# CS062

## DATA STRUCTURES AND ADVANCED PROGRAMMING

## 19: Binary Search Trees, 2-3 Search Trees

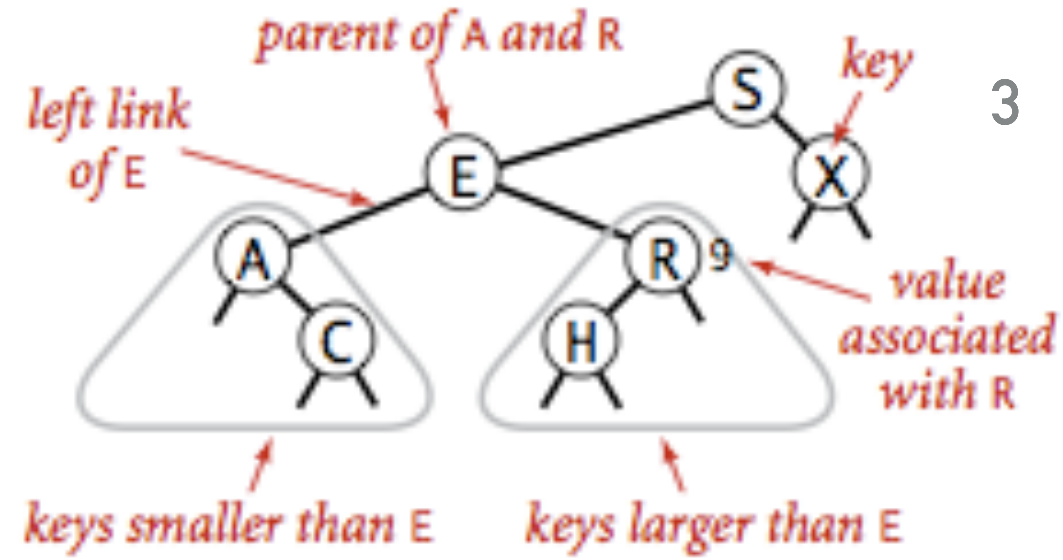
Alexandra Papoutsaki
she/her/hers

Tom Yeh
he/him/his

# Lecture 19: Binary Search Trees

- Binary Search Trees
- 2-3 Search Trees

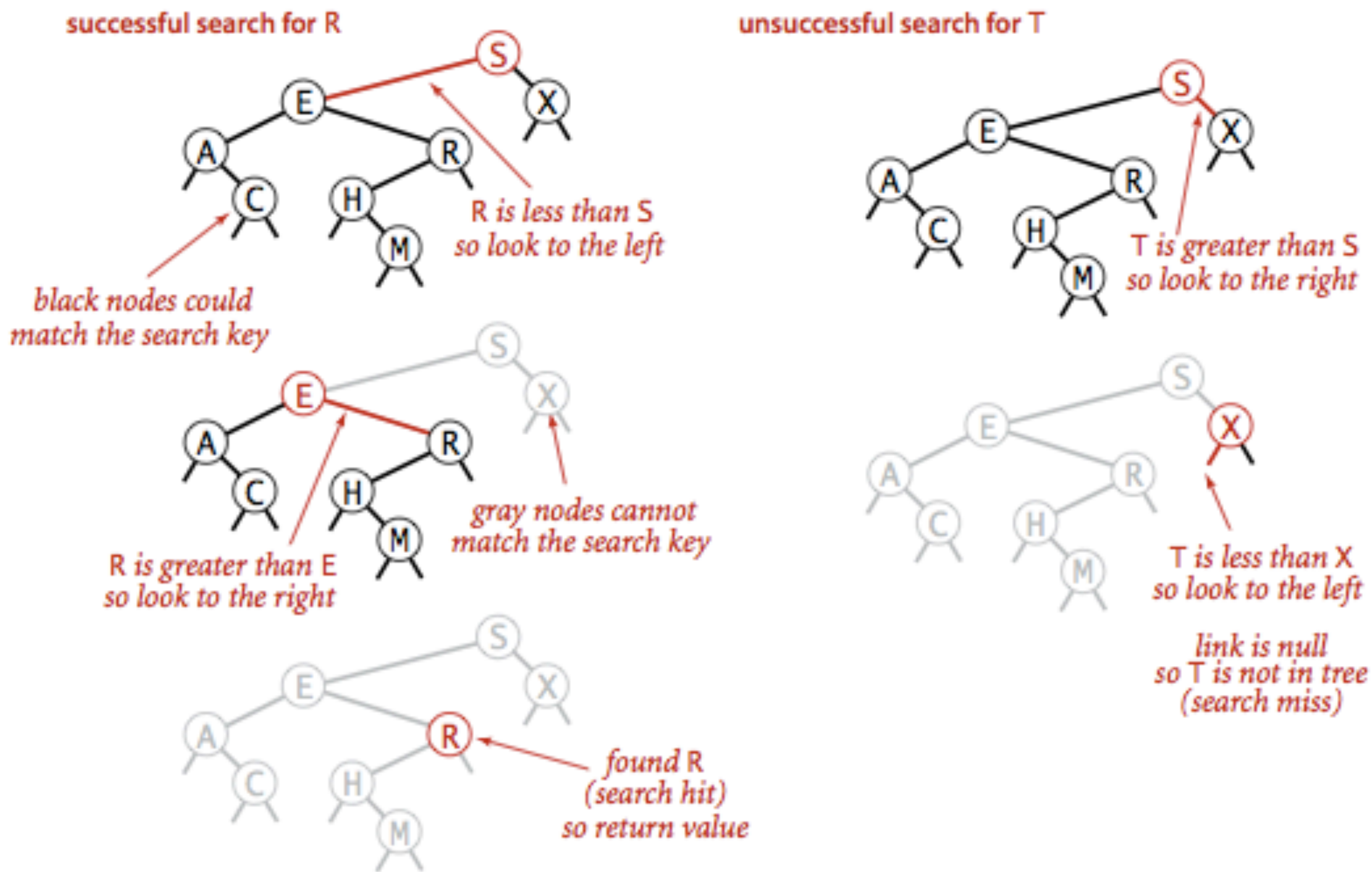
Some slides adopted from Algorithms 4th Edition or COS226




## Definitions

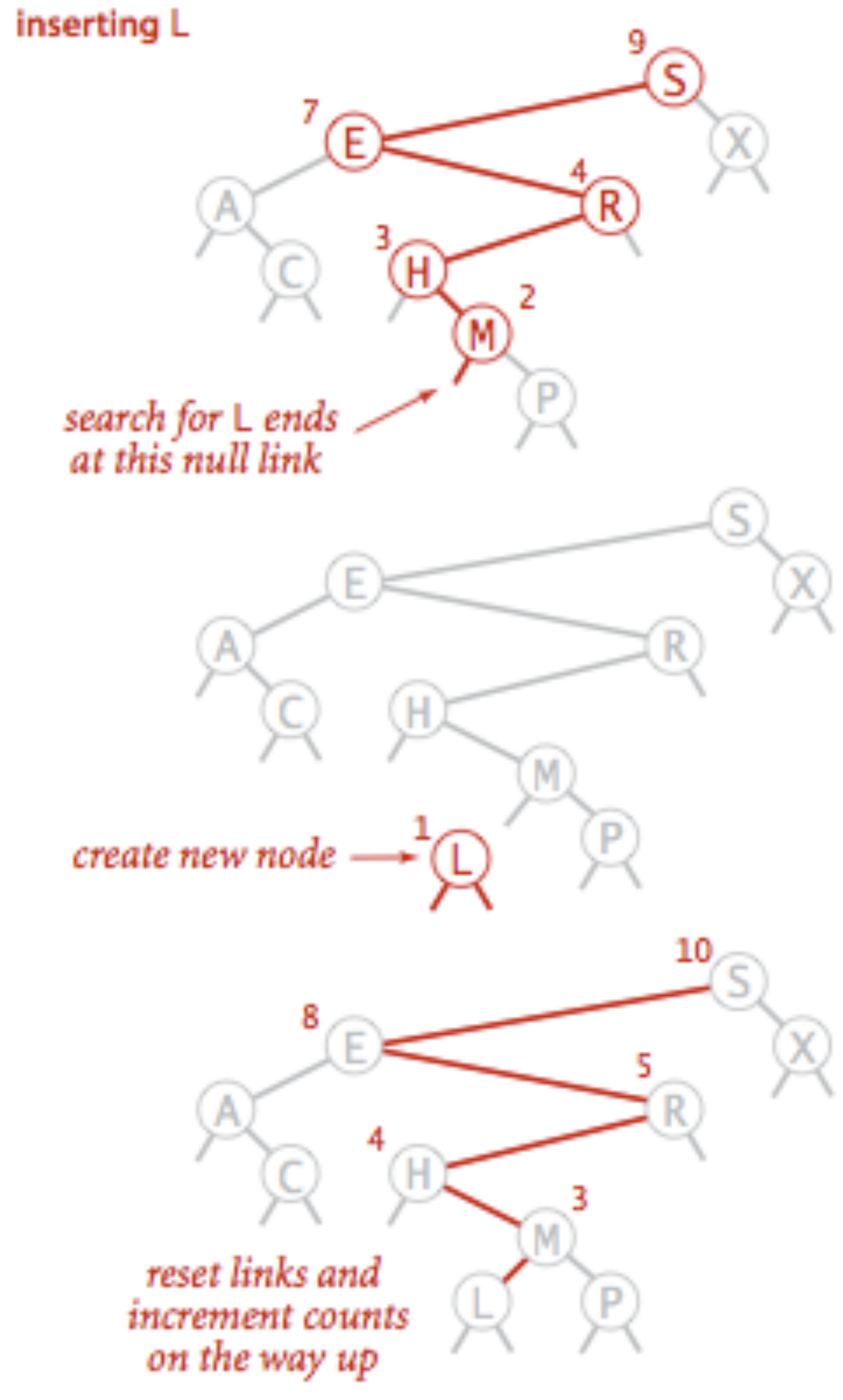
- Binary Search Tree: A binary tree in symmetric order.
- Symmetric order: Each node has a key, and every node's key is:
  - Larger than all keys in its left subtree.
  - Smaller than all keys in its right subtree.
-

# Search example




Successful (left) and unsuccessful (right ) search in a BST

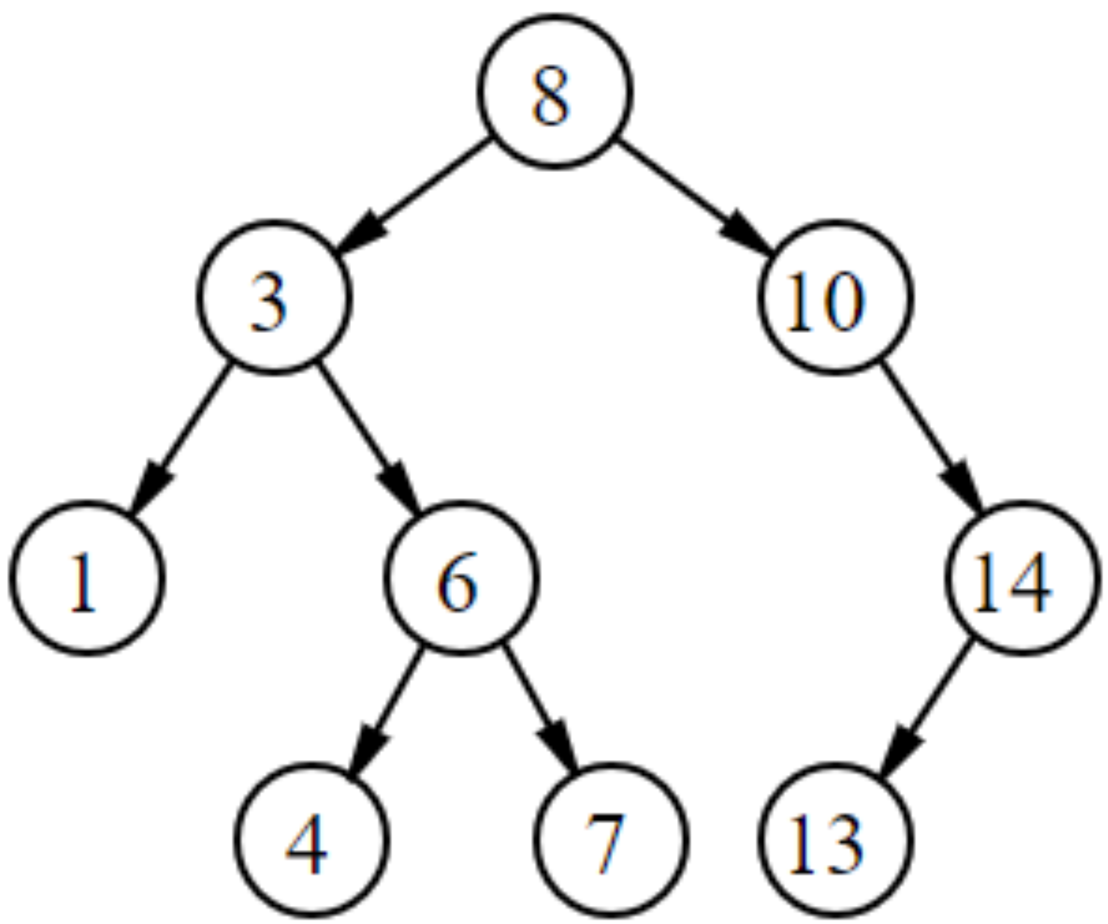
# Insert example




Insertion into a BST

# Practice Time

- Add the key-value pairs (4,3) and (9,2) in the following BST:

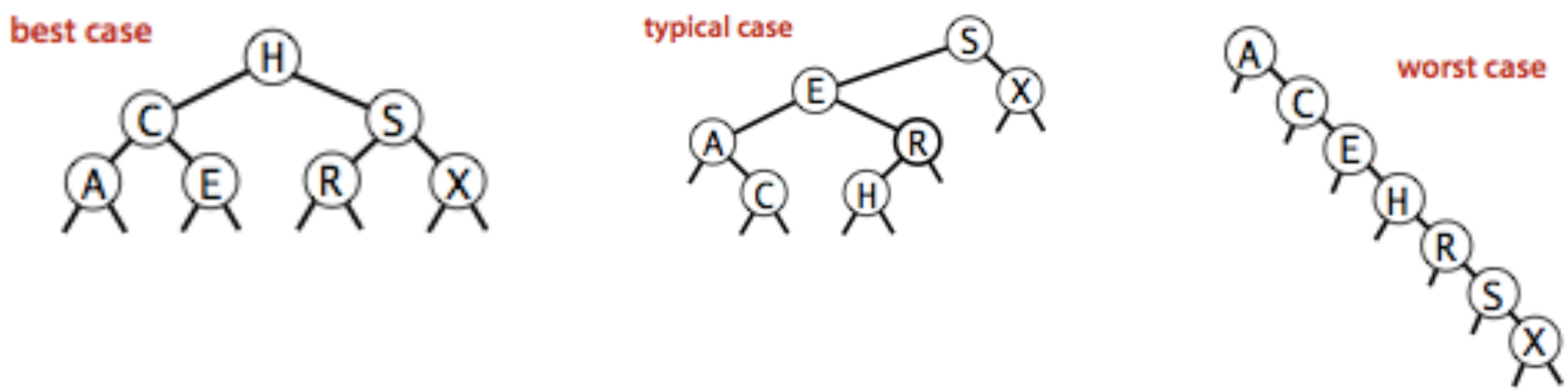



http://algs4.cs.princeton.edu

# 3.2 Binary Search Tree Demo

# Tree shape

- The same set of keys can result to different BSTs based on their order of insertion.
- Number of compares for search/insert is equal to depth of node +1.

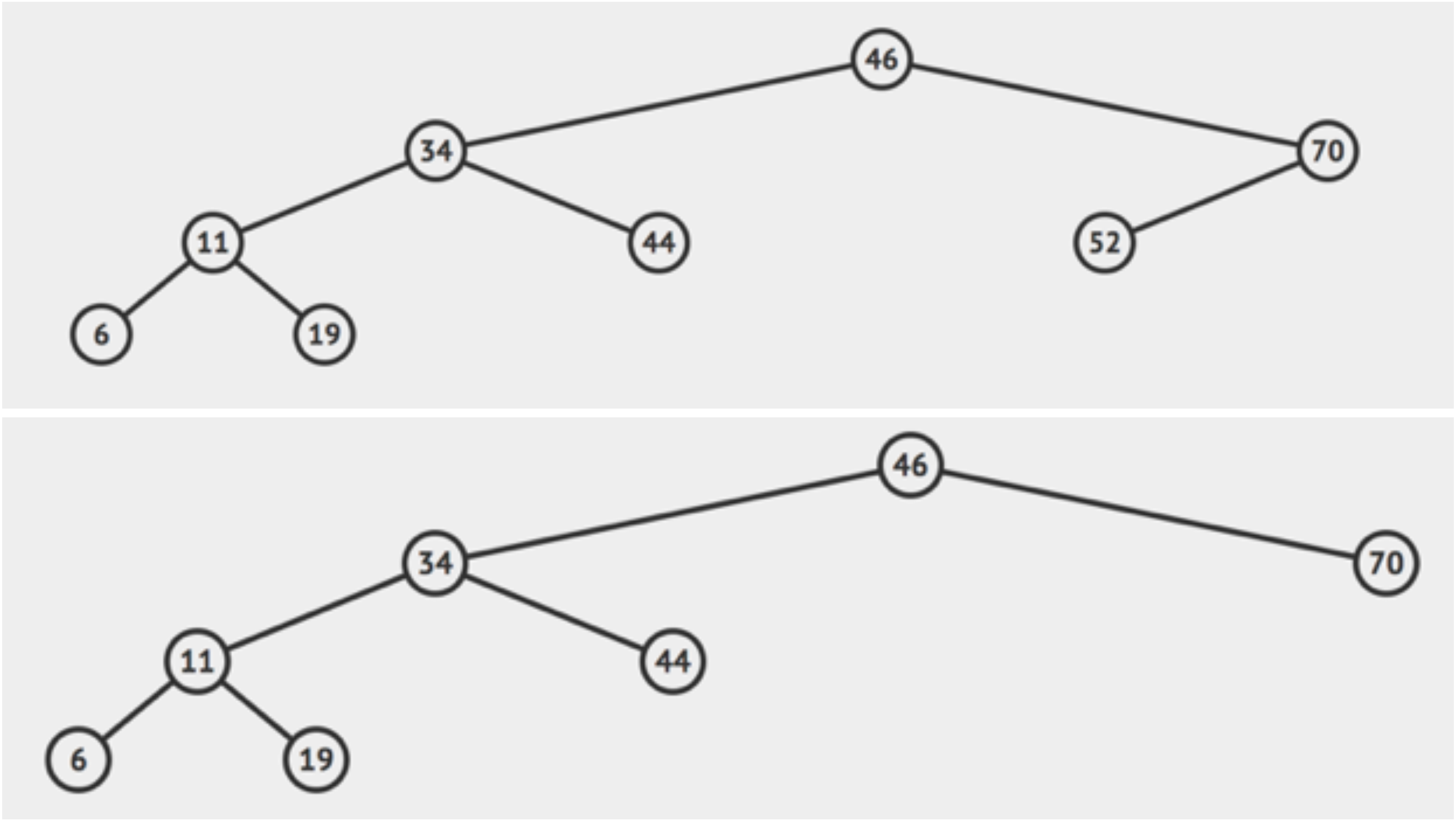


## BSTs mathematical analysis

- If $n$ distinct keys are inserted into a BST in random order, the expected number of compares of search/insert is $O(\log n)$.
    - If $n$ distinct keys are inserted into a BST in random order, the expected height of tree is $O(\log n)$. [Reed, 2003].
- Worst case height is $n$ but highly unlikely.
    - Keys would have to come (reversely) sorted!
- All ordered operations in a dictionary implemented with a BST depend on the height of the BST.

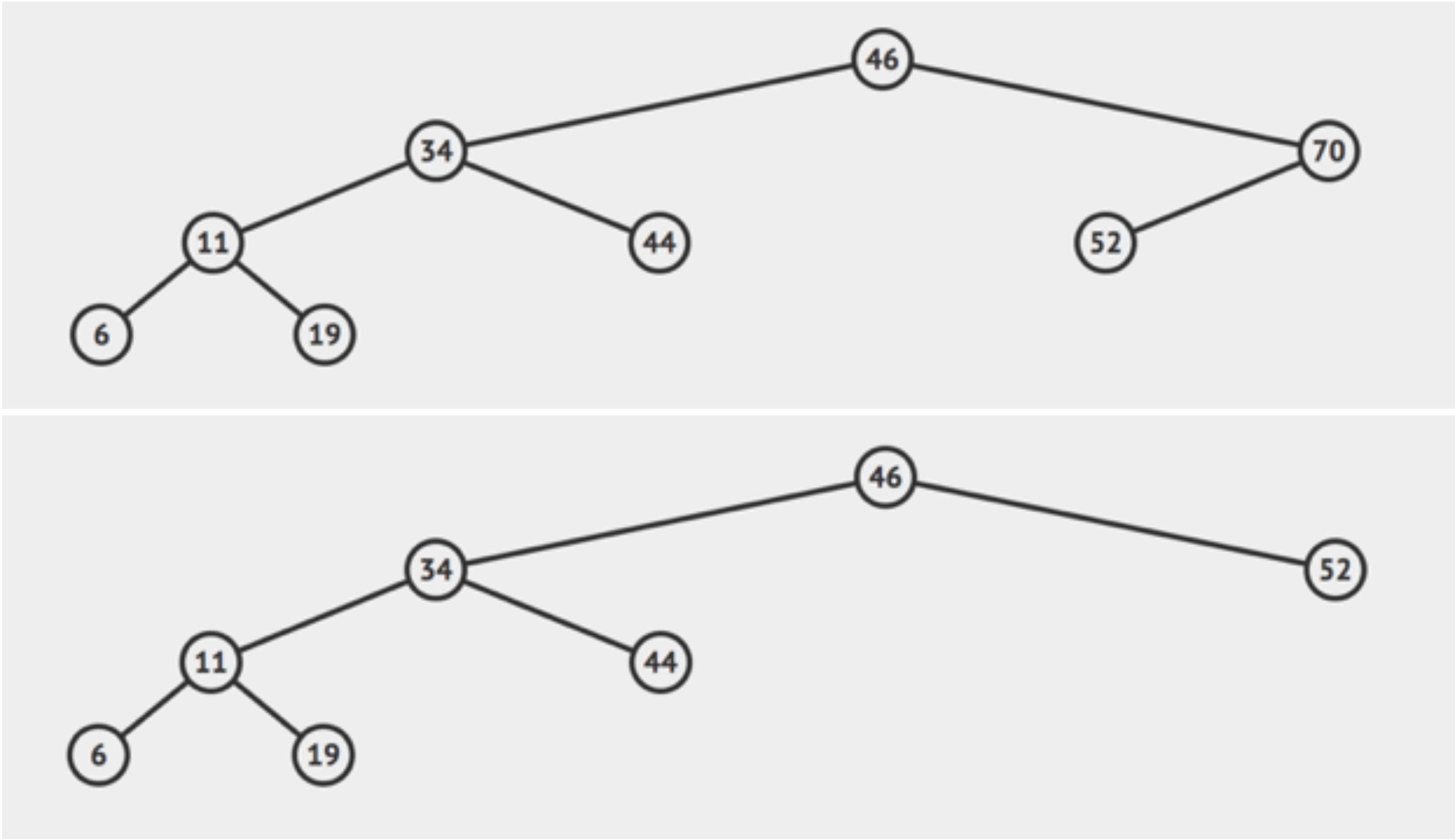
# Hibbard deletion: Delete node which is a leaf (case 0)

- Simply delete node.
- Example: delete 52 locates a node which is a leaf and removes it.



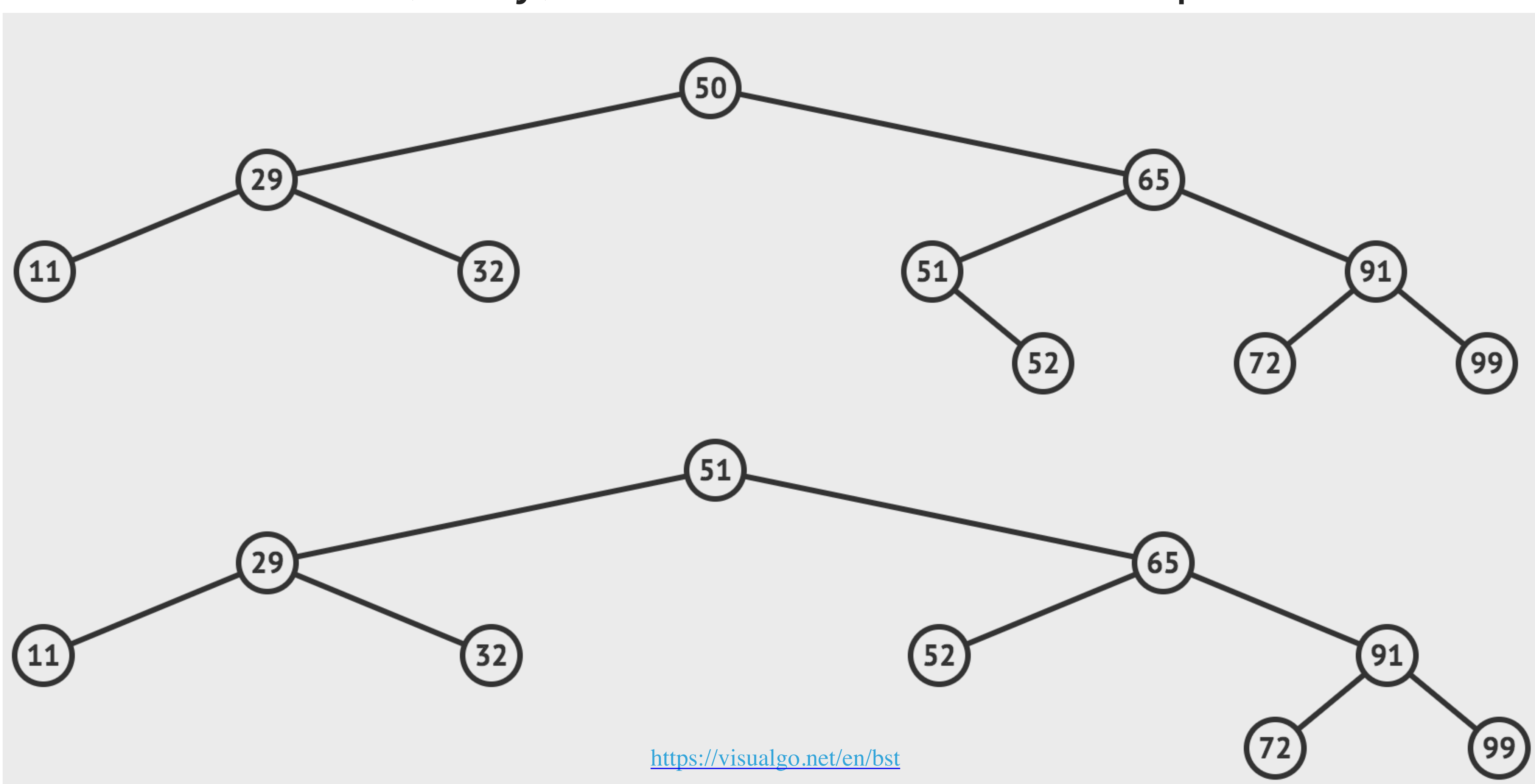
# Hibbard deletion: Delete node with one child (case 1)

- Delete node and replace it with its only child.
- Example: delete 70 locates a node which has one child and replaces it with the child.



# Hibbard deletion: Delete node with two children (case 2)

- Delete node and replace it with successor (node with smallest of the larger keys).
    - Where is the smallest node of the right subtree?
        - Left most node of right subtree
- Move successor's child (if any) where successor was. Example: Delete 50



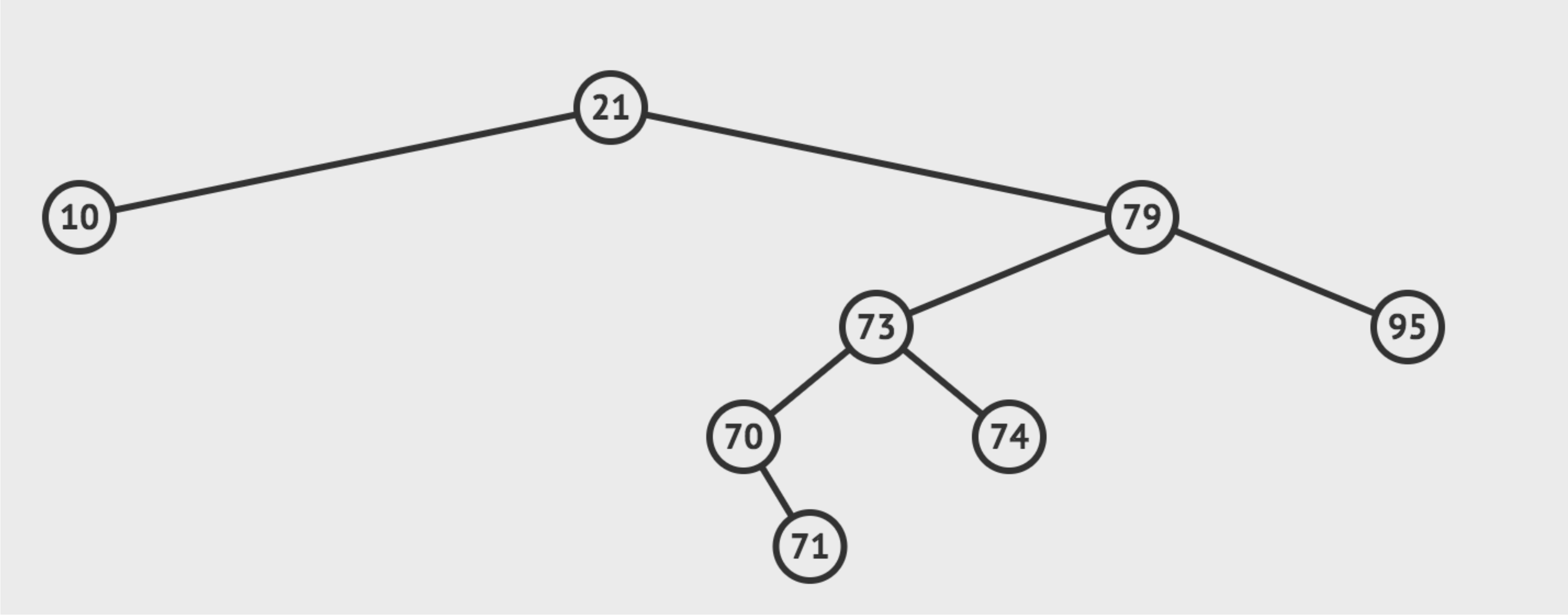
```
public void delete(Key key) {
    root = delete(root, key);
}

 private Node delete(Node x, Key key) {
     if (x == null) return null;

     int cmp = key.compareTo(x.key);    // compare key to node
     if (cmp < 0)
         x.left  = delete(x.left,  key);  // Search for key
     else if (cmp > 0)
         x.right = delete(x.right, key);
     else {                               // key found
         if (x.right == null)             // No right child
             return x.left;
         if (x.left  == null)             // No left child
             return x.right;
         Node t = x;                      // replace with successor
         x = min(t.right);                // find successor - min of x.right
         x.right = deleteMin(t.right);
         x.left = t.left;
     }
     x.size = size(x.left) + size(x.right) + 1;
     return x;
 }
```

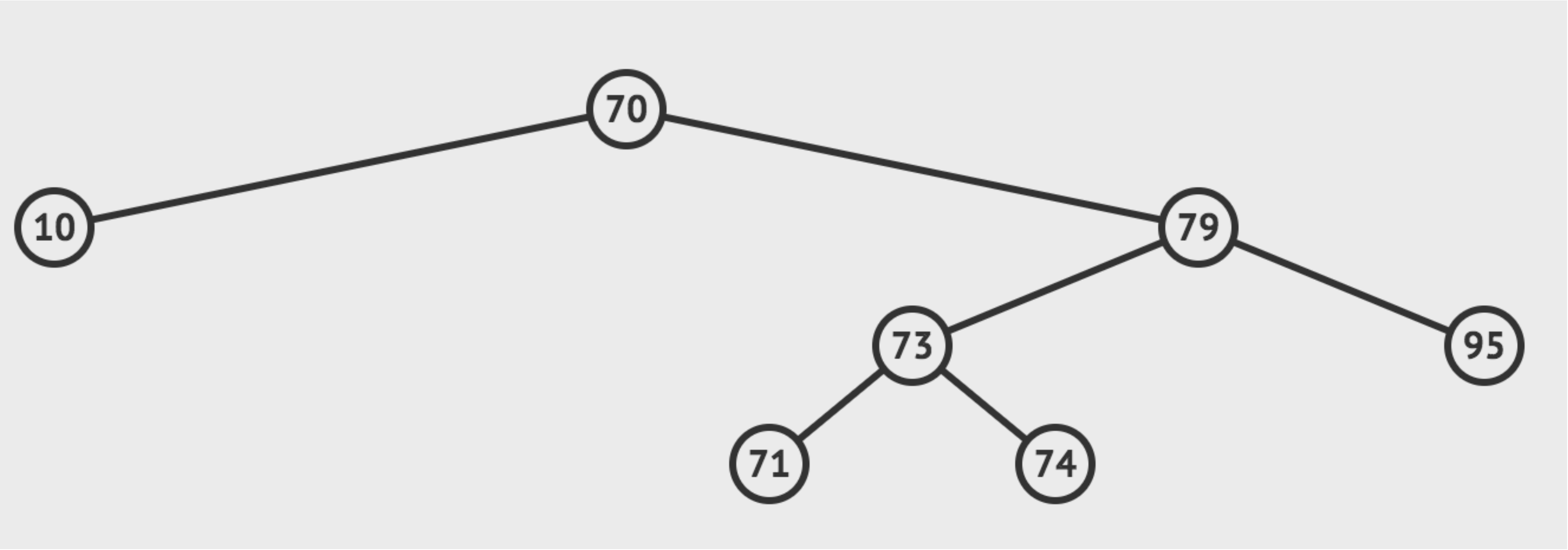
# Practice Time

- Delete the node 21 following Hibbard's deletion



# Answer

- Delete the node 21 following Hibbard's deletion



# Hibbard's deletion

- Unsatisfactory solution. If we were to perform many insertions and deletions the BST ends up being not symmetric and skewed to the left.
  - Extremely complicated analysis, but average cost of deletion ends up being $\sqrt{n}$. Let's simplify things by saying it stays $O(\log n)$.
  - No one has proven that alternating between the predecessor and successor will fix this.
- Hibbard devised the algorithm in 1962. Still no algorithm for efficient deletion in Binary Search Trees! Open problem.
- Overall, BSTs can have $O(n)$ worst-case for search, insert, and delete. We want to do better (see future lectures).

## Lecture 19: Binary Search Trees

- Binary Search Trees

# Readings:

- Textbook: Chapters 3.1 (Pages 362–386) and 3.2 (Pages 396–414)
- Website:
  - https://algs4.cs.princeton.edu/31elementary/
  - https://algs4.cs.princeton.edu/32bst/
- Visualization:
  - https://visualgo.net/en/bst

# Practice Problems:

- 3.1.1-3.1.6, 3.2.1-3.2.13

# Lecture 19: 2-3 Search Trees

- 2-3 Search Trees
- Search
- Insertion
- Construction
- Performance

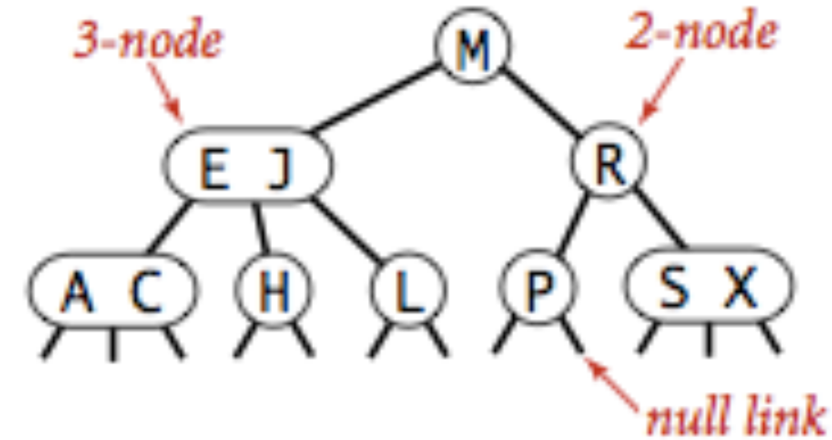
Some slides adopted from Algorithms 4th Edition or COS246

## The story so far

- The symbol table/dictionary is a fundamental data type.
- Naive implementations (arrays/linked lists sorted or unsorted) are way too slow.
- Binary search trees work well in the average case, but can grow too tall and imbalanced in the worst case.
- Question of the day: How to balance search trees?

# Order of growth for symbol table operations

| | Worst case | | | Average case | | |
|---|---|---|---|---|---|---|
| | **Search** | **Insert** | **Delete** | **Search** | **Insert** | **Delete** |
| **BST** | $n$ | $n$ | $n$ | $\log n$ | $\log n$ | $\sqrt{n}$ |
| **Goal** | $\log n$ | $\log n$ | $\log n$ | $\log n$ | $\log n$ | $\log n$ |



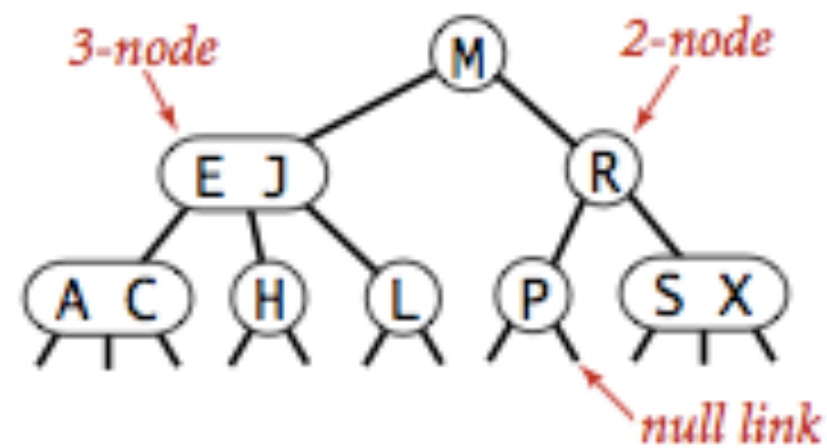

Anatomy of a 2-3 search tree

## 2-3 tree

- Definition: A 2-3 tree is either empty or a
  - 2-node: one key (and associated value) and two links, a left to a 2-3 search tree with smaller keys, and a right to a 2-3 search tree with larger keys (similarly to standard BSTs), or a
  - 3-node: two keys (and associated values) and three links, a left to a 2-3 search tree with smaller keys, a middle to a 2-3 search tree with keys between the node's keys, and a right to a 2-3 search tree with larger keys.
- Symmetric order: In-order traversal yields keys in ascending order.
- Perfect balance: Every path from root to null link (empty tree) has the same length.

# Example of a 2-3 tree

- 2-node, business as usual with BSTs.
  - (e.g.,  EJ are smaller than M and R is larger than M).
- In 3-node,
  - left link points to 2-3 search tree with smaller keys than first key,
    - (e.g., AC are smaller than E.)
  - middle link points to 2-3 search tree with keys between first and second key,
    - (e.g. H is between E and J.)
  - right link points to 2-3 search tree with keys larger than second key.
    - (e.g, L is larger than J).



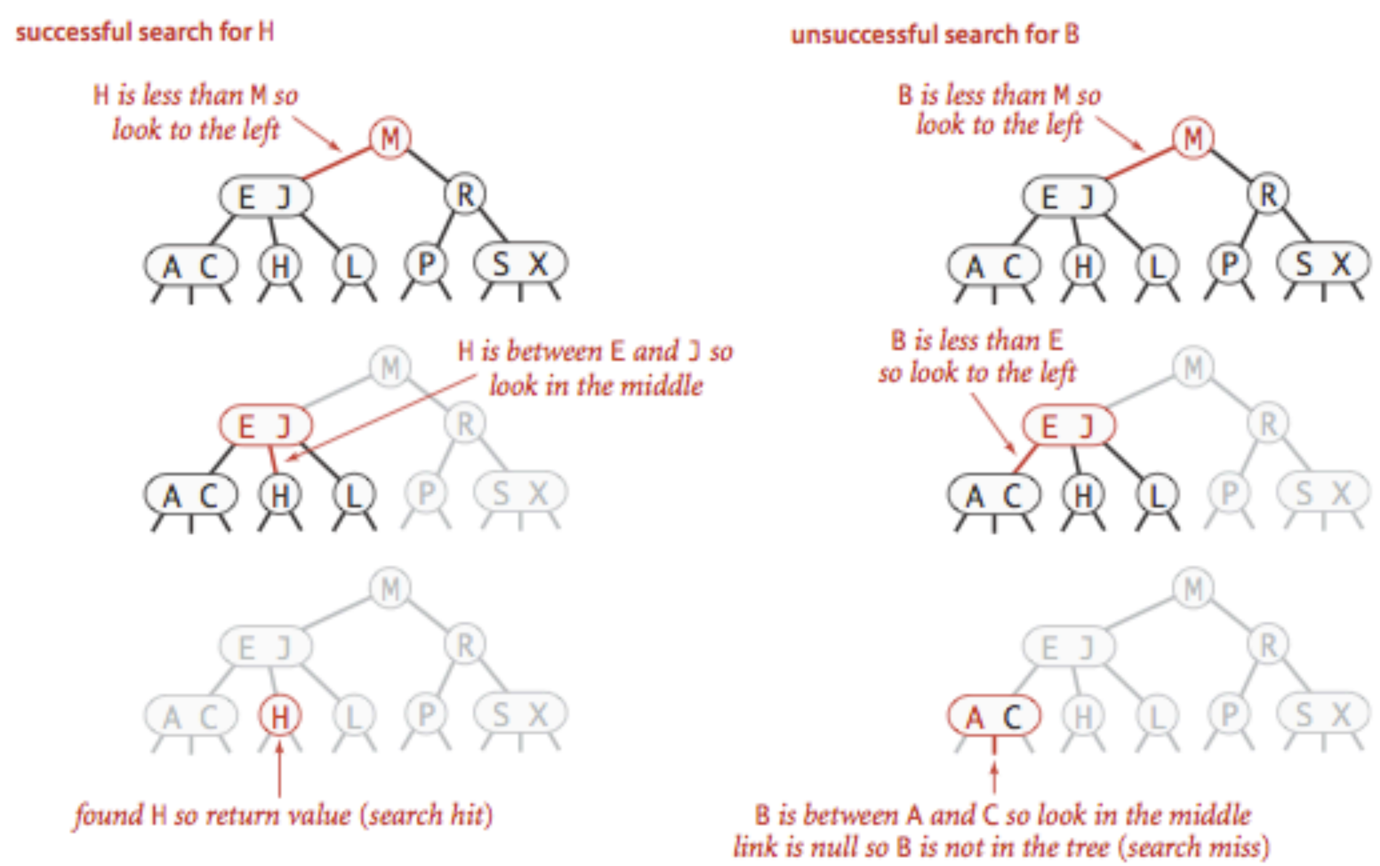

**Anatomy of a 2-3 search tree**

## Lecture 24: 2-3 Search Trees

- 2-3 Search Trees
- Search
- Insertion
- Construction
- Performance

# How to search for a key

- Compare search key against (every) key in node.
- Find interval containing search key (left, potentially middle, or right).
- Follow associated link, recursively.




Search hit (left) and search miss (right) in a 2-3 tree


http://algs4.cs.princeton.edu

# 3.3 2–3 Tree Demo

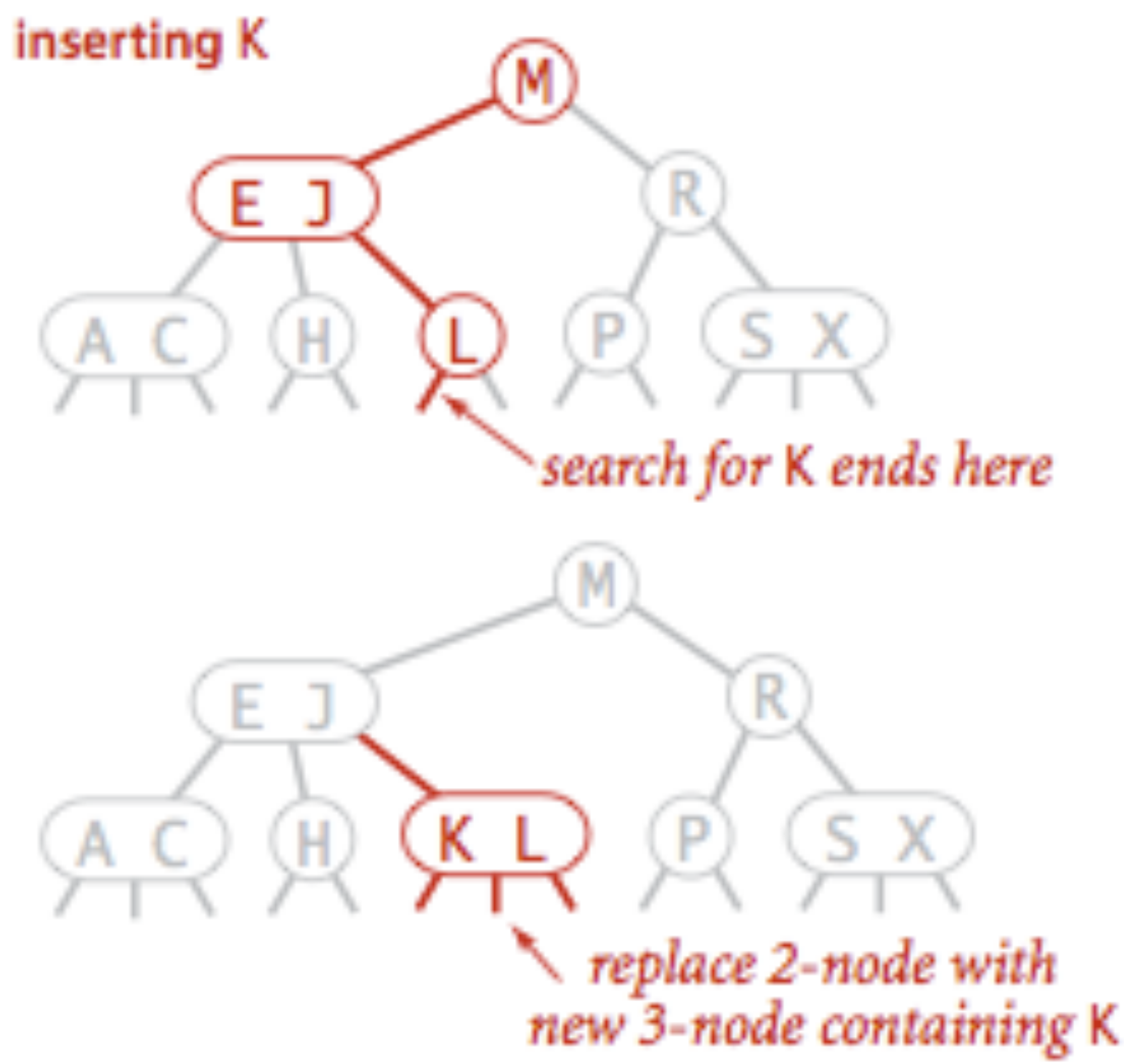
- *search*
- *insertion*
- *construction*

# Lecture 24: 2-3 Search Trees

- 2-3 Search Trees
- Search
- Insertion
- Construction
- Performance

# How to insert into a 2-node

- Add new key to 2-node to create a 3-node.



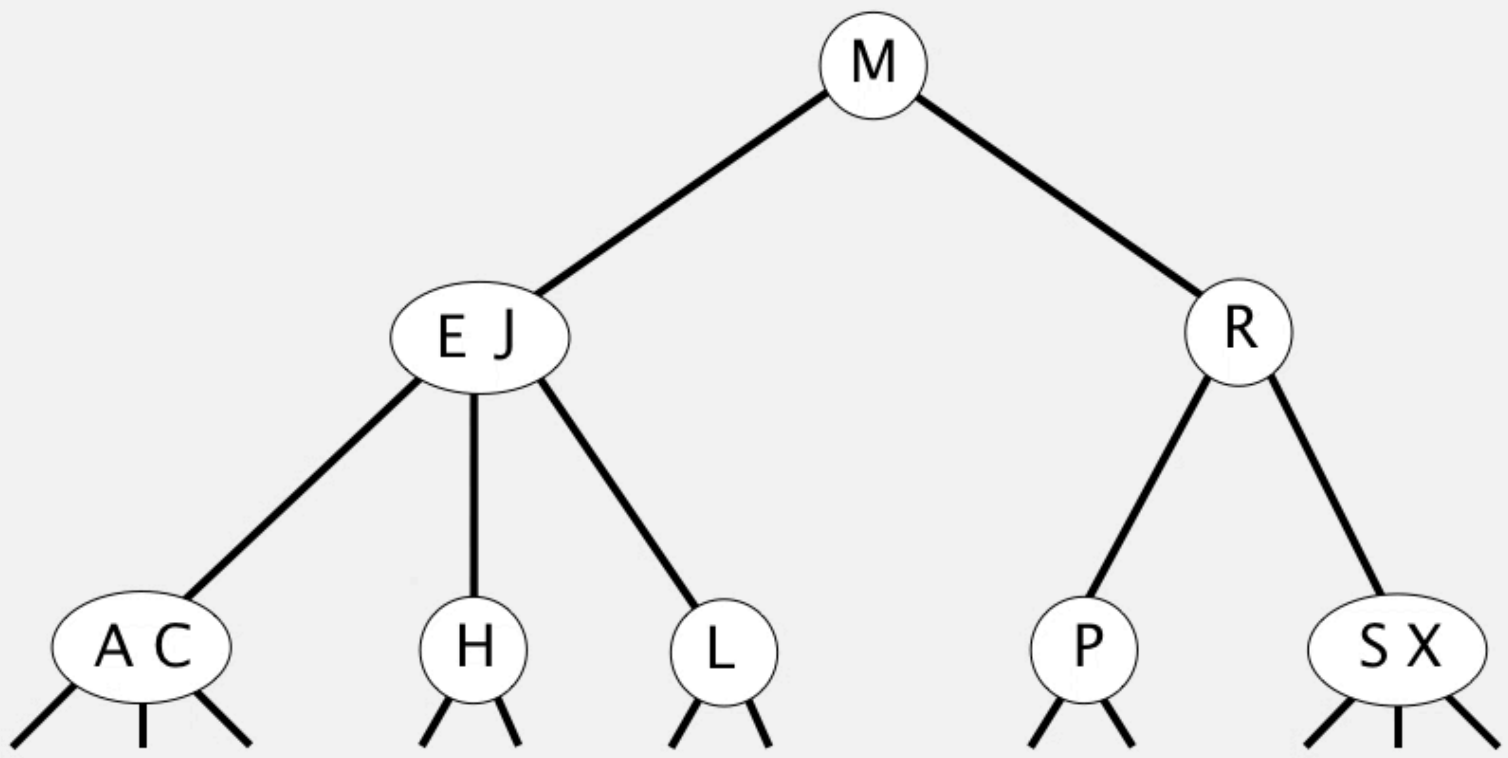

**Insert into a 2-node**

# 2-3 tree demo: insertion

Insert into a 2-node at bottom.

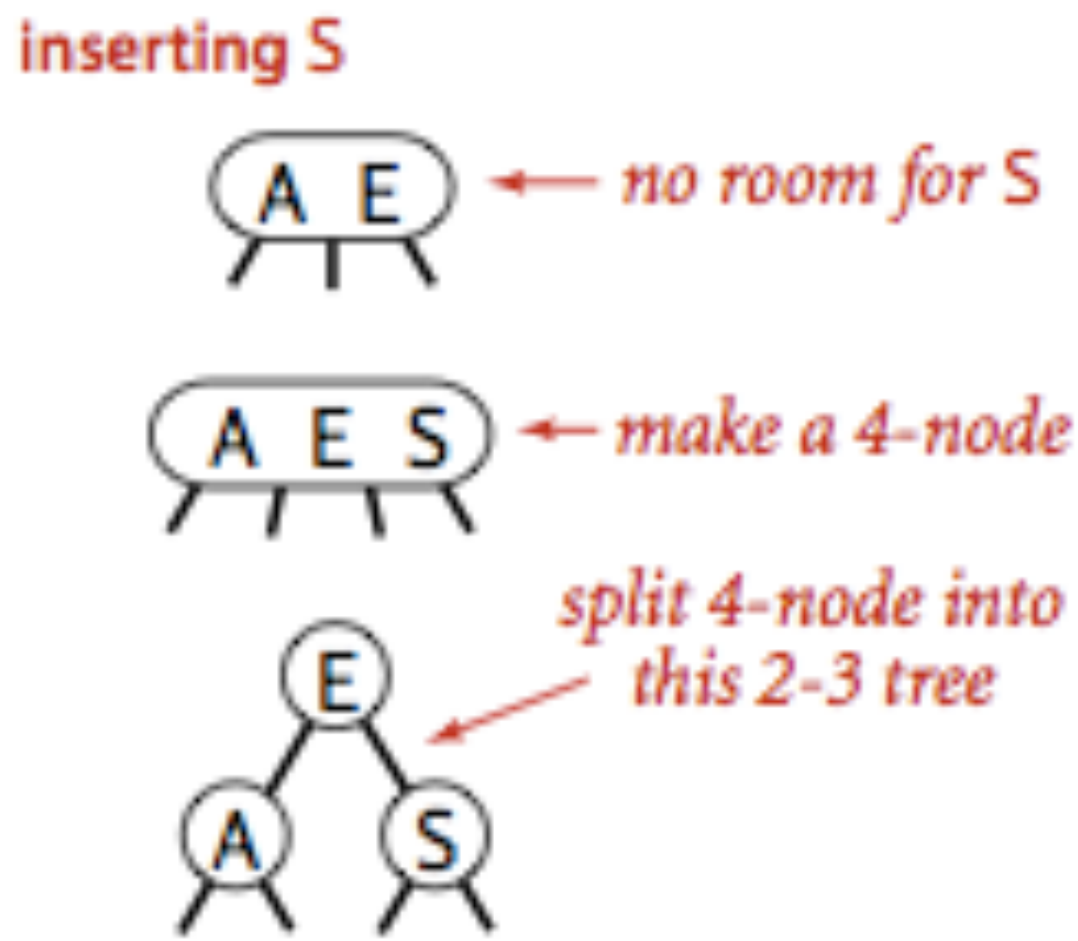
- Search for key, as usual.
- Replace 2-node with 3-node.

insert K



## How to insert into a tree consisting of a single 3-node

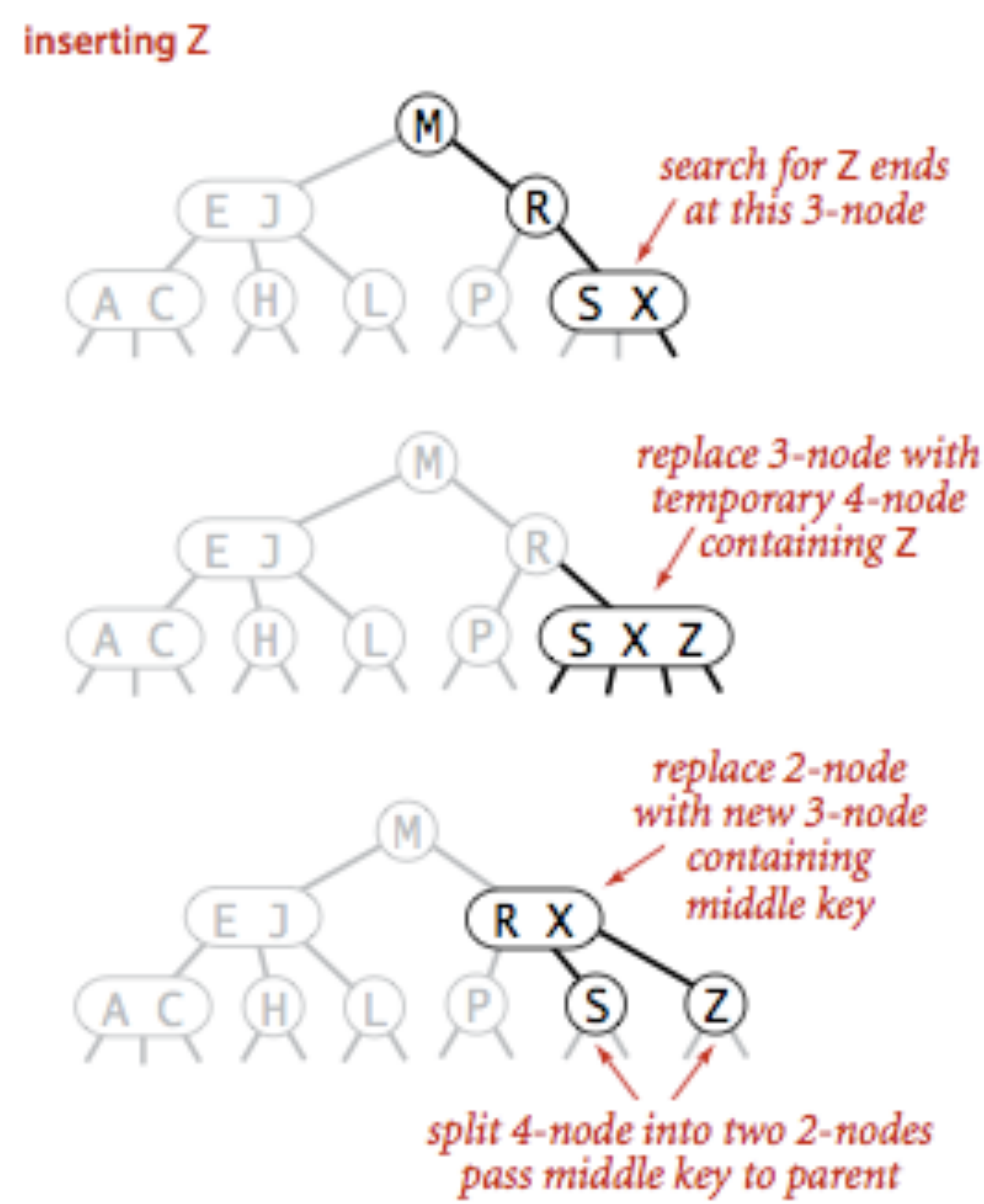
- Add new key to 3-node to create a temporary 4-node.
- Move middle key in 4-node into parent.
- Split 4-node into two 2-nodes.
- Height went up by 1.




Insert into a single 3-node

# How to insert into a 3-node whose parent is a 2-node

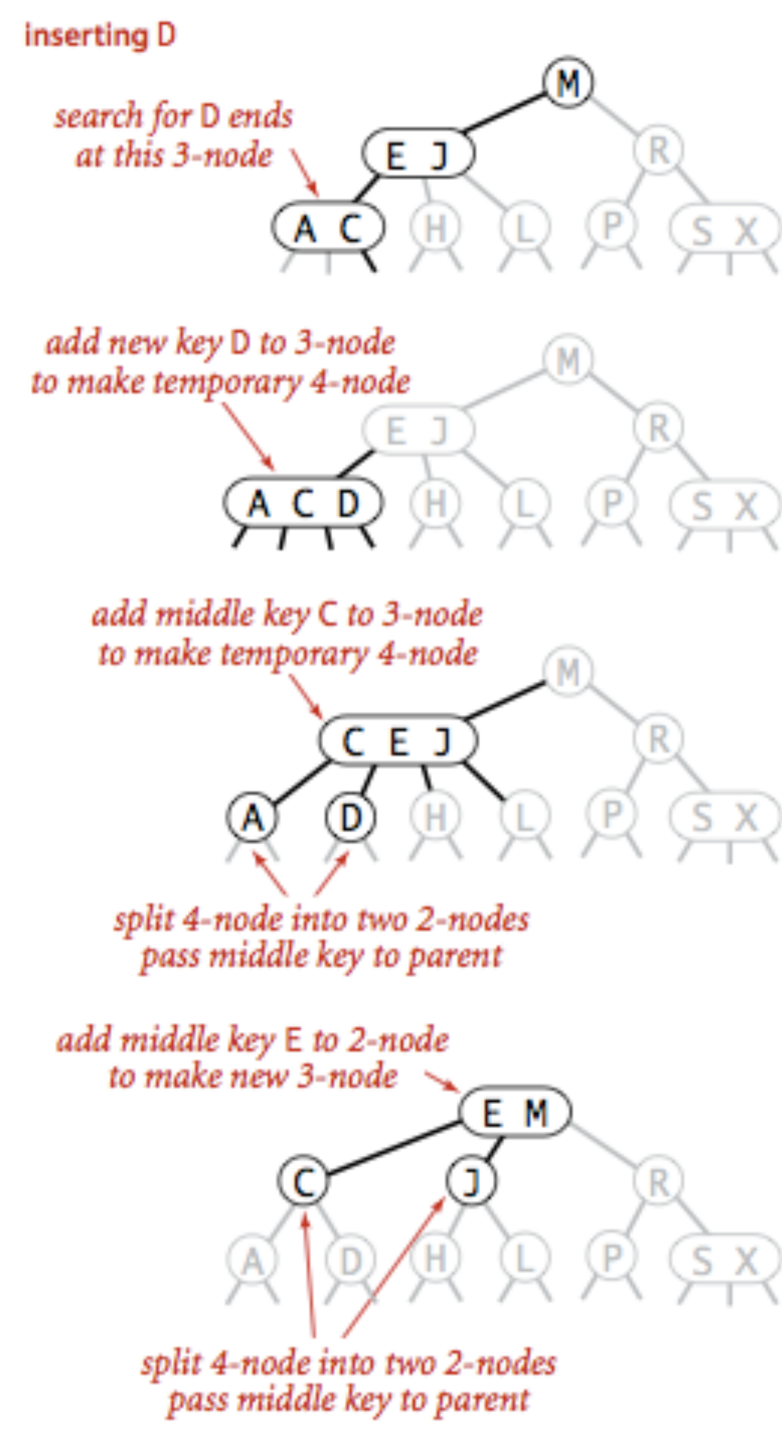
- Add new key to 3-node to create a temporary 4-node.
- Split 4-node into two 2-nodes and pass middle key to parent.
- Replace 2-node parent with 3-node.




Insert into a 3-node whose parent is a 2-node

# How to insert into a 3-node whose parent is a 3-node

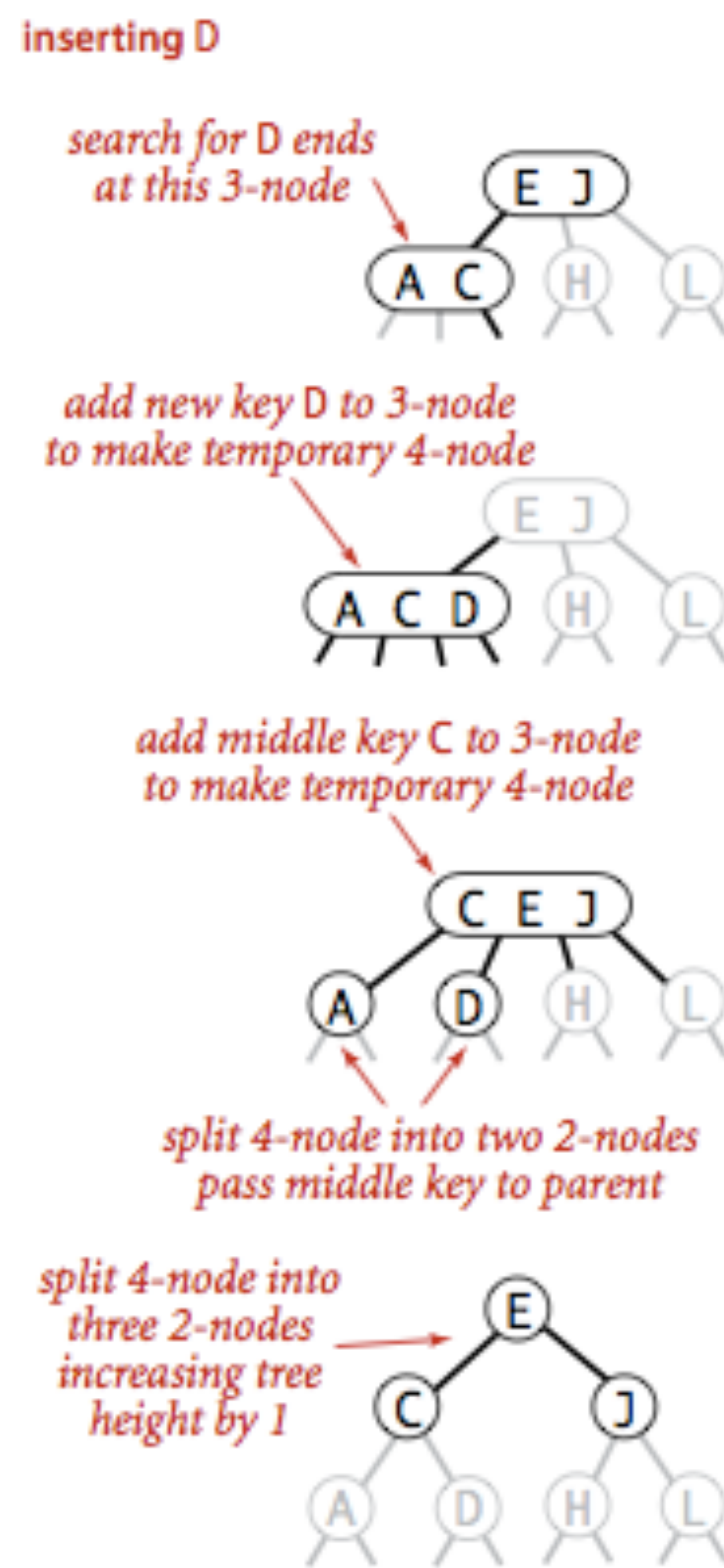
- Add new key to 3-node to create a temporary 4-node.
- Split 4-node into two 2-nodes and pass middle key to parent creating a temporary 4-node.
- Split 4-node into two 2-nodes and pass middle key to parent.
- Repeat up the tree, as necessary.




Insert into a 3-node whose parent is a 3-node

## Splitting the root

- If end up with a temporary 4-node root, split into three 2-nodes.
- Increases height by 1 but perfect balance is preserved.



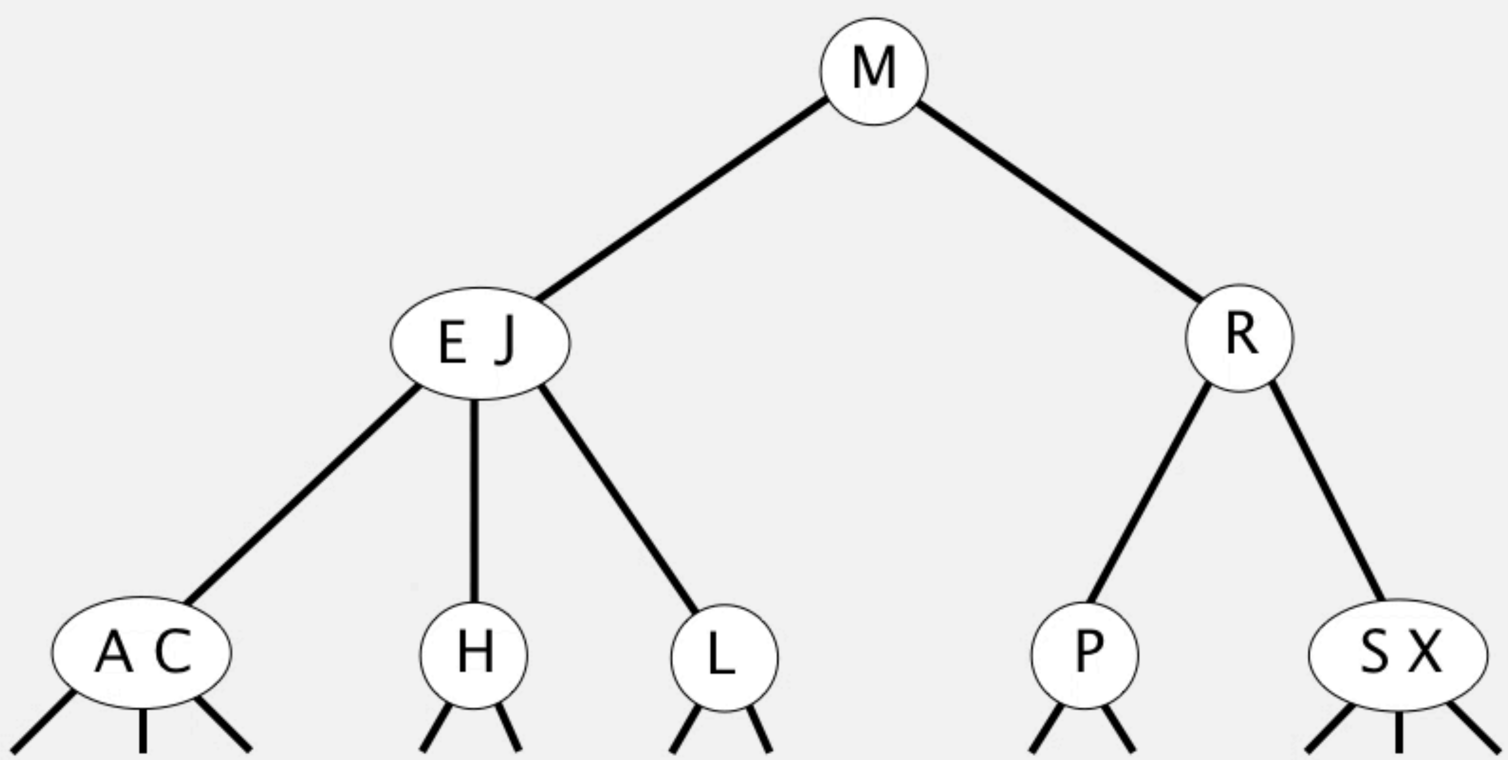

Splitting the root

# 2-3 tree demo: insertion

Insert into a 2-node at bottom.

- Search for key, as usual.
- Replace 2-node with 3-node.

insert K

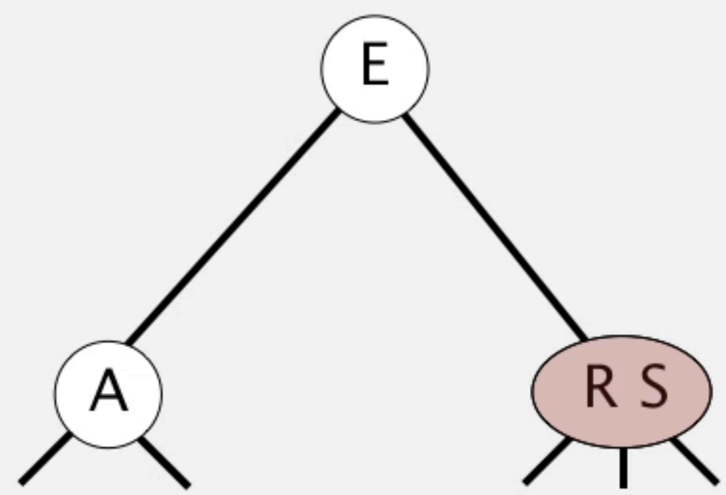


## Lecture 24: 2-3 Search Trees

- 2-3 Search Trees
- Search
- Insertion
- Construction
- Performance

# 2-3 tree demo: construction

insert R

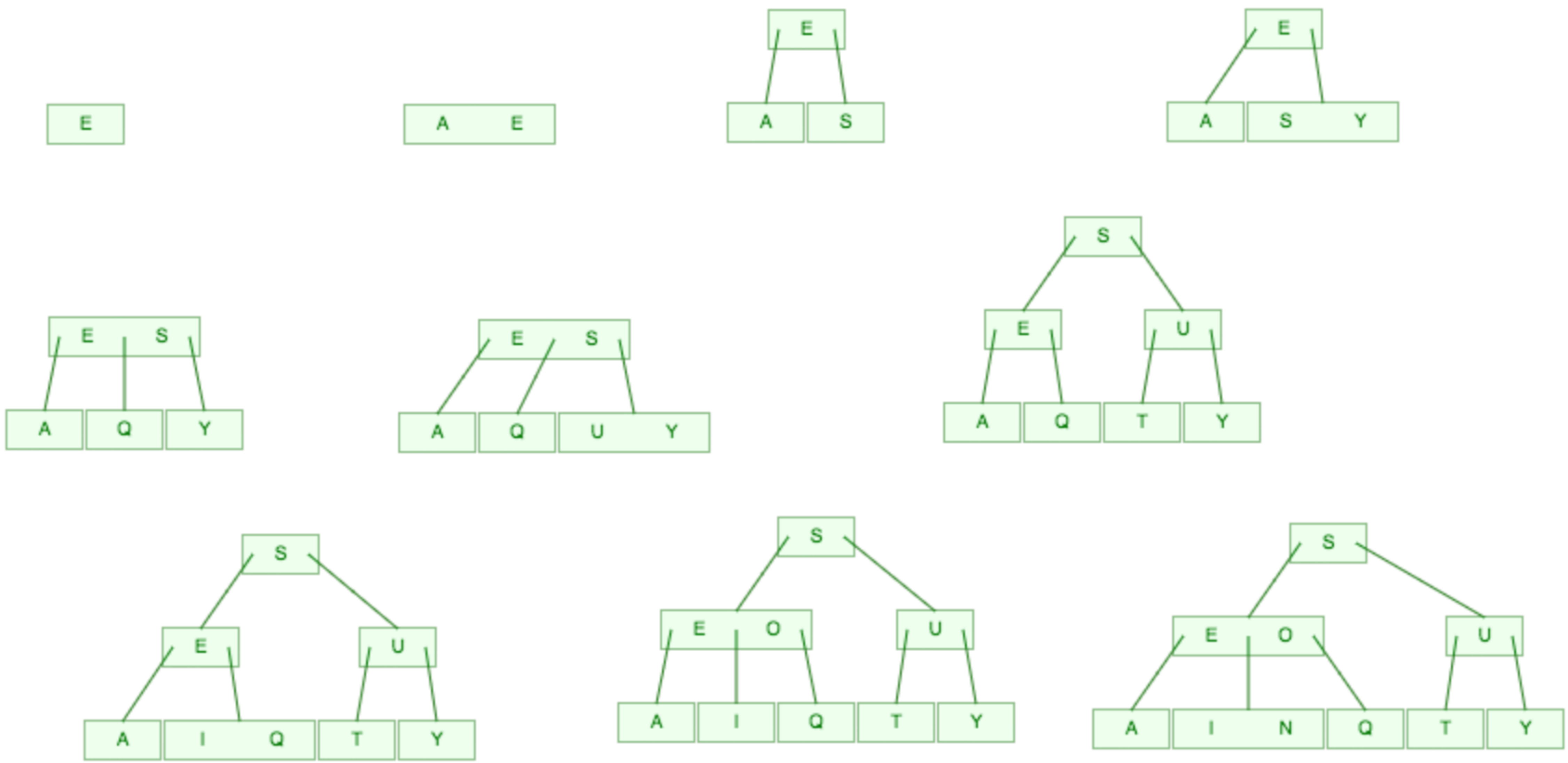


## Practice Time

- Draw the 2-3 tree that results when you insert the keys:
  E A S Y Q U T I O N in that order in an initially empty tree.

# Answer

- E A S Y Q U T I O N




https://www.cs.usfca.edu/~galles/visualization/BTree.html

## Lecture 24: 2-3 Search Trees

- 2-3 Search Trees
- Search
- Insertion
- Construction
- Performance

## Height of 2-3 search trees

- Worst case: $\log n$ (all 2-nodes).
- Best case: $\log_3 n = 0.631 \log n$ (all 3-nodes)
  - That means that storing a million nodes will lead to a tree with height between 12 and 20, and storing a billion nodes to a tree with height between 18 and 30 (not bad!).
- Search and insert are $O(\log n)$!
- But implementation is a pain and the overhead incurred could make the algorithms slower than standard BST search and insert.
- We did provide insurance against a worst case but we would prefer the overhead cost for that insurance to be low. Stay tuned! We will see a much easier way.

# Summary for symbol table/dictionary operations

| | Worst case | | | Average case | | |
|---|---|---|---|---|---|---|
| | **Search** | **Insert** | **Delete** | **Search** | **Insert** | **Delete** |
| **BST** | $n$ | $n$ | $n$ | $\log n$ | $\log n$ | $\sqrt{n}$ |
| **2-3 search trees** | $\log n$ | $\log n$ | $\log n$ | $\log n$ | $\log n$ | $\log n$ |

## Lecture 24: 2-3 Search Trees

- 2-3 Search Trees
- Search
- Insertion
- Construction
- Performance

## Readings:

- Textbook: Chapter 3.3 (Pages 424-431)
- Website:
  - https://algs4.cs.princeton.edu/33balanced/

## Practice Problems:

- 3.3.2-3.3.5