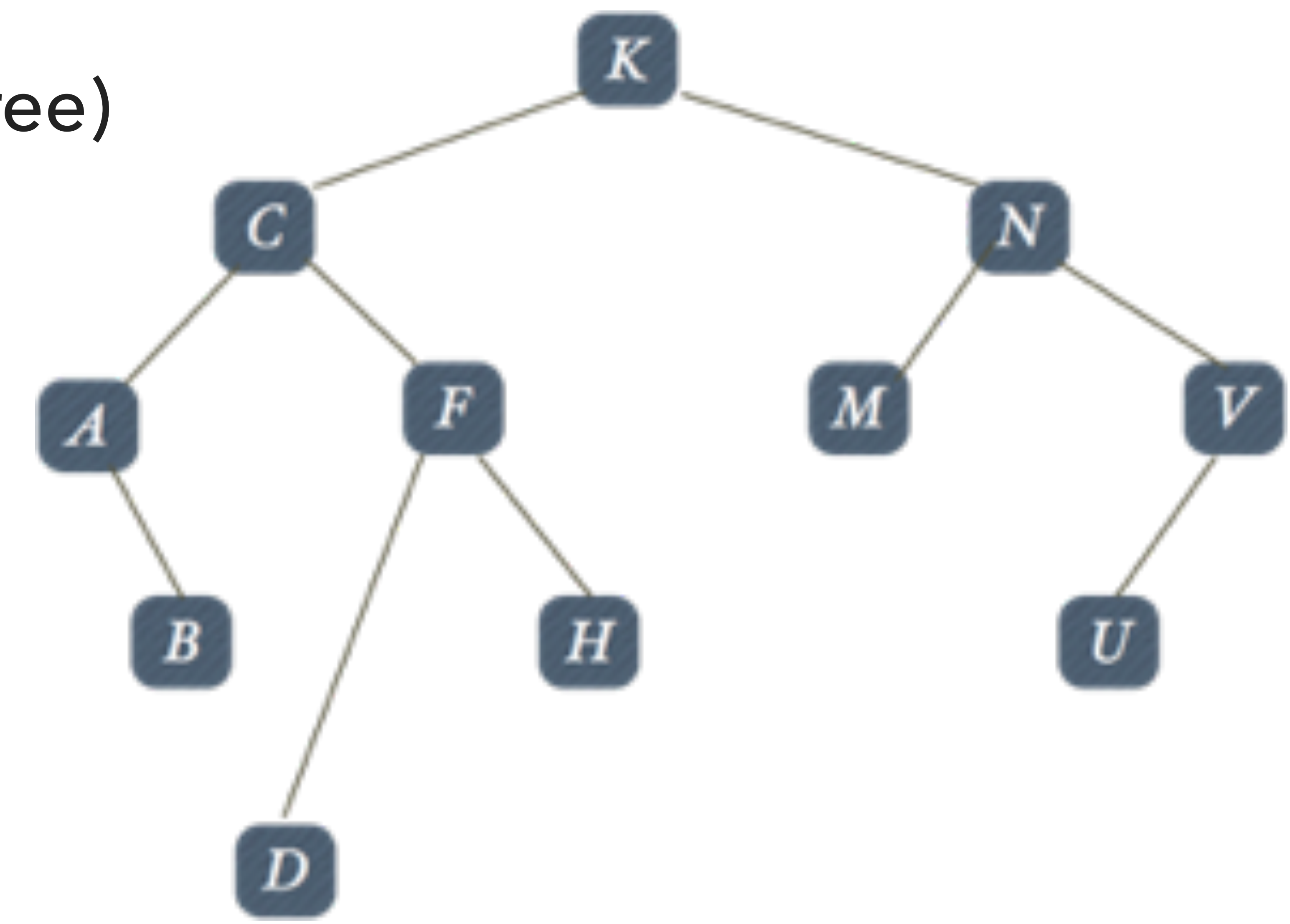
In-order traversal

- Inorder(Tree)
  - Inorder(Left Subtree)
  - Mark root as visited
  - Inorder(Right Subtree)
- A B C D F H K M N U V



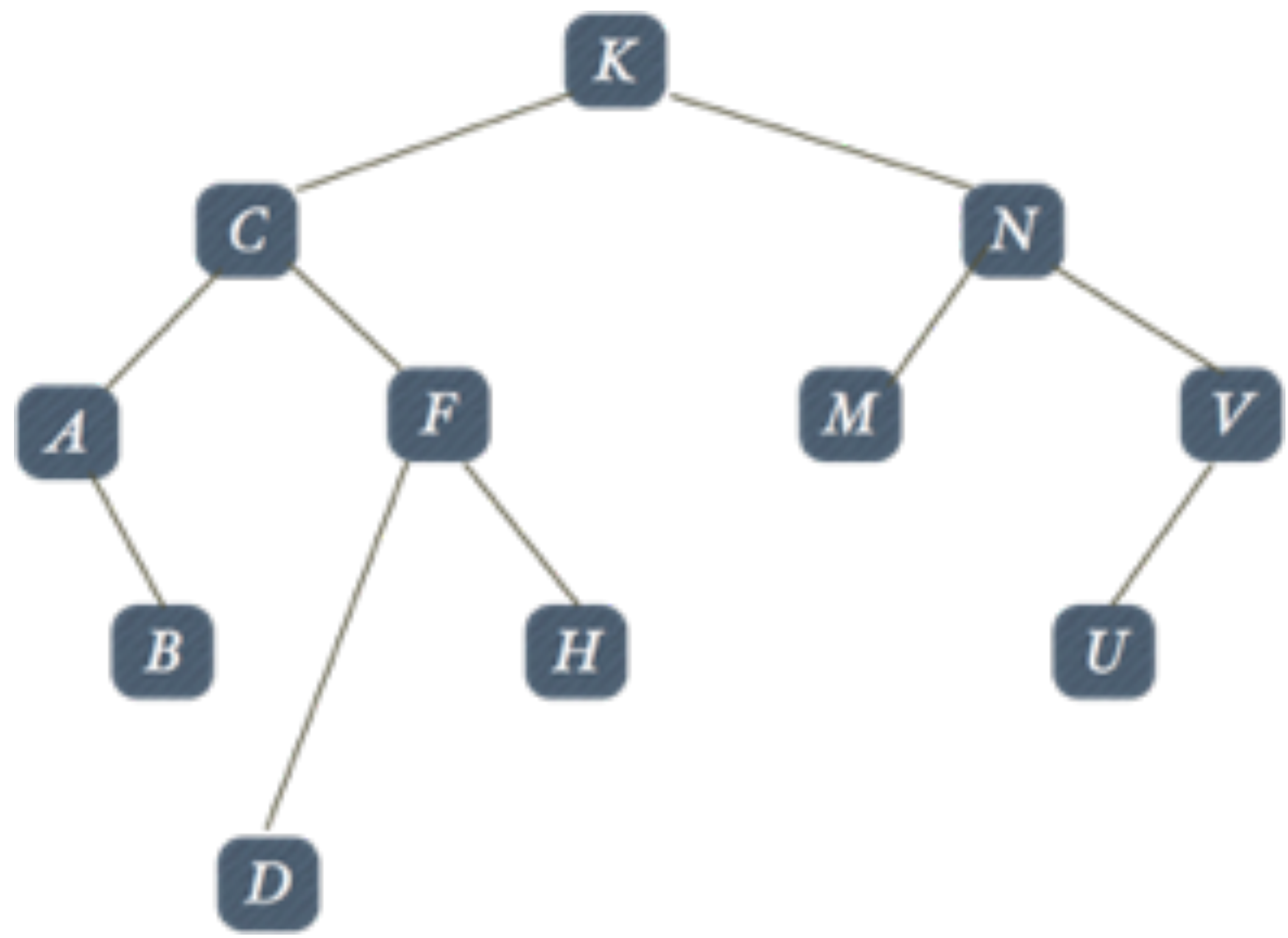
Post-order traversal

- Postorder(Tree)
  - Postorder(Left Subtree)
  - Postorder(Right Subtree)
  - Mark root as visited
- B A D H F C M U V N K



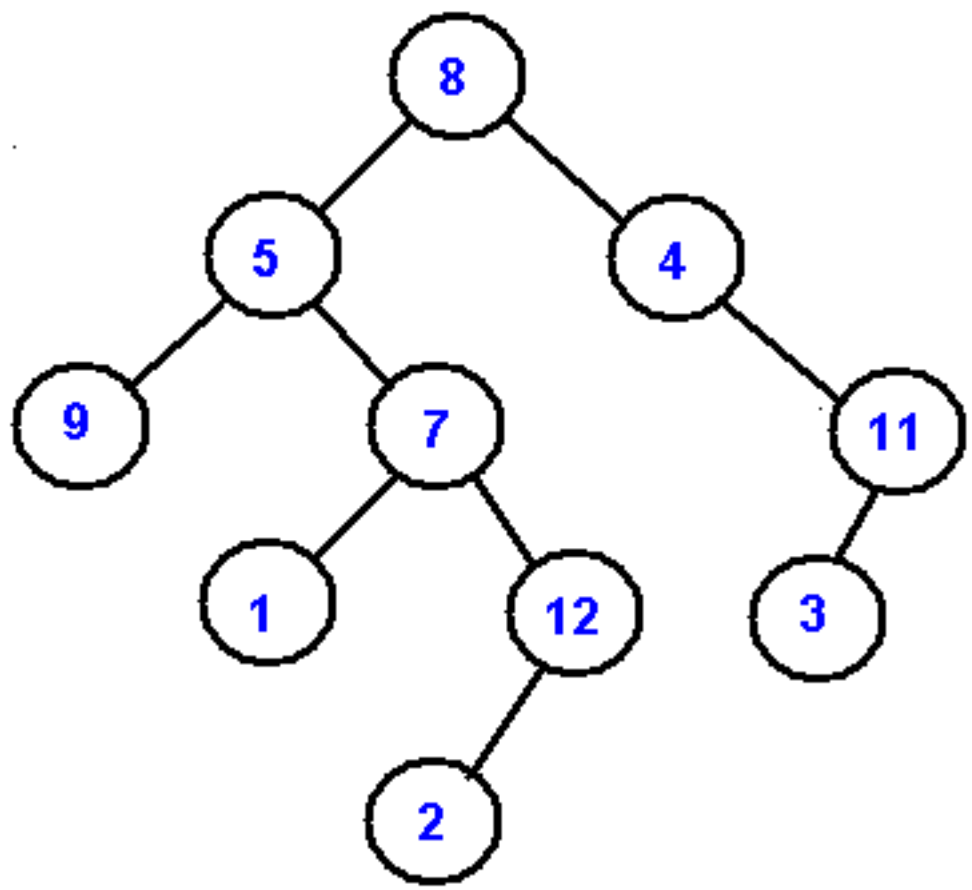
Level-order traversal

- From left to right, mark nodes of level i as visited before nodes in level $i + 1$. Start at level 0.
- K C N A F M V B D H U

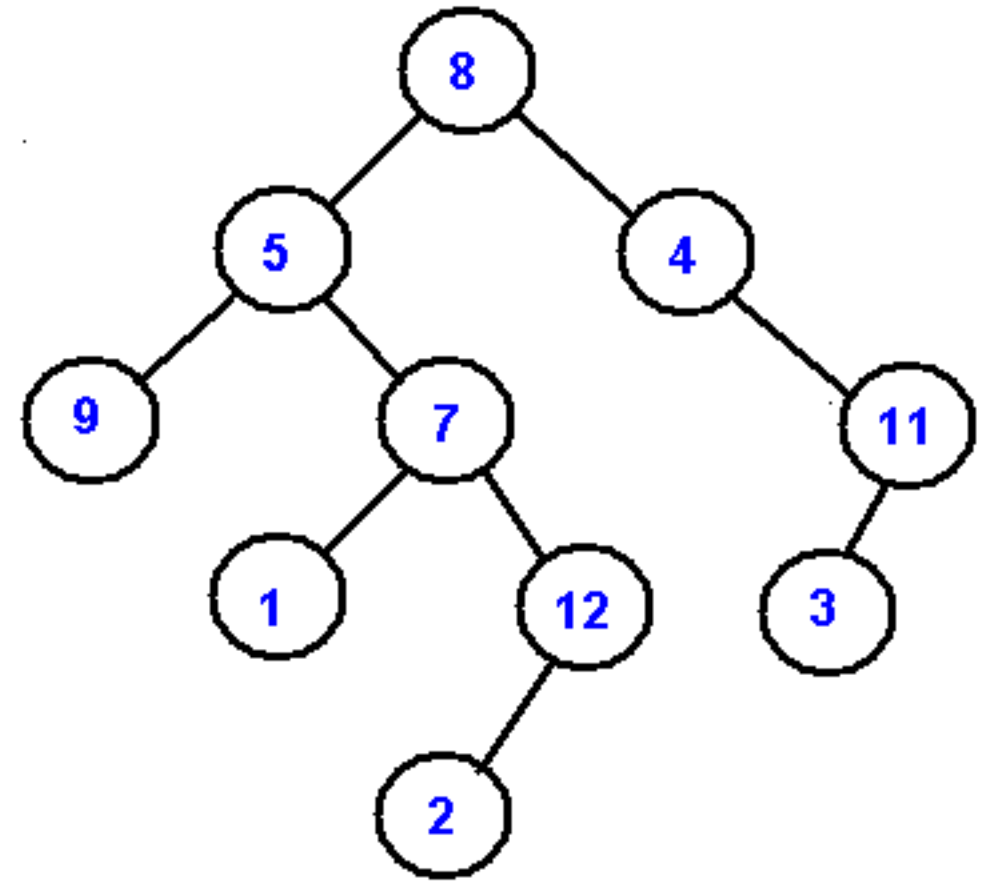


Practice Time

- List the nodes in pre-order, in-order, post-order, and level order:



Answer



- Pre-order: 8, 5, 9, 7, 1, 12, 2, 4, 11, 3
- In-order: 9, 5, 1, 7, 2, 12, 8, 4, 3, 11
- Post-order: 9, 1, 2, 12, 7, 5, 3, 11, 4, 8
- Level-order: 8, 5, 4, 9, 7, 11, 1, 12, 3, 2

Lecture 21: Binary Trees and Heaps

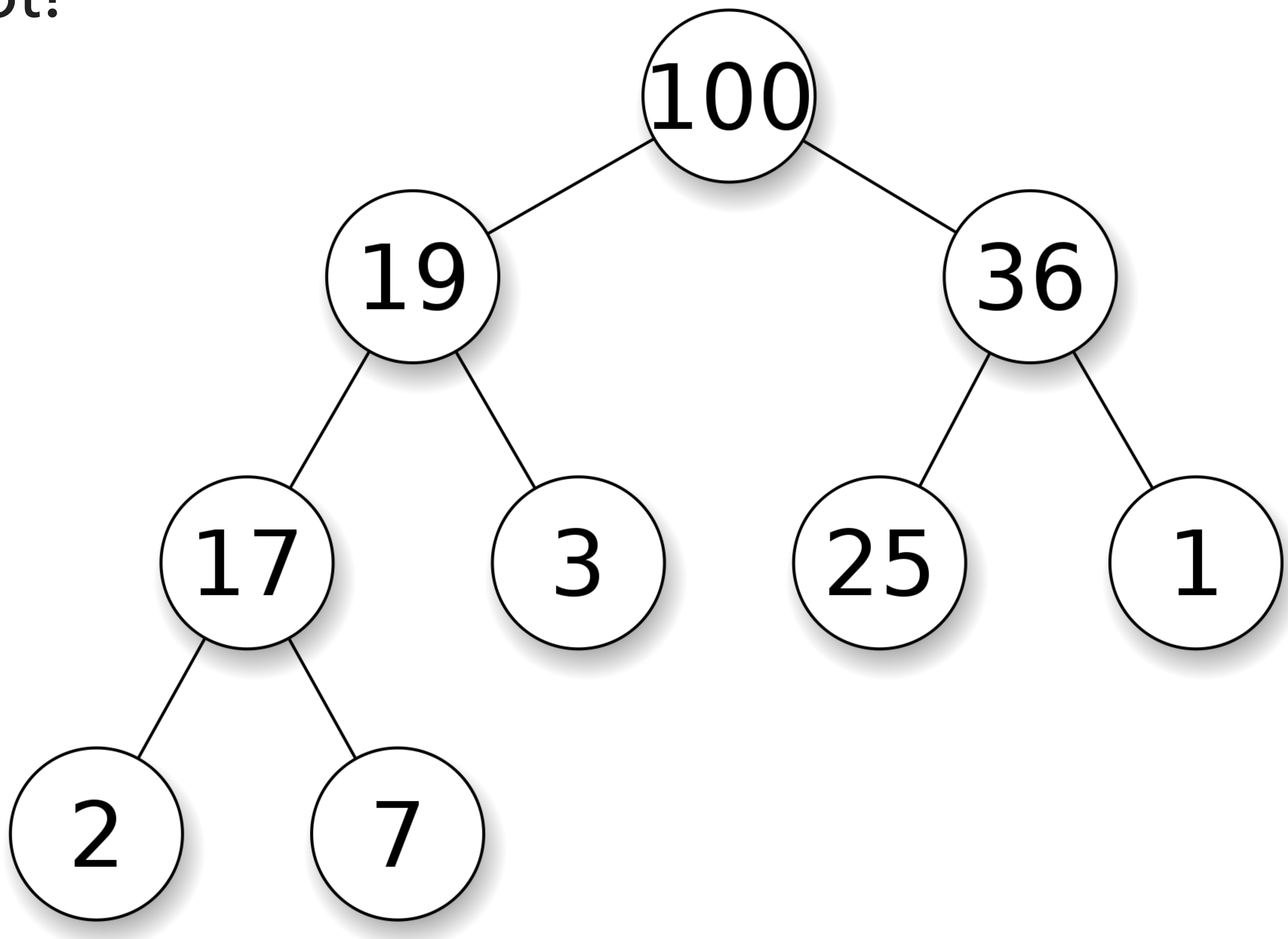
- Binary Trees
- Tree traversals
- Binary Heaps

Heap-ordered binary trees

- A binary tree is heap-ordered if the key in each node is larger than or equal to the keys in that node's two children (if any).
- Equivalently, the key in each node of a heap-ordered binary tree is smaller than or equal to the key in that node's parent (if any).
- No assumption of which child is smaller.
- Moving up from any node, we get a non-decreasing sequence of keys.
- Moving down from any node we get a non-increasing sequence of keys.

Heap-ordered binary trees

- The largest key in a heap-ordered binary tree is found at the root!



Binary heap representation

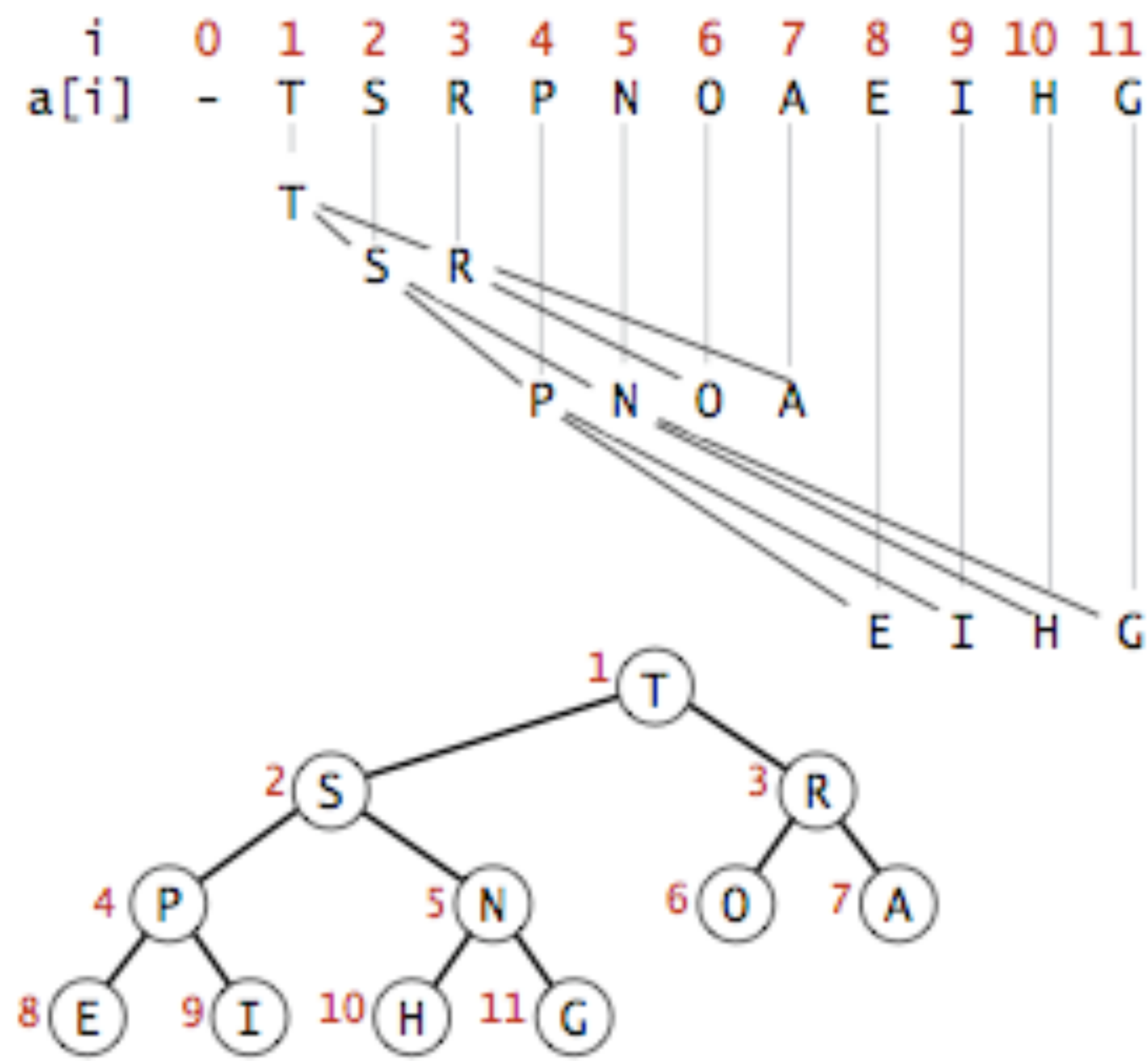
- We could use a linked representation but we would need three links for every node (one for parent, one for left subtree, one for right subtree).
- If we use complete binary trees, we can use instead an array.
  - Compact arrays vs explicit links means memory savings!

Binary heaps

- Binary heap: the array representation of a complete heap-ordered binary tree.
  - Items are stored in an array such that each key is guaranteed to be larger (or equal to) than the keys at two other specific positions (children).
- Max-heap but there are min-heaps, too.

Array representation of heaps

- Nothing is placed at index 0.
- Root is placed at index 1.
- Rest of nodes are placed in level order.
- No unnecessary indices and no wasted space because it's complete.



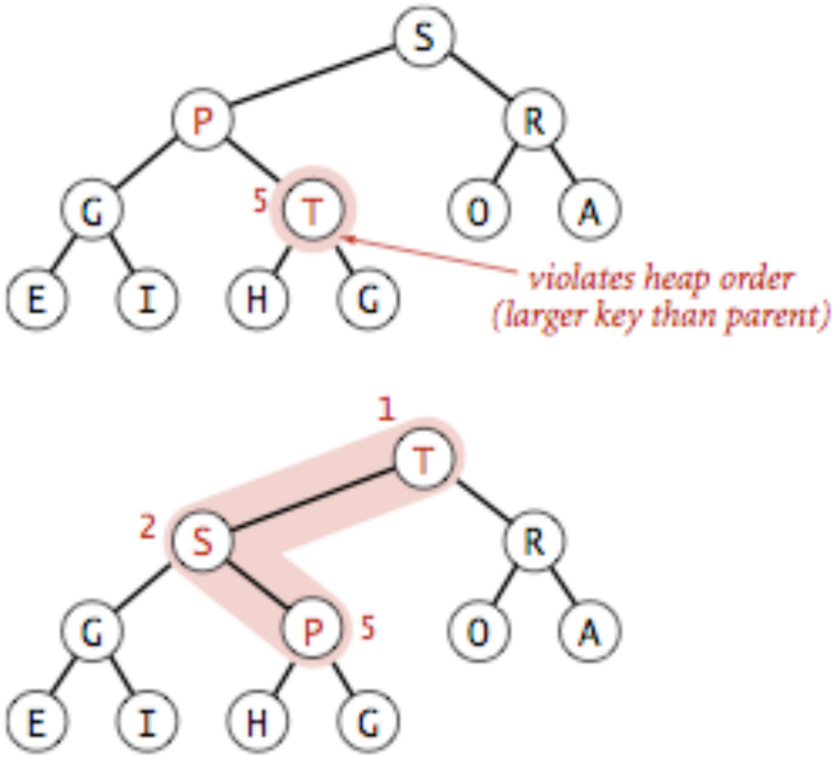
Heap representations

Reuniting immediate family members.

- For every node at index k, its parent is at index $\lfloor k/2 \rfloor$.
- Its two children are at indices $2k$ and $2k + 1$.
- We can travel up and down the heap by using this simple arithmetic on array indices.

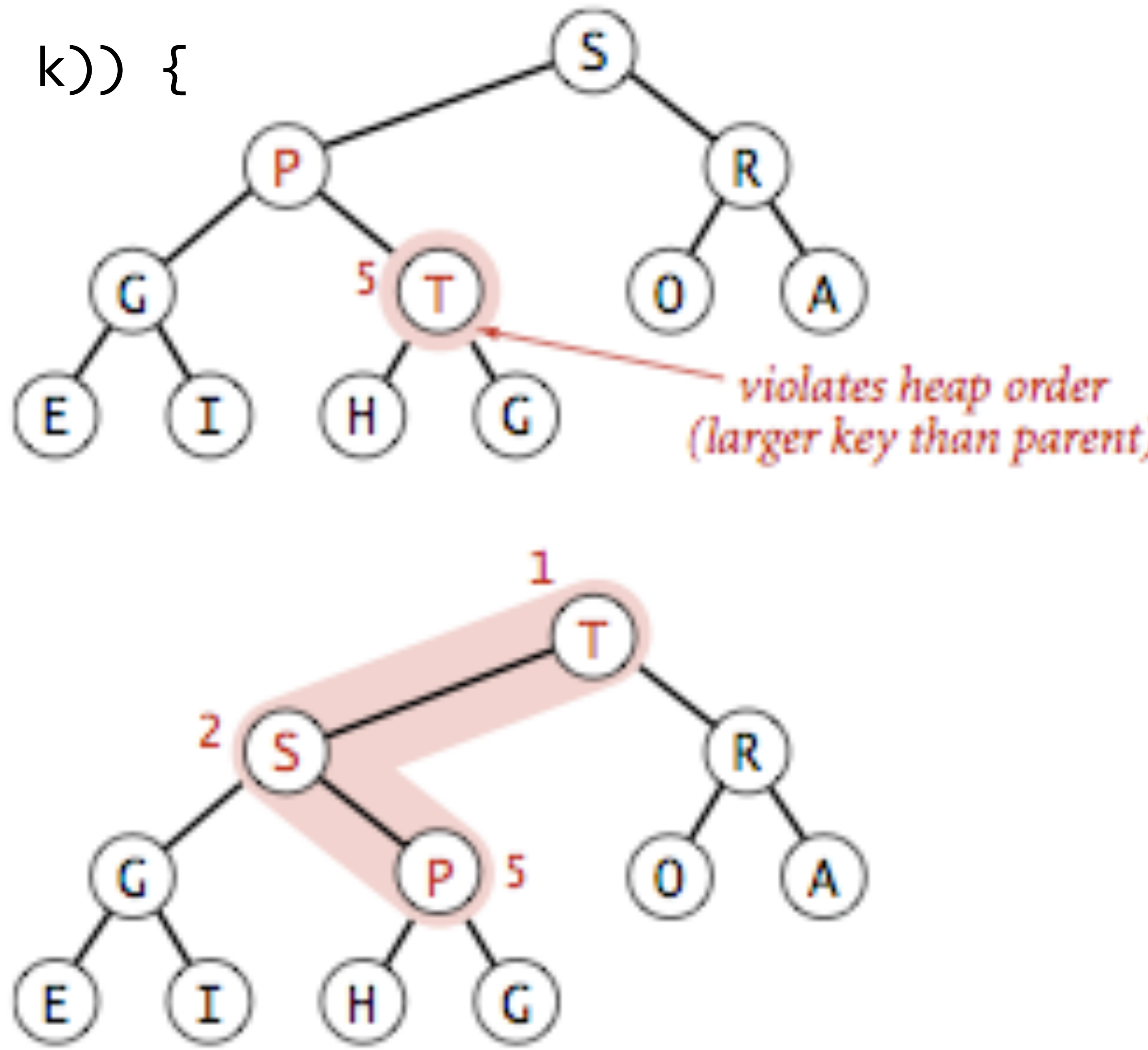
Swim/promote/percolate up/bottom up reheapify

- Scenario: a key becomes larger than its parent therefore it violates the heap-ordered property.
- To eliminate the violation:
  - Exchange key in child with key in parent.
  - Repeat until heap order restored.



Swim/promote/percolate up

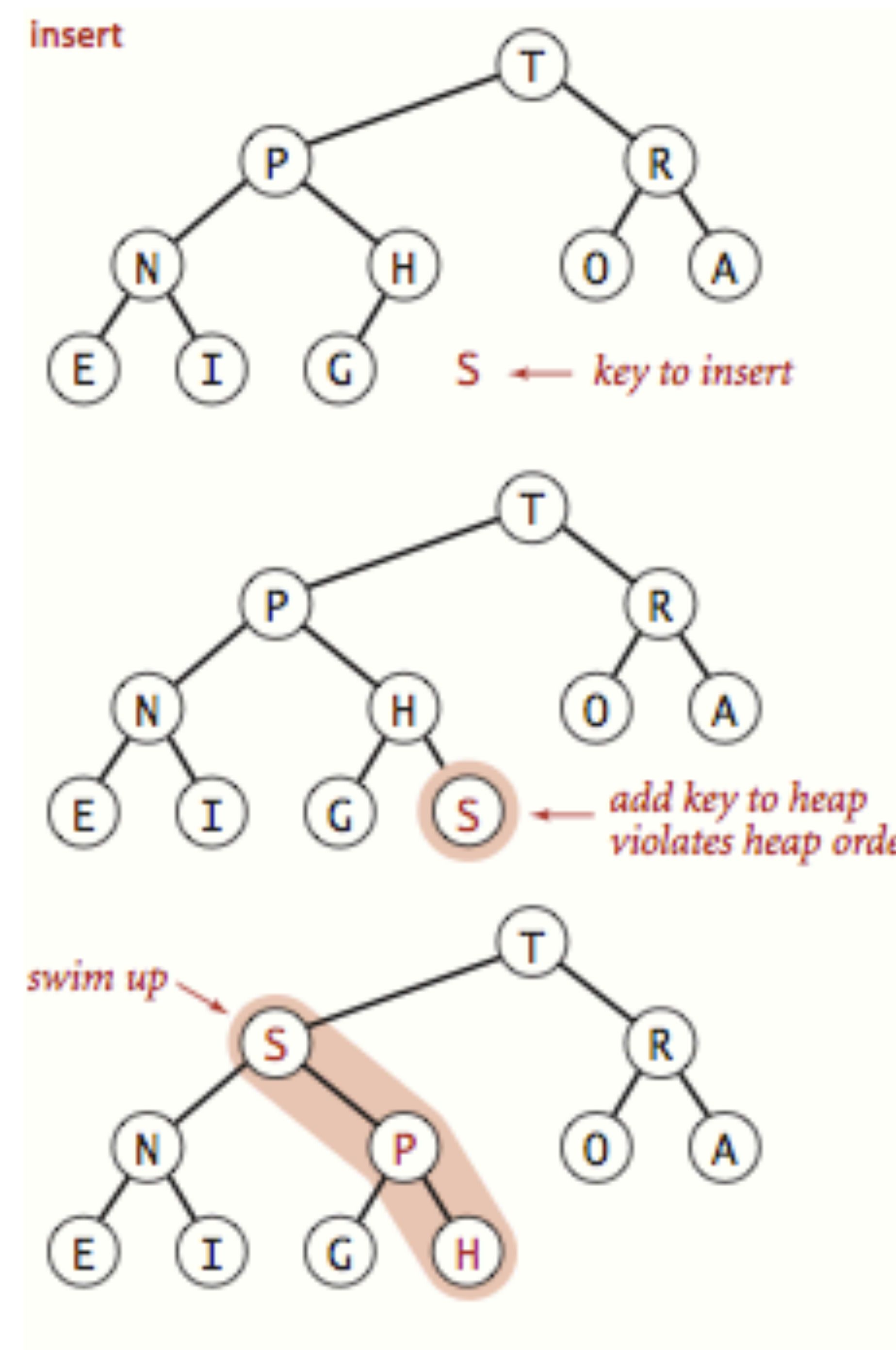
```
private void swim(int k) {
   while (k > 1 && less(k/2, k)) {
      exch(k, k/2);
      k = k/2;
   }
}
```



Binary heap: insertion

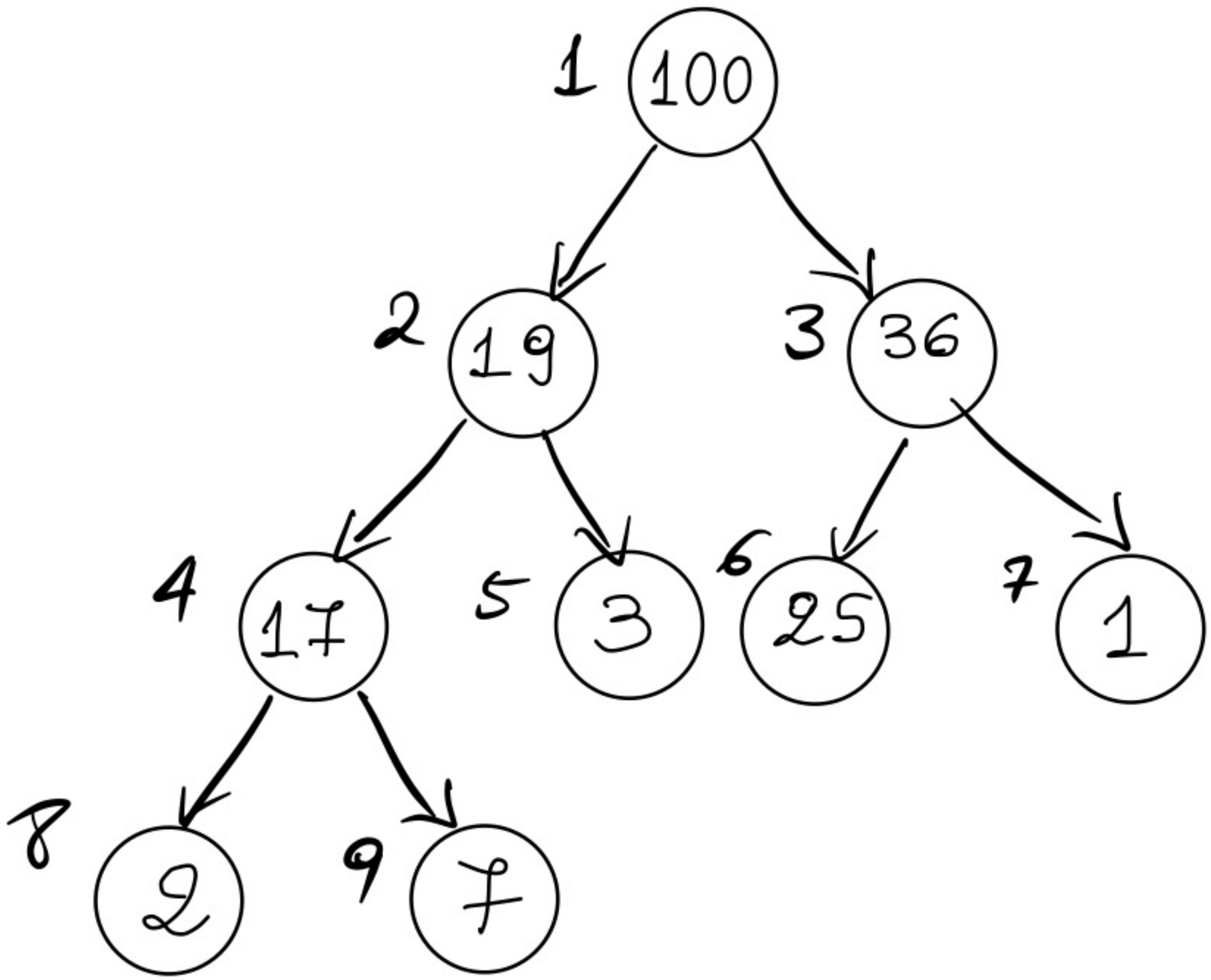
- Insert: Add node at end in bottom level, then swim it up.
- Cost: At most $\log n + 1$ compares.

```
public void insert(Key x) {
    pq[++n] = x;
    swim(n);
}
```

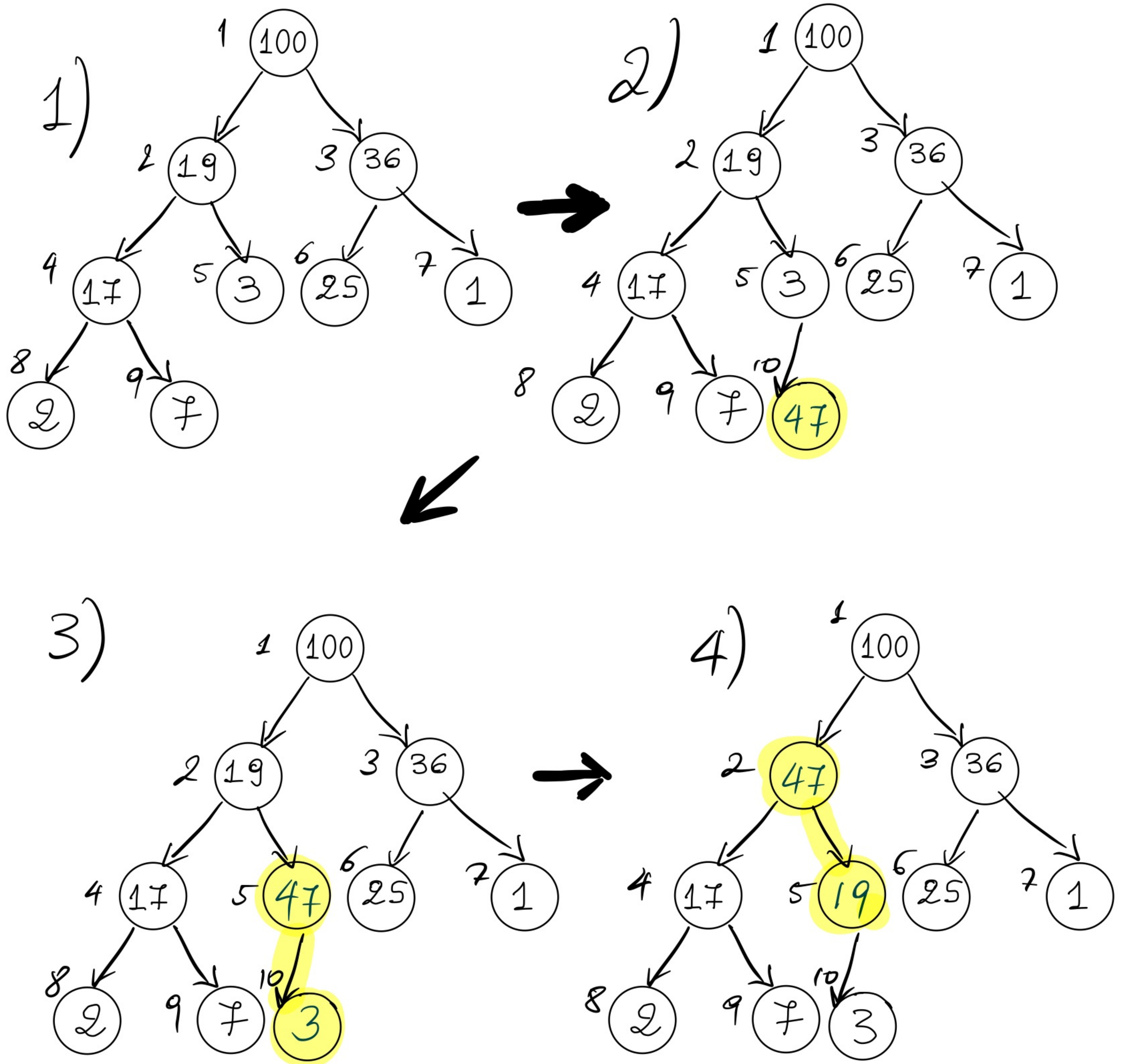


Practice Time

- Insert 47 in this binary heap.

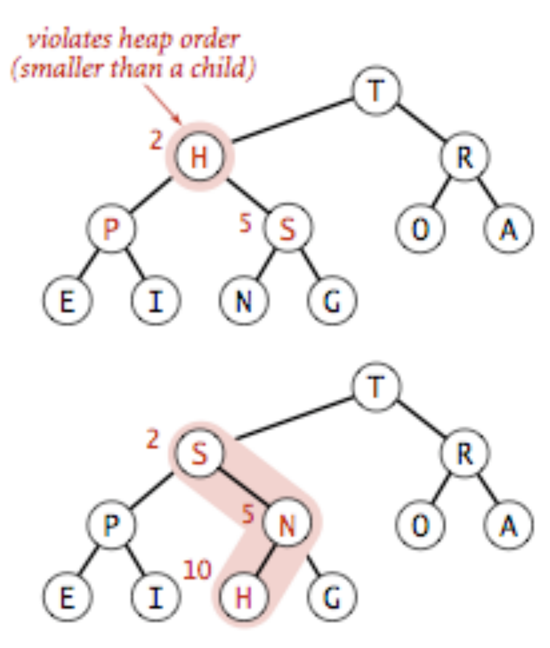


Answer



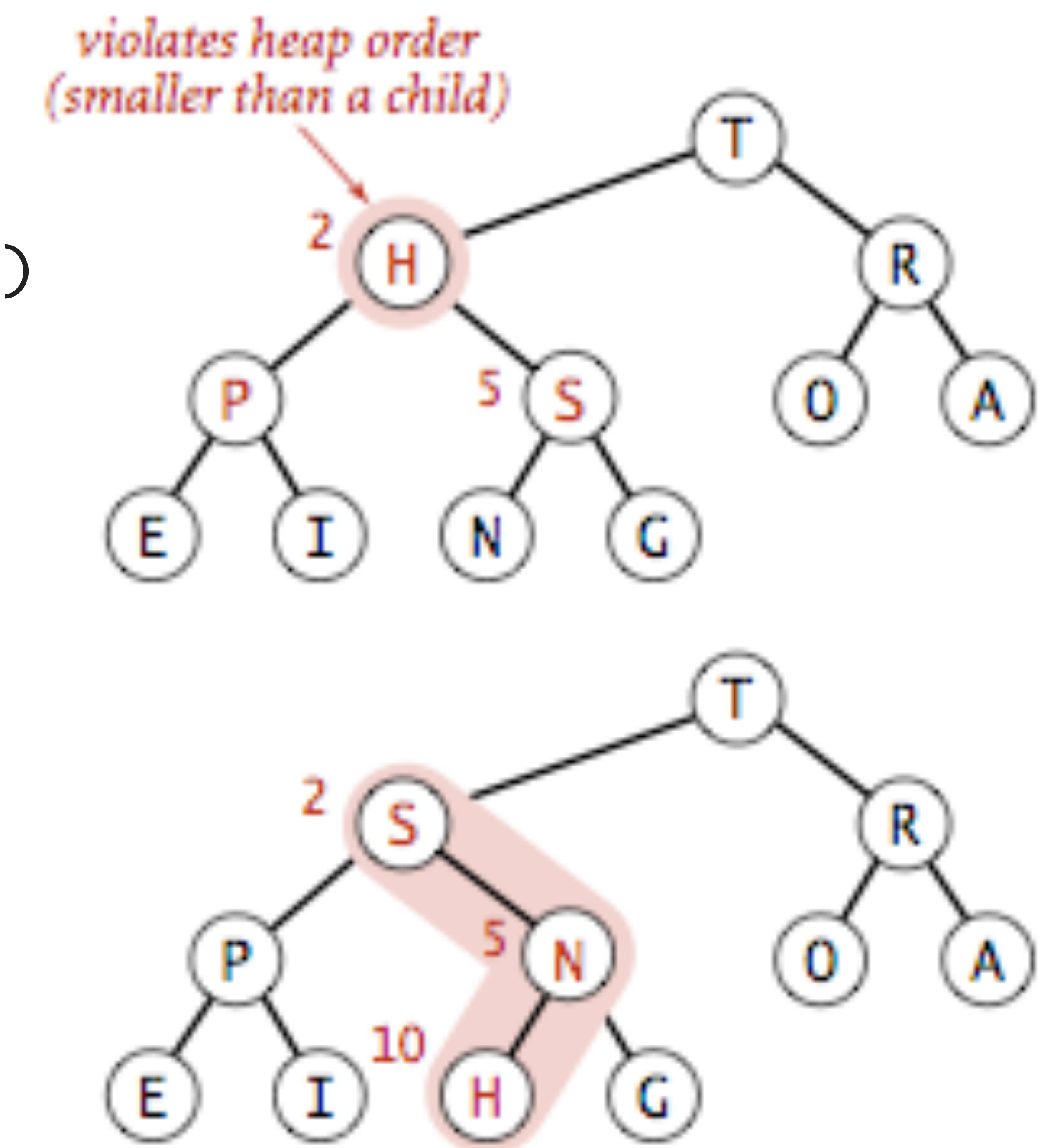
Sink/demote/top down heapify

- Scenario: a key becomes smaller than one (or both) of its children's keys.
- To eliminate the violation:
  - Exchange key in parent with key in **larger** child.
  - Repeat until heap order is restored.



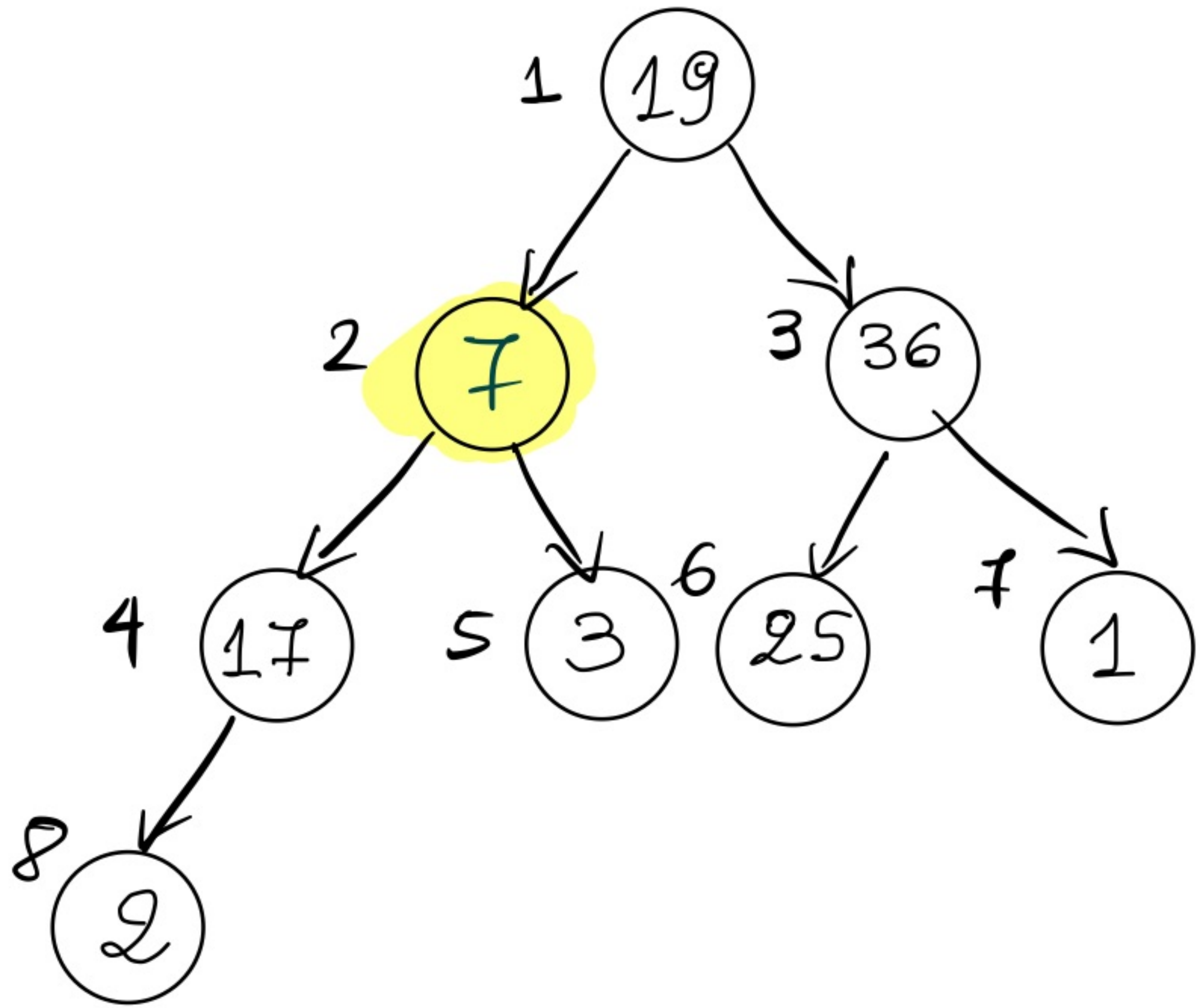
Sink/demote/top down heapify

```
private void sink(int k) {
    while (2*k <= n) {
        int j = 2*k;
        if (j < n && less(j, j+1))
            j++;
        if (!less(k, j))
            break;
        exch(k, j);
        k = j;
    }
}
```

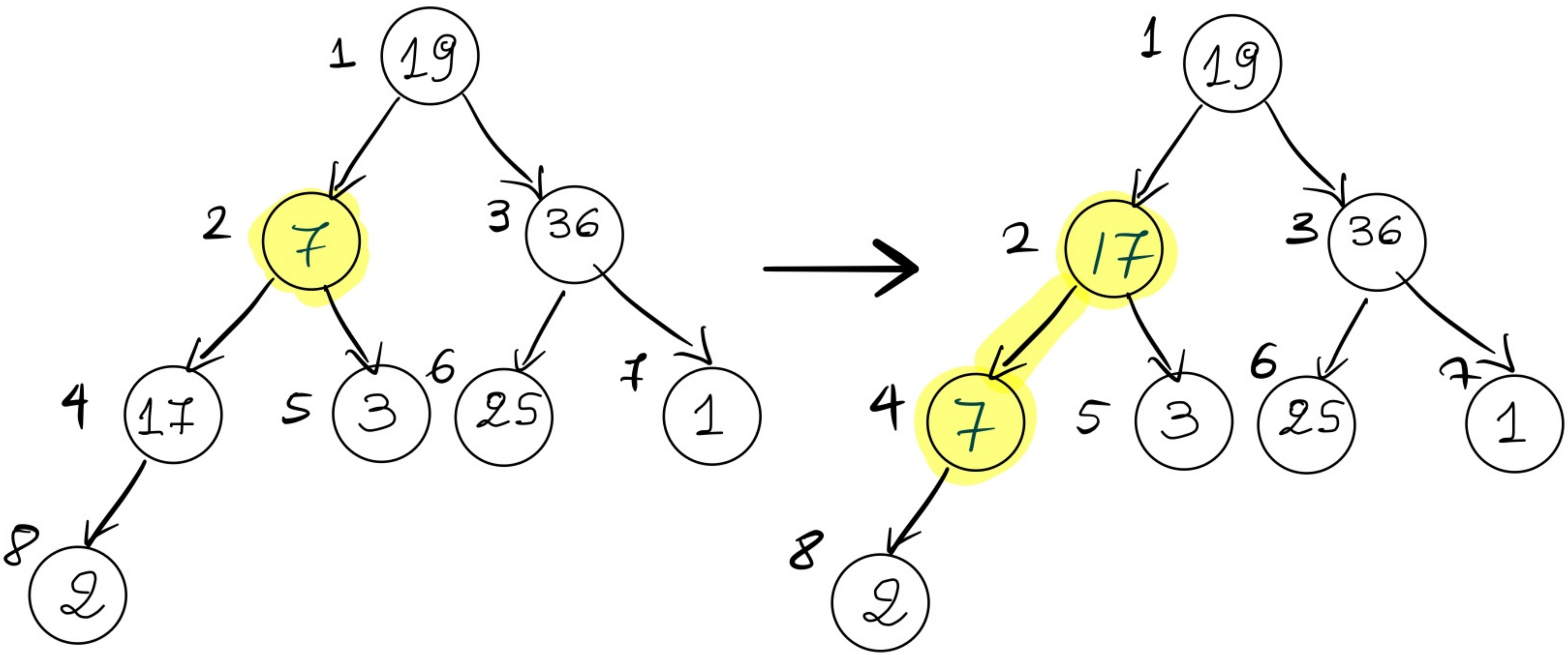


Practice Time

- Sink 7 to its appropriate place in this binary heap.



Answer

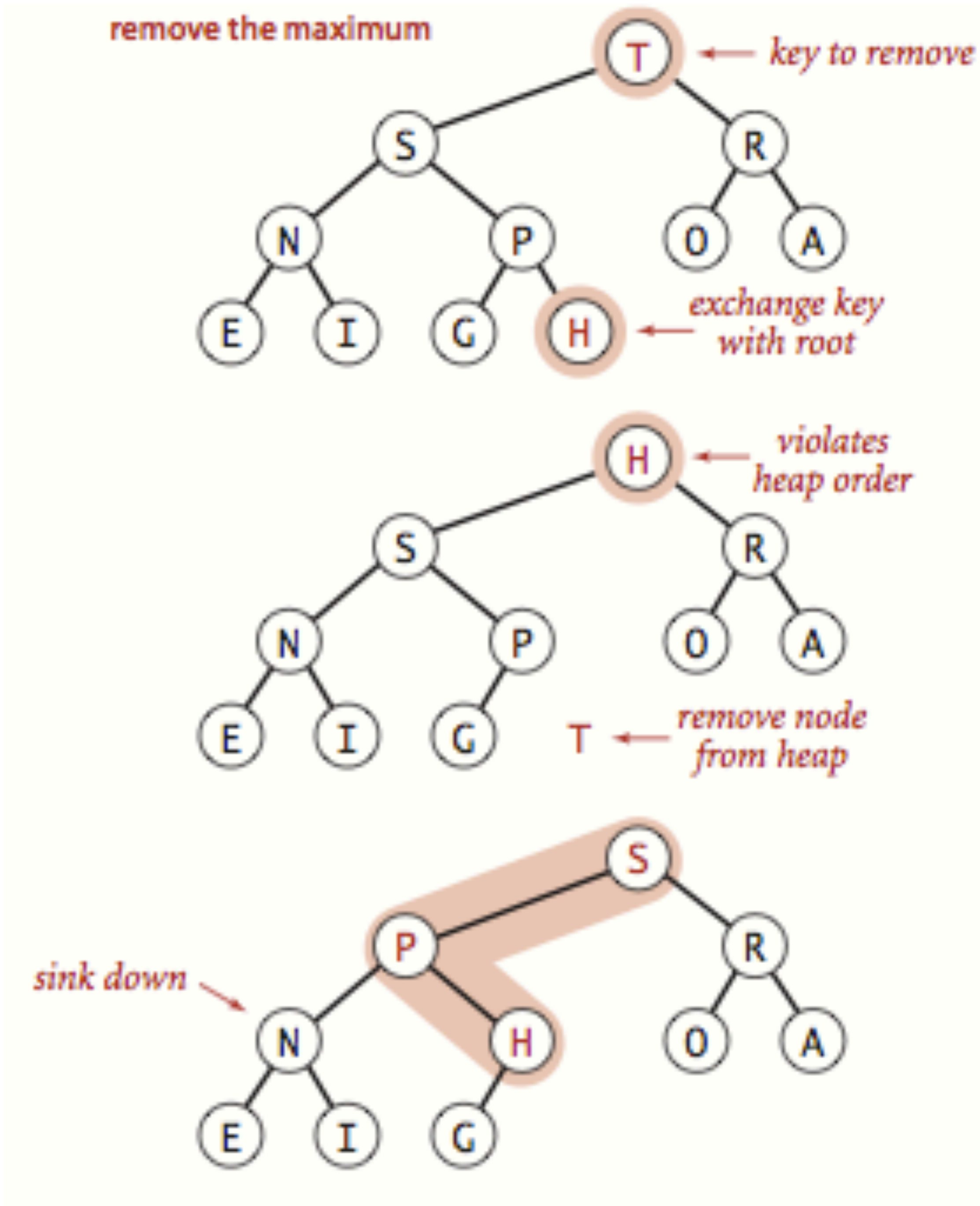


Binary heap: return (and delete) the maximum

- Delete max: Exchange root with node at end. Return it and delete it. Sink the new root down.
- Cost: At most $2 \log n$ compares.

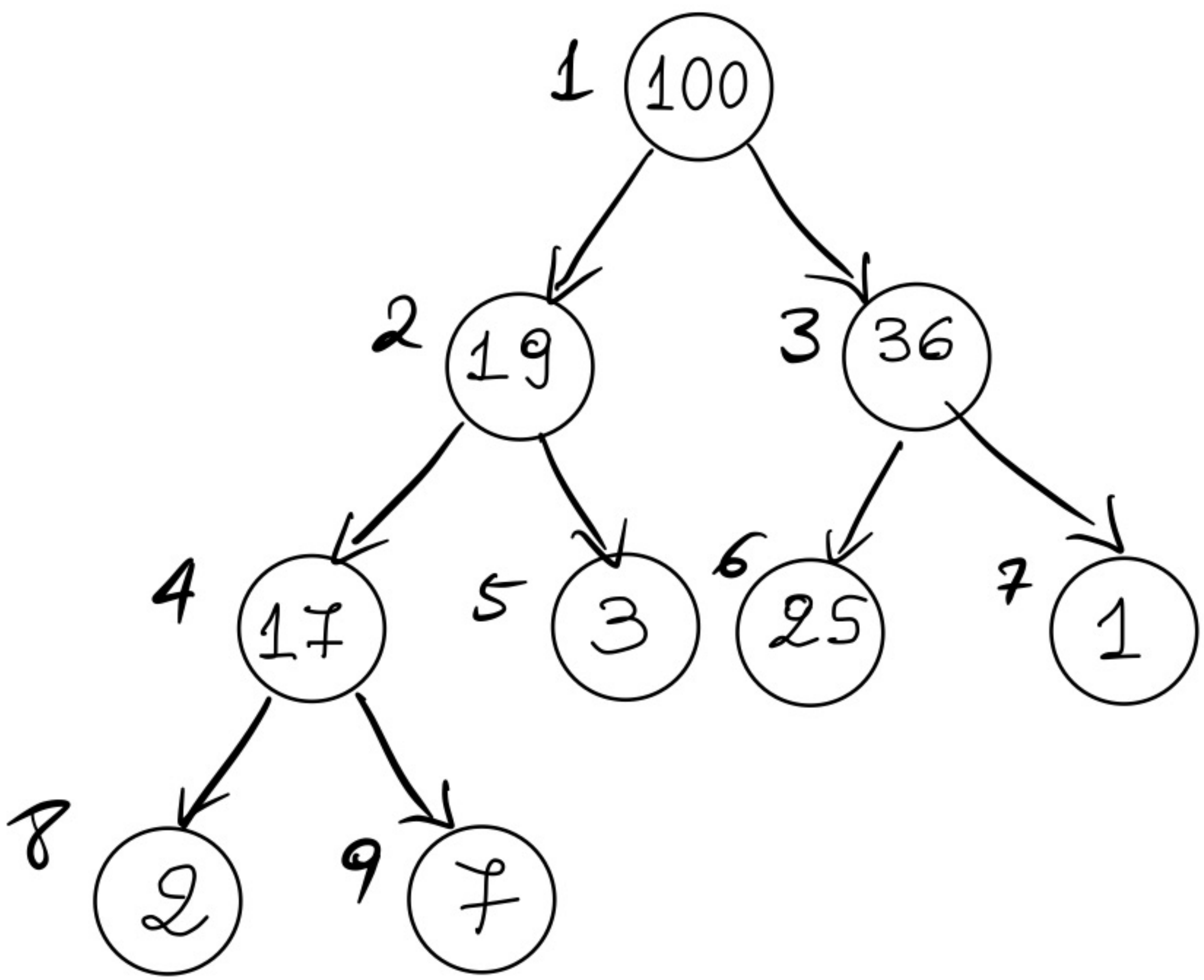
```
public Key delMax() {
    Key max = pq[1];
    exch(1, n--);
    sink(1);
    pq[n+1] = null;
    return max;
}
```

Binary heap: delete and return maximum



Practice Time

- Delete max (and return it!)



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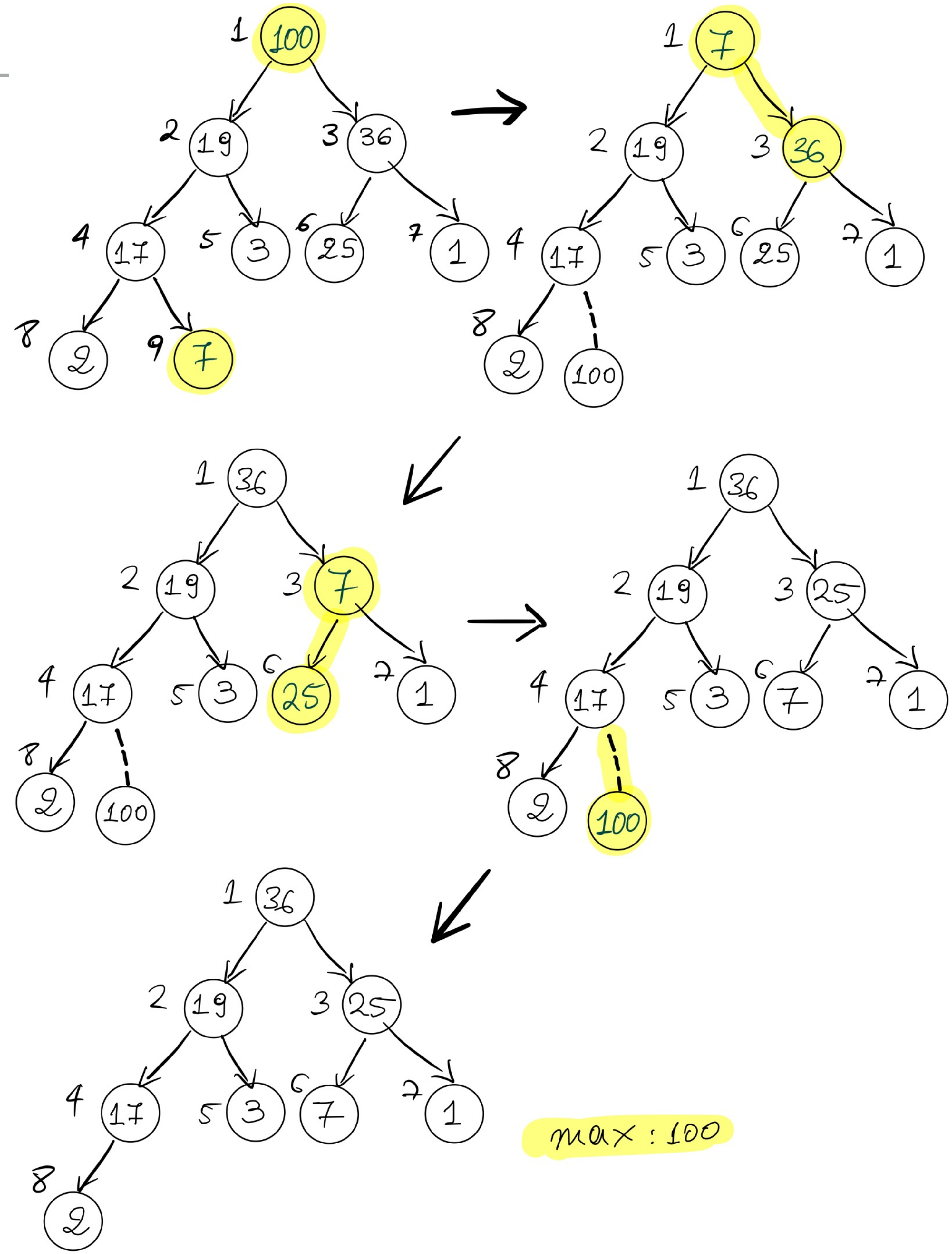

Things to remember about runtime complexity of heaps

- Insertion is $O(\log n)$.
- Delete max is $O(\log n)$.
- Space efficiency is $O(n)$.

http://algs4.cs.princeton.edu

2.4 BINARY HEAP DEMO

Lecture 21: Binary Trees and Heaps

- Binary Trees
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- Binary Heaps

Readings:

- Textbook:
  - Chapter 2.4 (Pages 308-327)
- Website:
  - Priority Queues: https://algs4.cs.princeton.edu/24pq/
- Visualization:
  - Insert and ExtractMax: https://visualgo.net/en/heap

Practice Problems:

- Practice with traversals of trees and insertions and deletions in binary heaps