SORTING

CS062

DATA STRUCTURES AND ADVANCED PROGRAMMING

18: Binary Search Trees

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Lecture 18: Binary Search Trees

- Heapsort
- Dictionaries
- Binary Search Trees

Some slides adopted from Algorithms 4th Edition or COS226

Option 3: Binary heap

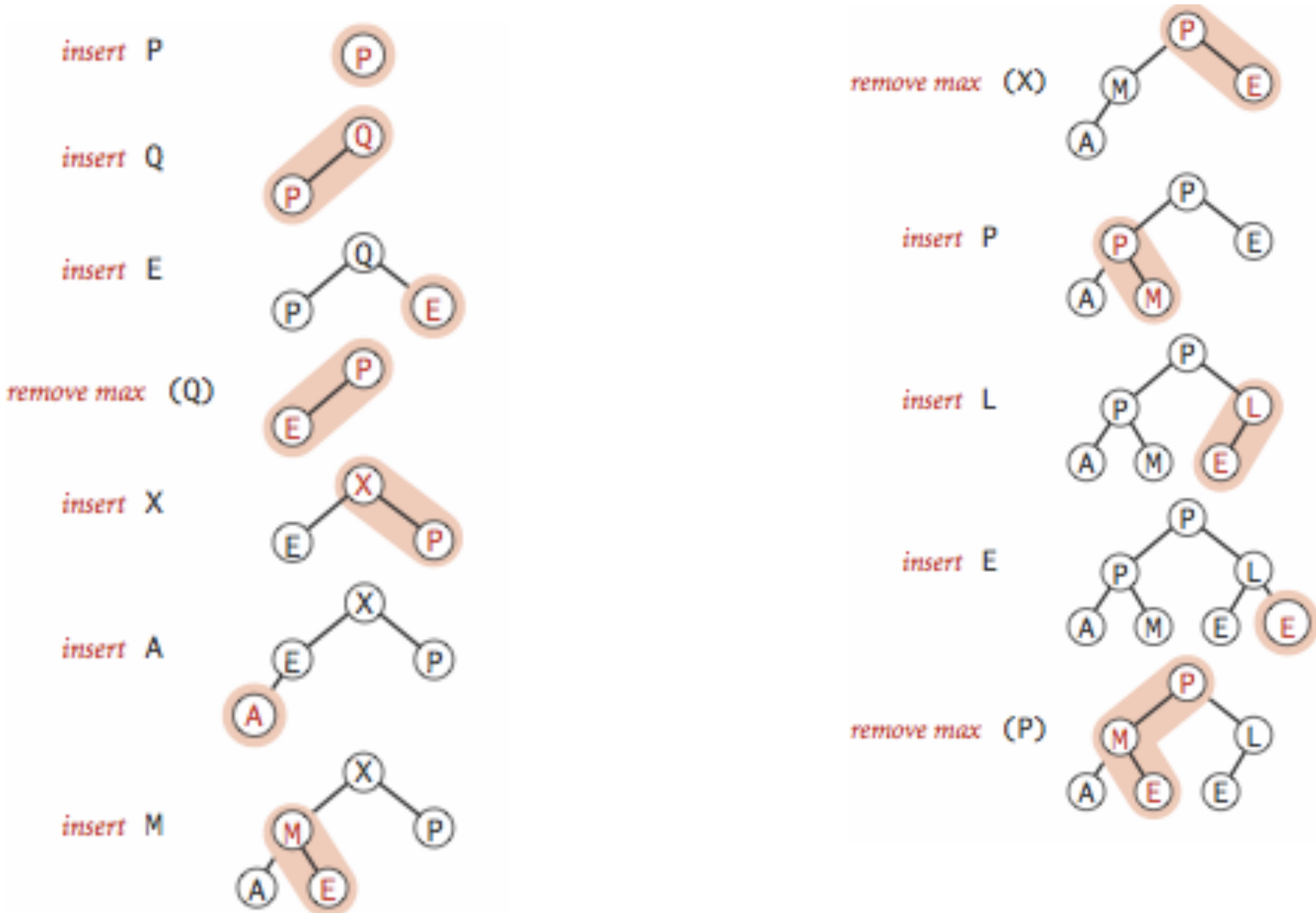
- Will allow us to both insert and delete max in $O(\log n)$ running time.
- There is no way to implement a priority queue in such a way that insert and delete max can be achieved in $O(1)$ running time.
- Priority queues are synonyms to binary heaps.

Practice Time

- Given an empty binary heap that represents a priority queue, perform the following operations:

1. Insert P
2. Insert Q
3. Insert E
4. Delete max
5. Insert X
6. Insert A
7. Insert M
8. Delete max
9. Insert P
10. Insert L
11. Insert E
12. Delete max

Answer


insert P
insert Q
insert E
remove max (Q)
insert X
insert A
insert M
remove max (X)
insert P
insert L
insert E
remove max (P)

Lecture 18: Heapsort, Dictionary, BST

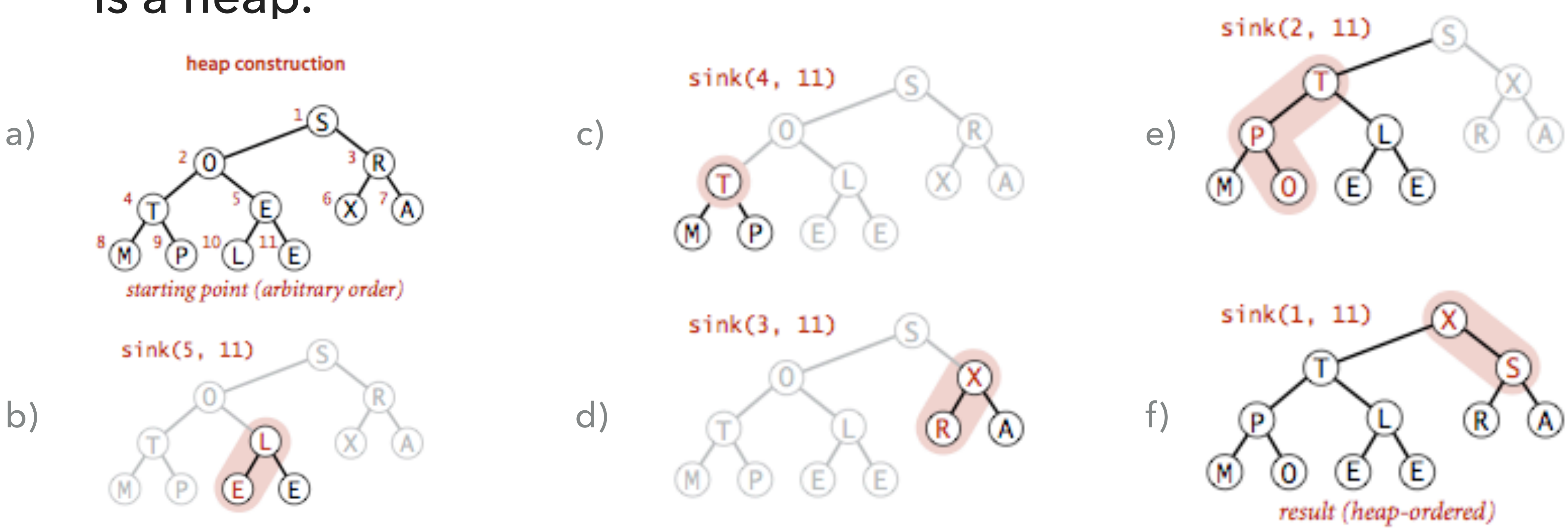
- Heapsort

Basic plan for heap sort

- Use a priority queue to develop a sorting method that works in two steps:
- 1) Heap construction: build a binary heap with all n keys that need to be sorted.
- 2) Sortdown: repeatedly remove and return the maximum key.

$O(n)$ Heap construction

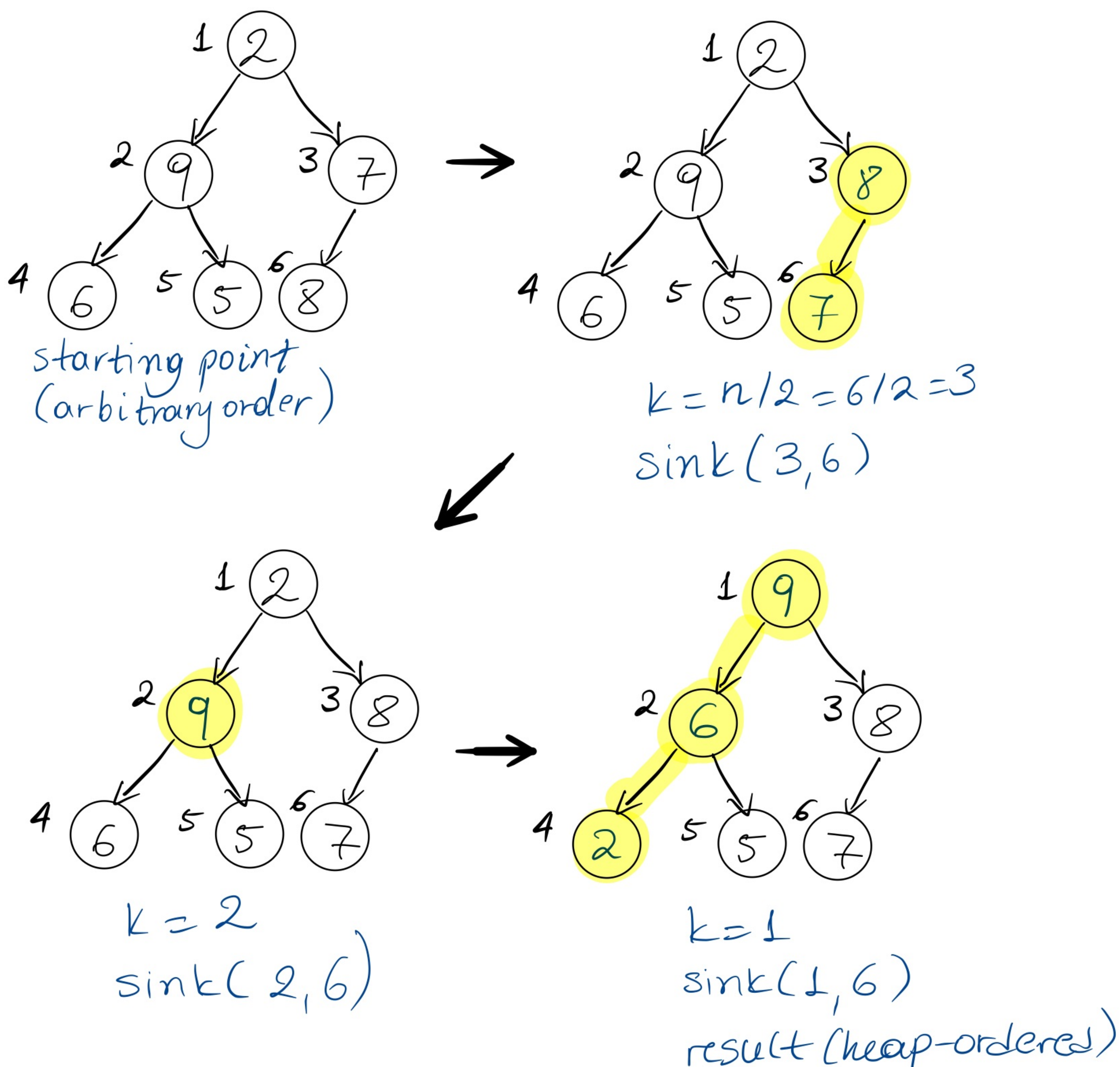
- Construct complete binary tree with elements
- Ignore all leaves (indices n/2+1,…,n).
- ```
  for(int k = n/2; k >= 1; k--)
          sink(a, k, n);
  ```
- Key insight: After `sink(a,k,n)` completes, the subtree rooted at k is a heap.


  ```

Practice Time

- Run the first step of heapsort, heap construction, on the array [2,9,7,6,5,8].

Answer: Heap construction

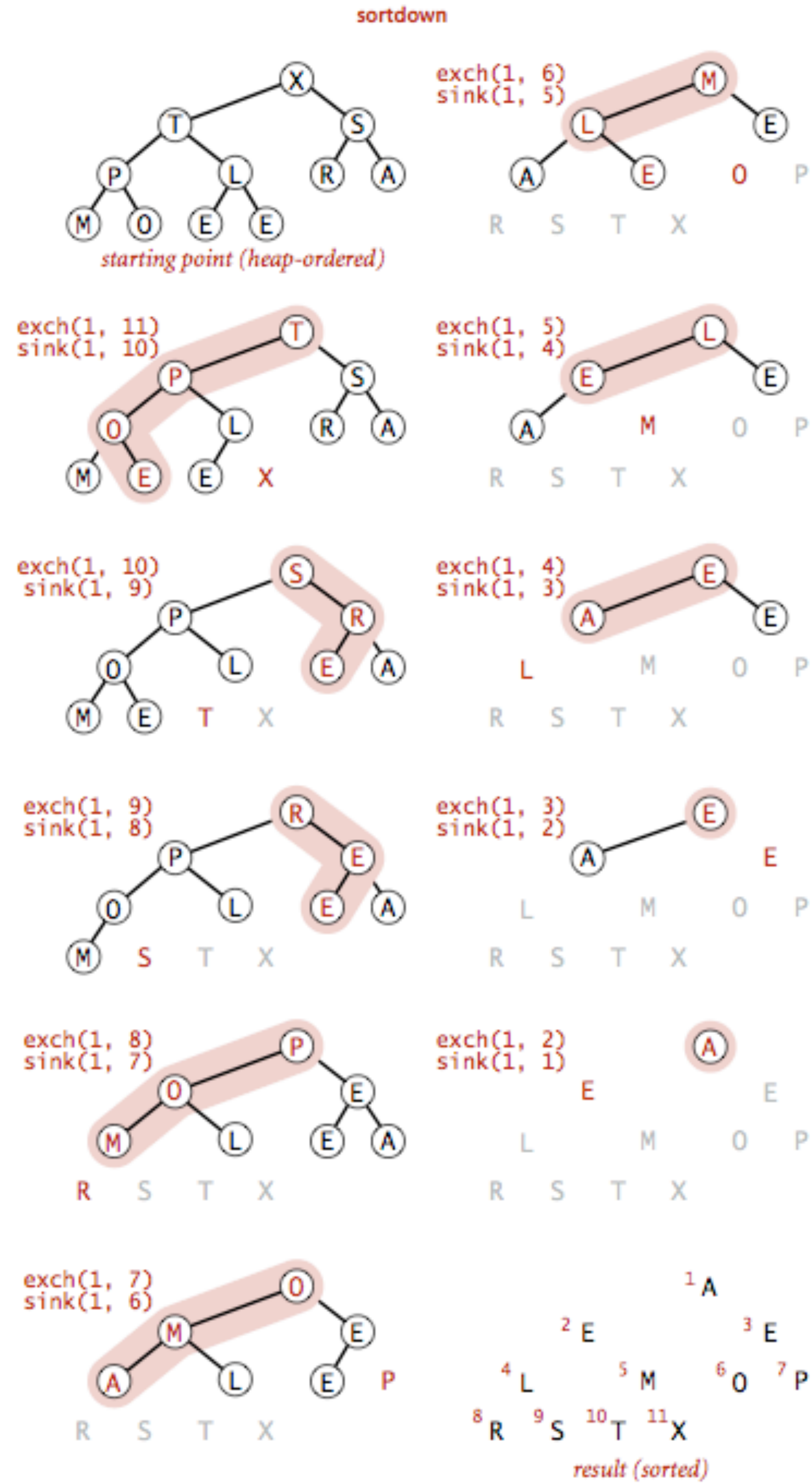


Sortdown

- Remove the maximum, one at a time, but leave in array instead of nulling out.
- ```
  while(n>1){
     exch(a, 1, n--);
     sink(a, 1, n);
  }
  ```
- Key insight: After each iteration the array consists of a heap-ordered subarray followed by a sub-array in final order.
  ```

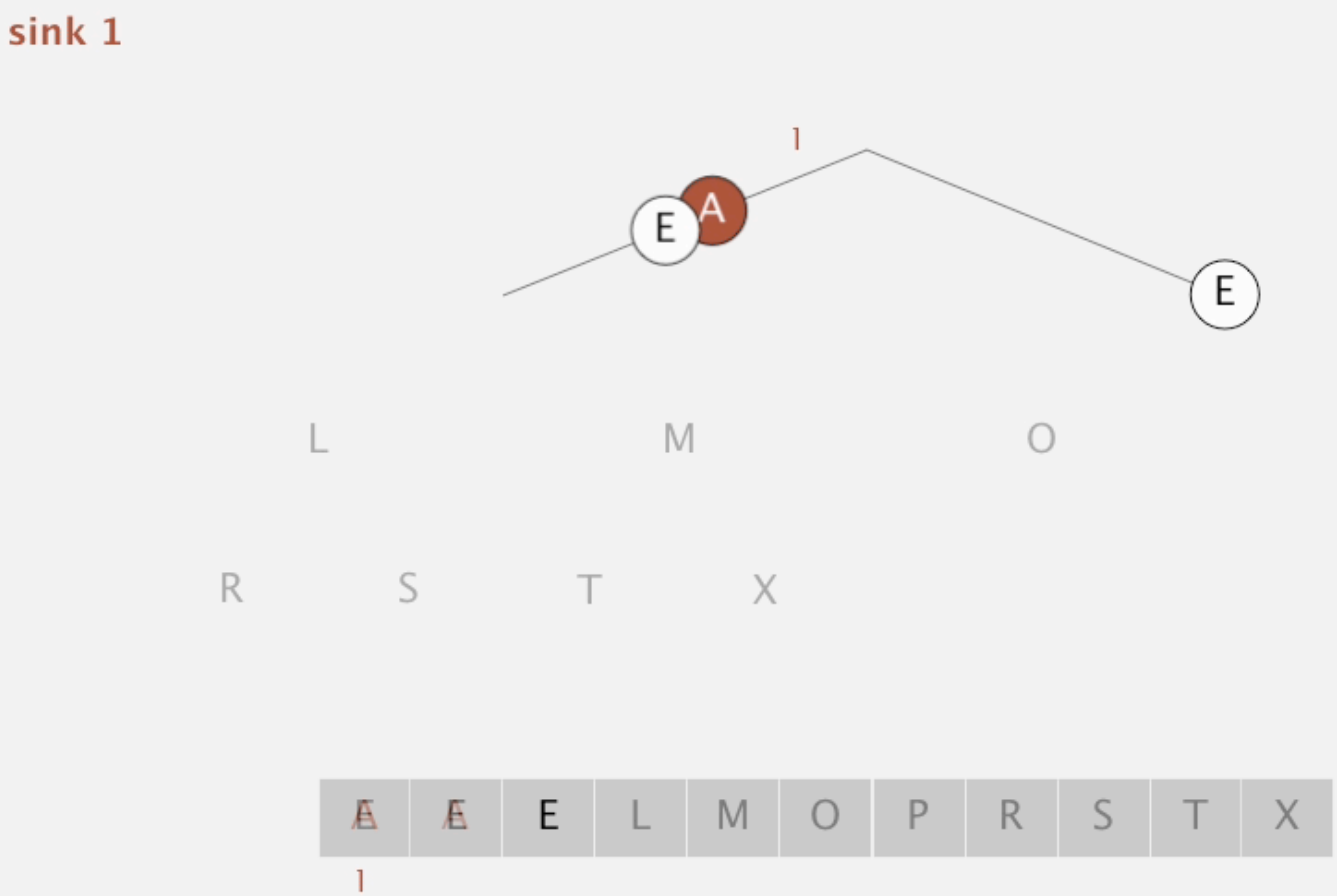
Sortdown

- ```
  while(n>1){
     exch(a, 1, n--);
     sink(a, 1, n);
  }
  ```


  ```

Heapsort demo

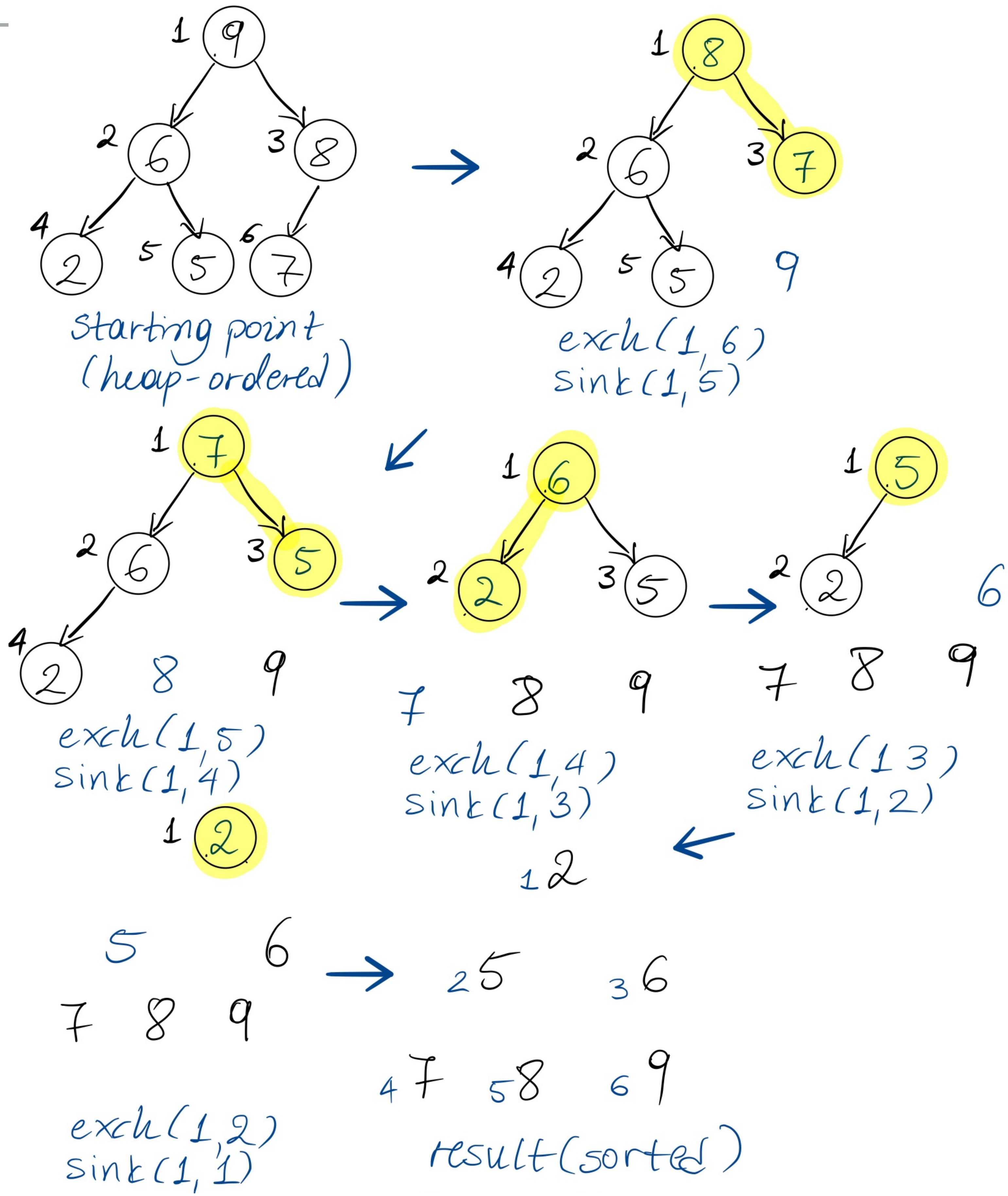
Sortdown. Repeatedly delete the largest remaining item.



Practice Time

- Given the heap you constructed before, run the second step of heapsort, sortdown, to sort the array [2,9,7,6,5,8].

Answer: Sortdown



Heapsort analysis

- Heap construction makes $O(n)$ exchanges and $O(n)$ compares.
- Sortdown and therefore the entire heap sort $O(n \log n)$ exchanges and compares.
- In-place sorting algorithm with $O(n \log n)$ worst-case!
- Remember:
  - mergesort: not in place, requires linear extra space.
  - quicksort: quadratic time in worst case.
- Heapsort is optimal both for time and space in terms of Big-O, but:
  - Inner loop longer than quick sort.
  - Poor use of cache. Why?
  - Not stable.

Sorting: Everything you need to remember about it!

Which Sort	In place	Stable	Best	Average	Worst	Remarks
Selection	X		$O(n^2)$	$O(n^2)$	$O(n^2)$	n exchanges
Insertion	X	X	$O(n)$	$O(n^2)$	$O(n^2)$	Use for small arrays or partially ordered
Merge		X	$O(n \log n)$	$O(n \log n)$	$O(n \log n)$	Guaranteed performance; stable
Quick	X		$O(n \log n)$	$O(n \log n)$	$O(n^2)$	$n \log n$ probabilistic guarantee; fastest!
Heap	X		$O(n \log n)$	$O(n \log n)$	$O(n \log n)$	Guaranteed performance; in place

Lecture 22: Priority Queues and Heapsort

- Priority Queue
- Heapsort

Readings:

- Textbook:
  - Chapter 2.4 (Pages 308-327), 2.5 (336-344)
- Website:
  - Priority Queues: https://algs4.cs.princeton.edu/24pq/
- Visualization:
  - Create (nlogn) and heapsort: https://visualgo.net/en/heap

Practice Problems:

- 2.4.1-2.4.11. Also try some creative problems.

Readings:

- Textbook:
  - Chapter 2.4 (Pages 308-327)
- Website:
  - Priority Queues: https://algs4.cs.princeton.edu/24pq/
- Visualization:
  - Insert and ExtractMax: https://visualgo.net/en/heap

Practice Problems:

- Practice with traversals of trees and insertions and deletions in binary heaps

Lecture 18: Binary Search Trees

- Dictionaries
- Binary Search Trees

Some slides adopted from Algorithms 4th Edition or COS226

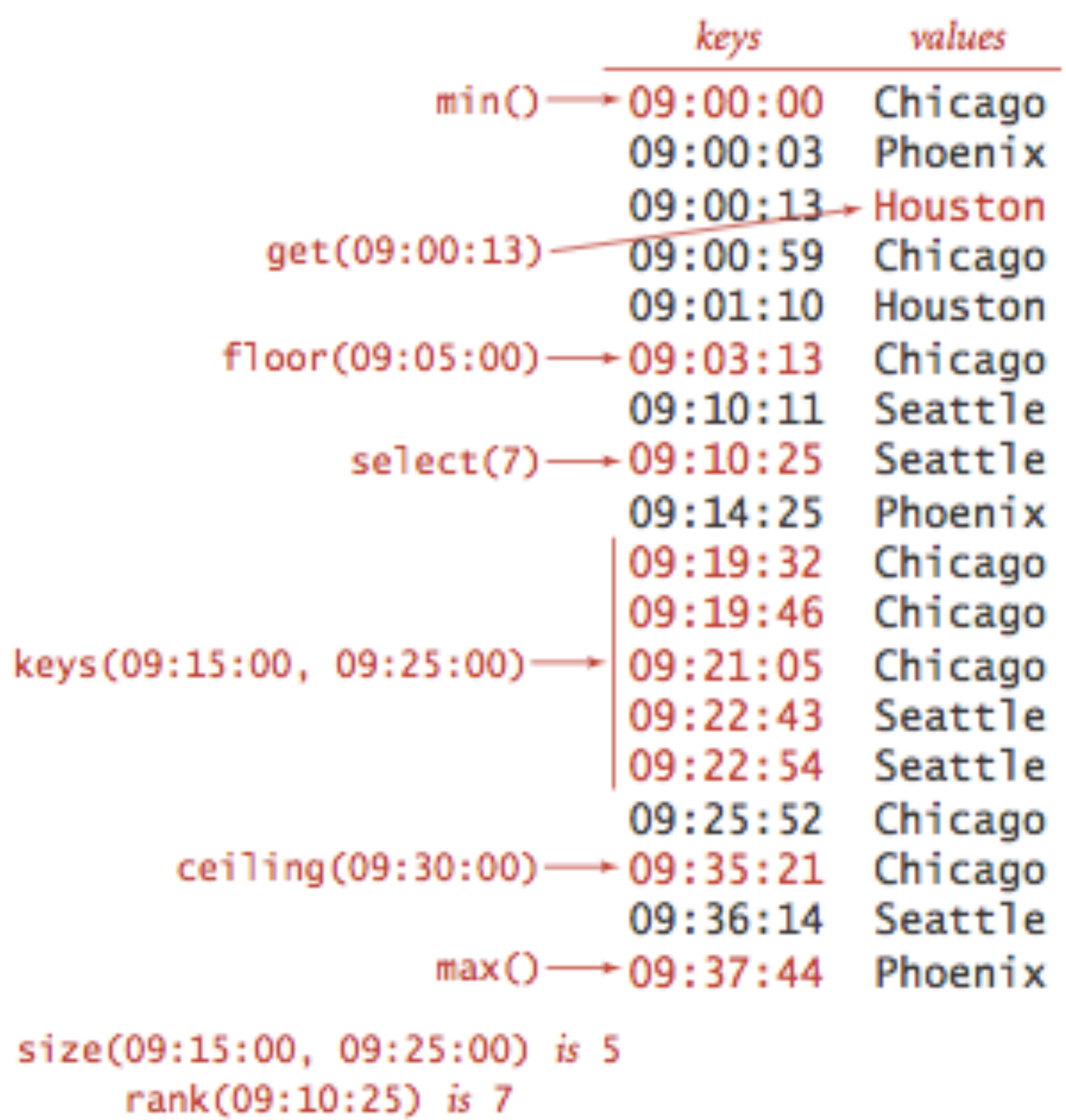
Dictionaries

- Also known as: symbol tables, maps, indices, associative arrays.
- Key-value pair abstractions that support two operations:
  - Insert a key-value pair.
  - Given a key, search for the corresponding value.
- Supported either with built-in or external libraries by the majority of programming languages.

Basic symbol table API

- `public class ST <Key extends Comparable<Key>, Value>`
- `ST()`: create an empty symbol table. By convention, values are not `null`.
- `void put(Key key, Value val)`: insert key-value pair.
    - Overwrites old value with new value if key already exists.
- `Value get(Key key)`: return value associated with key.
    - Returns `null` if key not present.
- `boolean contains(Key key)`: is there a value associated with key?
- `Iterable keys()`: all the keys in the symbol table.
- `void delete(Key key)`: delete key and associated value.
- `boolean isEmpty()`: is the symbol table empty?
- `int size()`: number of key-value pairs.

Ordered symbol tables

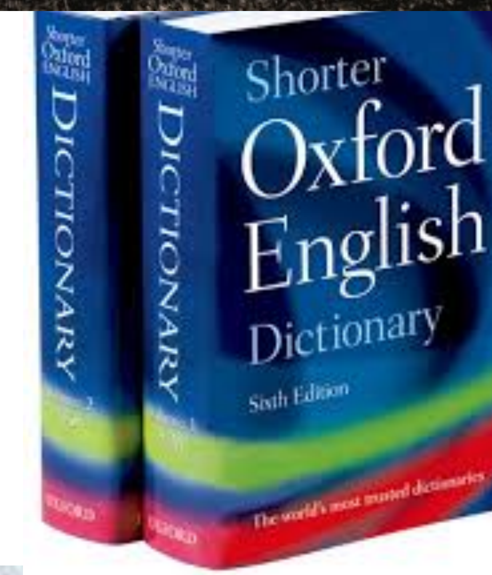


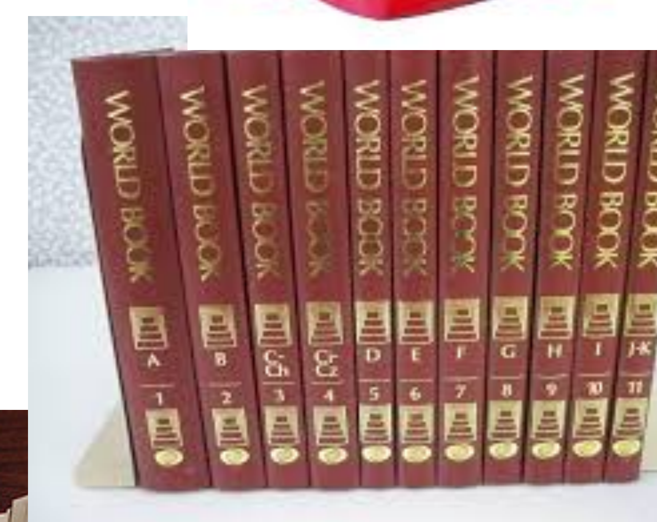
Ordered symbol table API

- `Key min()`: smallest key.
- `Key max()`: largest key.
- `Key floor(Key key)`: largest key less than or equal to given key.
- `Key ceiling(Key key)`: smallest key greater than or equal to given key.
- `int rank(Key key)`: number of keys less that given key.
- `Key select(int k)`: key with rank `k`.
- `Iterable keys()`: all keys in symbol table in sorted order.
- `Iterable keys(int lo, int hi)`: keys in `[lo, …, hi]` in sorted order.

Printed symbol tables are all around us

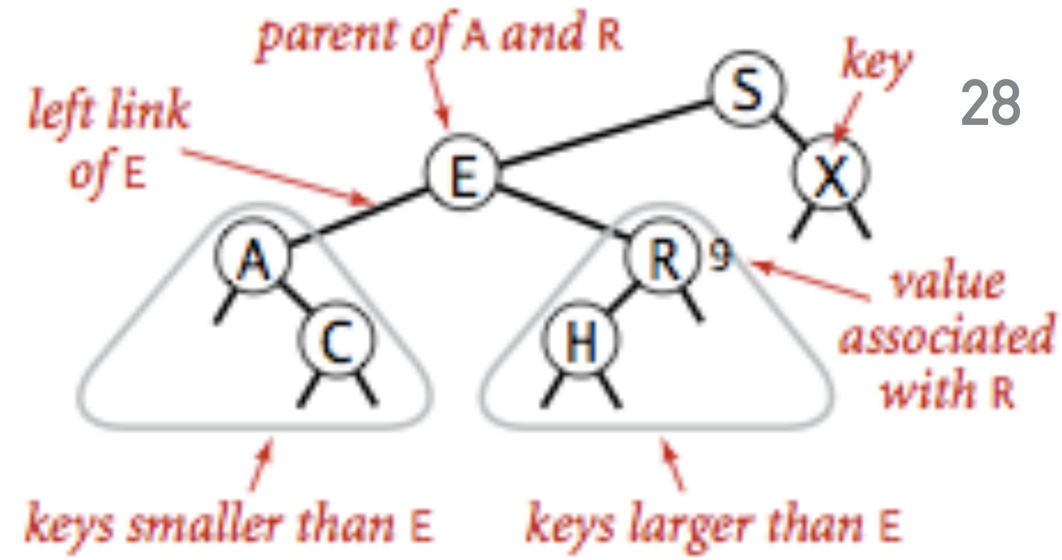
- Dictionary: key = word, value = definition.
- Encyclopedia: key = term, value = article.
- Phonebook: key = name, value = phone number.
- Math table: key = math functions and input, value = function output.
- Unsupported operations:
  - Add a new key and associated value.
  - Remove a given key and associated value.
  - Change value associated with a given key.





Lecture 23: Binary Search Trees

- Dictionaries
- Binary search Trees



Definitions

- Binary Search Tree: A binary tree in symmetric order.
- Symmetric order: Each node has a key, and every node's key is:
  - Larger than all keys in its left subtree.
  - Smaller than all keys in its right subtree.
- Our textbook uses BSTs to implement dictionaries, therefore each node holds a key-value pair. Other implementations hold only a key.

Differences between heaps and BSTs

	Heap	BST
Used to implement	Priority queues	Dictionaries
Supported operations	Insert, delete max	insert, search, delete, ordered operations
What is inserted	Keys	Key-value pairs
Underlying data structure	(Resizing) array	Linked nodes
Tree shape	Complete binary tree	Depends on data
Ordering of keys	Heap-ordered	Symmetrically-ordered
Duplicate keys allowed?	Yes	No*

*: when BSTs used to implement dictionaries.

BST representation of dictionaries

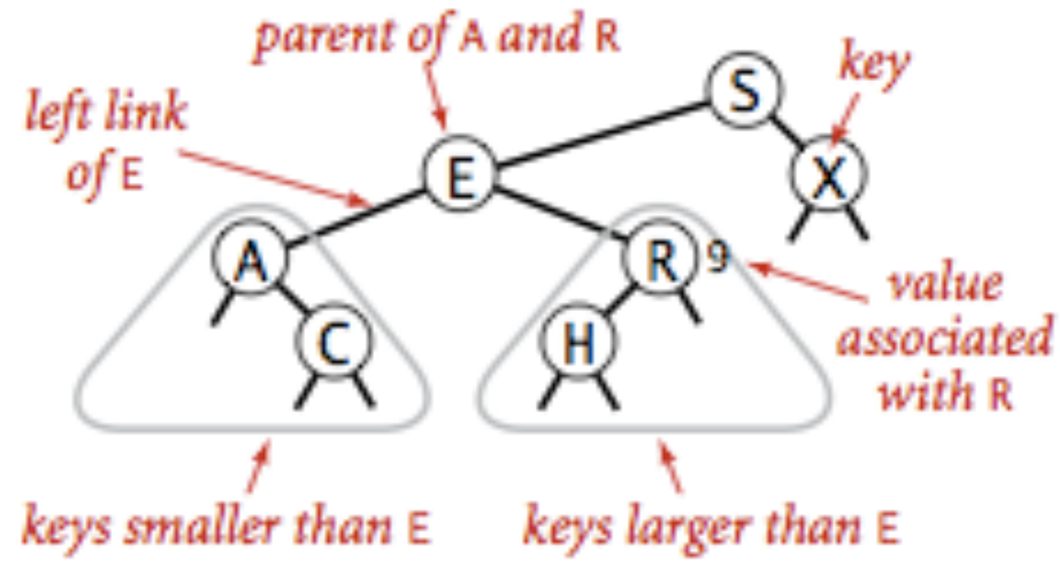
- We will use an inner class `Node` that is composed by:
    - A `Key` that is comparable and a `Value`
    - A reference to the root nodes of the left (smaller keys) and right (larger keys) subtrees.
    - Potentially, the total number of nodes in the subtree that has root at this node.
- A BST has a reference to a `Node root`.

BST and Node implementation

```
public class BST<Key extends Comparable<Key>, Value> {
   private Node root;                  // root of BST

   private class Node {
        private Key key;               // sorted by key
        private Value val;             // associated value
        private Node left, right;      // roots of left and right subtrees
        private int size;              // #nodes in subtree rooted at this

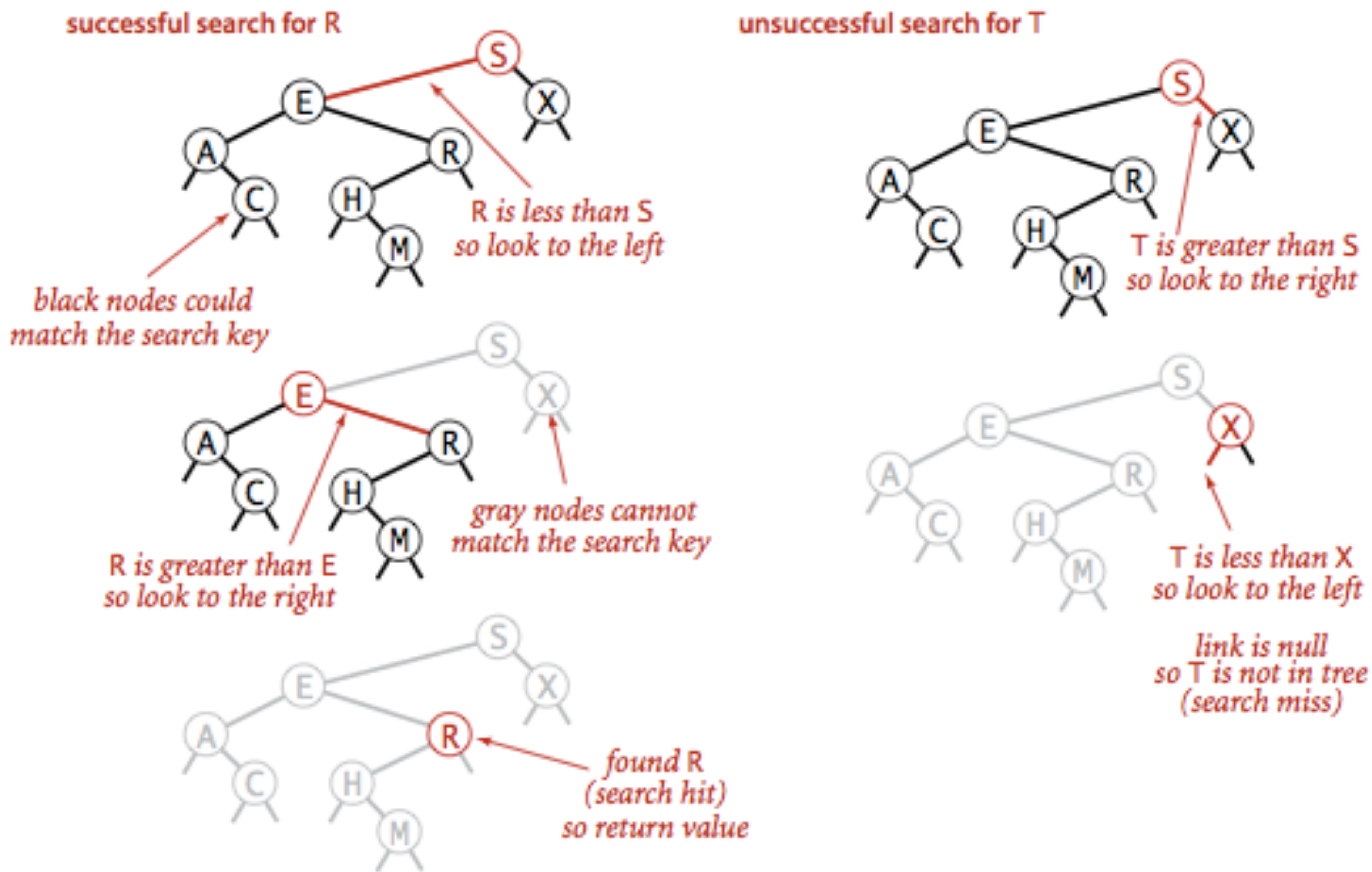
        public Node(Key key, Value val, int size) {
            this.key = key;
            this.val = val;
            this.size = size;
        }
    }
```



Search for a key

- If less than key in node go to left subtree.
- If greater than key in node go to right subtree.
- If given key and key at examined node are equal, search hit.
- Return value corresponding to given key, or `null` if no such key.
  - In other implementations, you return the last node you reached.
- Number of compares is equal to the depth of the node + 1.

Search example



Successful (left) and unsuccessful (right) search in a BST

Search - iterative implementation

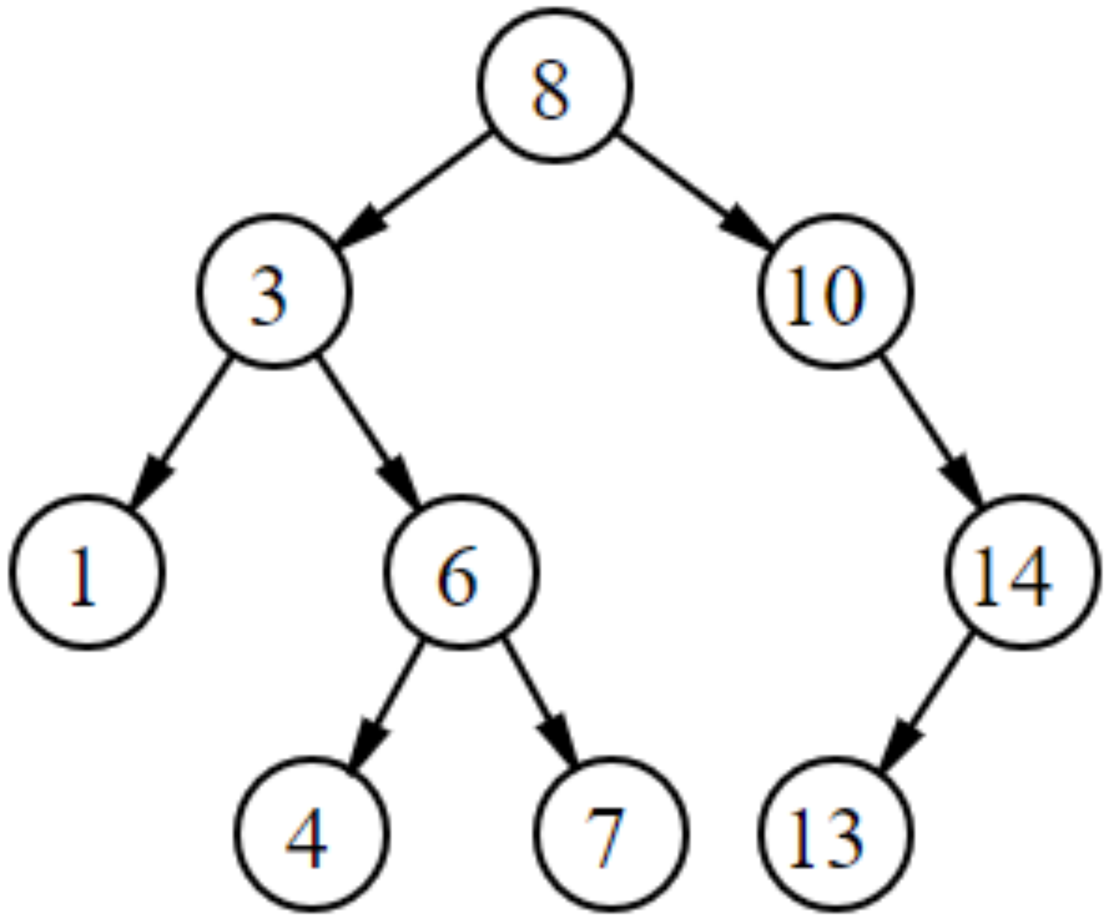
- ```
  public Value get(Key key) {
      Node x = root;
      while (x != null) {
          int cmp = key.compareTo(x.key);
          if (cmp < 0)
              x = x.left;
          else if (cmp > 0)
              x = x.right;
          else if (cmp == 0)
              return x.val;
      }
      return null;
  }
  ```
  ```

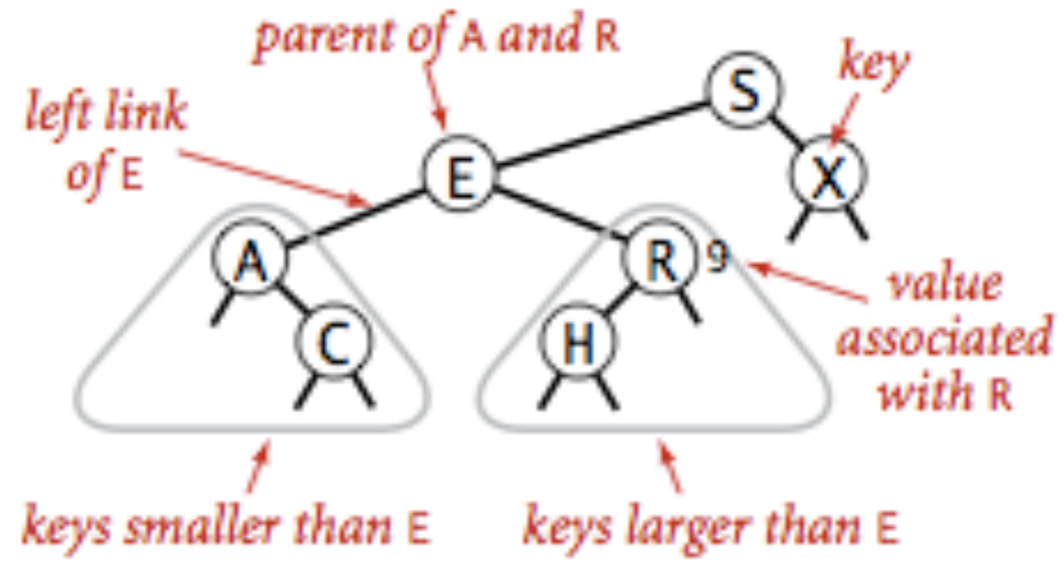
Search - recursive implementation

- ```
  public Value get(Key key) {
      return get(root, key);
  }
  ```
- ```
  private Value get(Node x, Key key) {
      if (x == null)
          return null;
      int cmp = key.compareTo(x.key);
      if (cmp < 0)
          return get(x.left, key);
      else if (cmp > 0)
          return get(x.right, key);
      else
          return x.val;
  }
  ```

Practice Time

- Search for the keys 4 and 9 in the following BST:

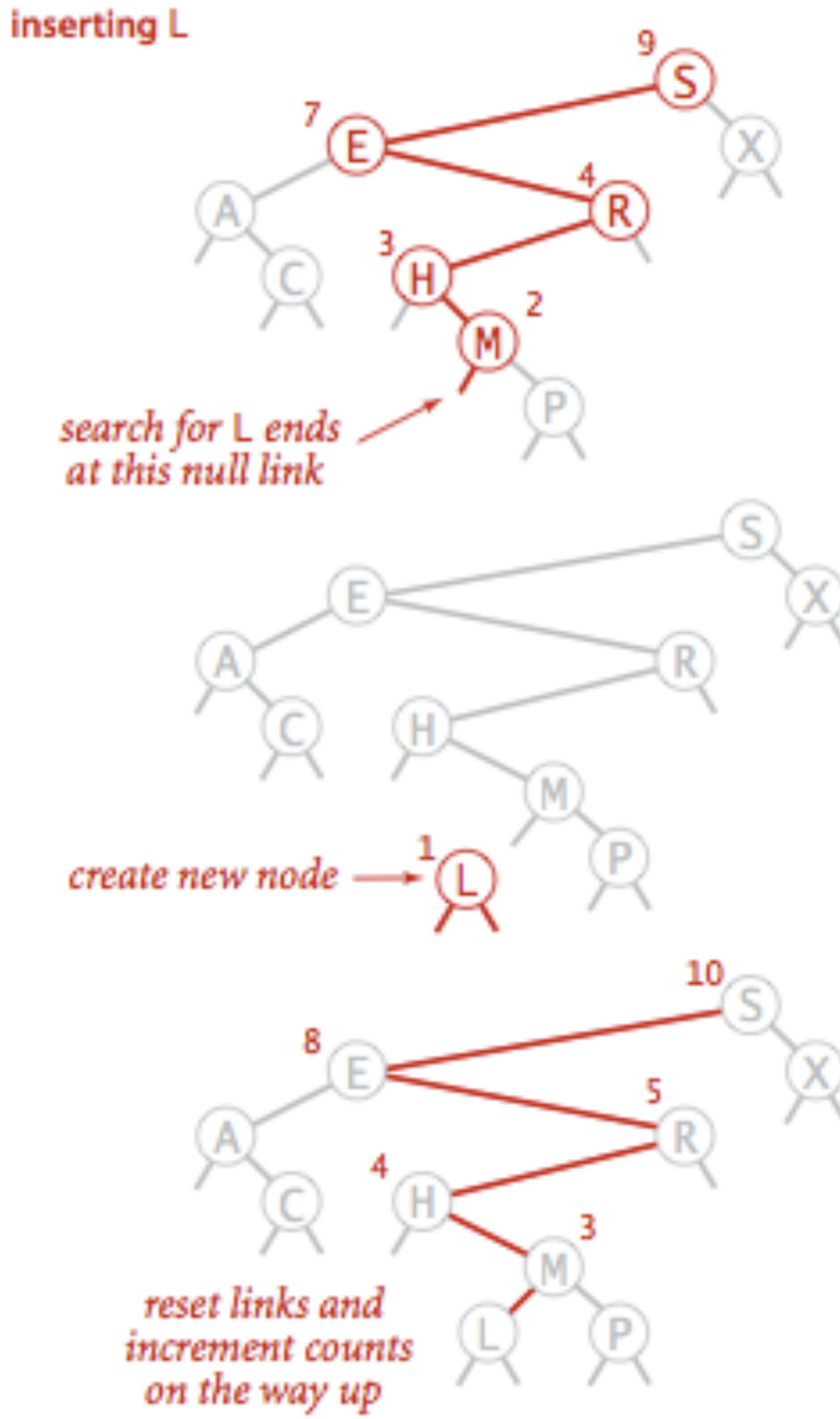




Insert

- If less than key in node go left.
- If greater than key in node go right.
- If null, insert.
- If already exists, update value.
- Number of compares is equal to the depth of the node + 1.

Insert example



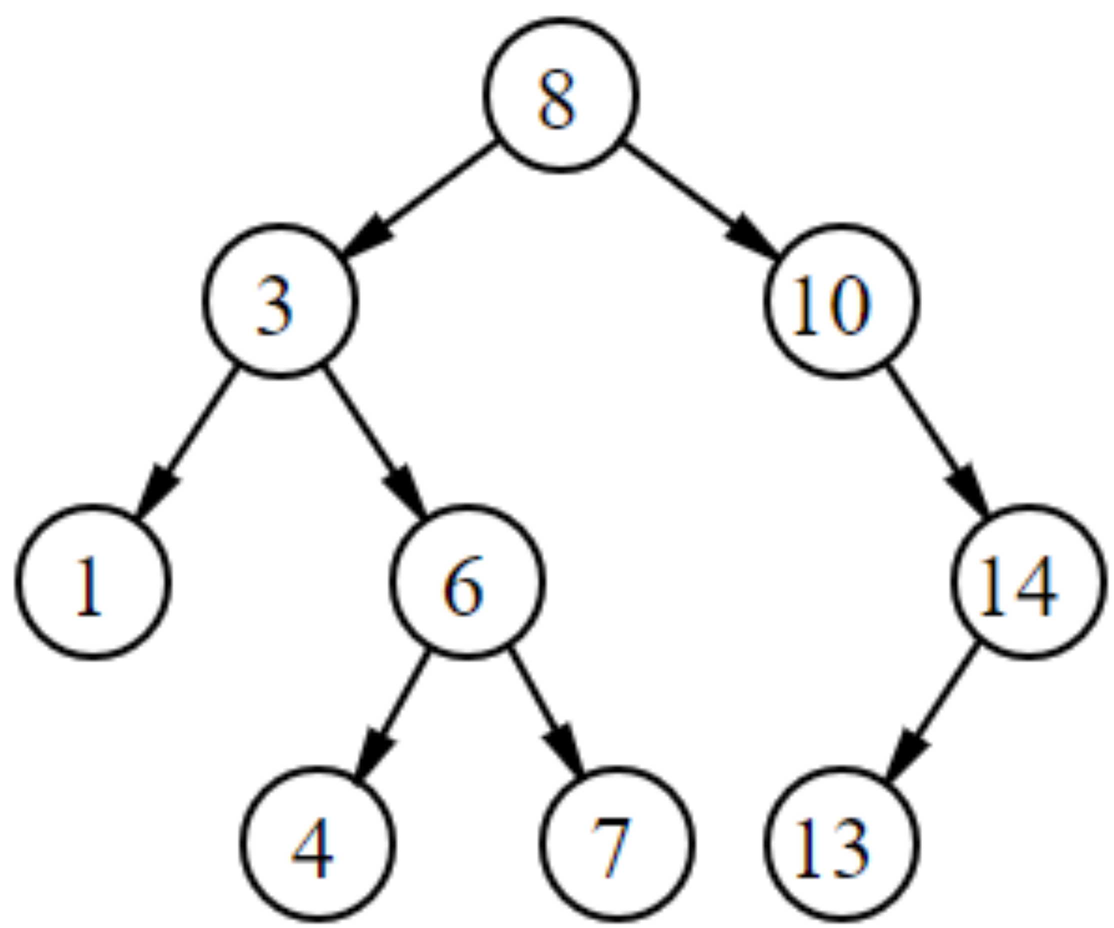
Insertion into a BST

Insert

- ```
  public void put(Key key, Value val) {
        root = put(root, key, val);
  }
  private Node put(Node x, Key key, Value val) {
        if (x == null)
              return new Node(key, val, 1);
        int cmp = key.compareTo(x.key);
        if (cmp < 0)
            x.left = put(x.left, key, val);
        else if (cmp > 0)
            x.right = put(x.right, key, val);
        else
            x.val = val;
        x.size = 1 + size(x.left) + size(x.right);
        return x;
  }
  ```
  ```

Practice Time

- Add the key-value pairs (4,3) and (9,2) in the following BST:

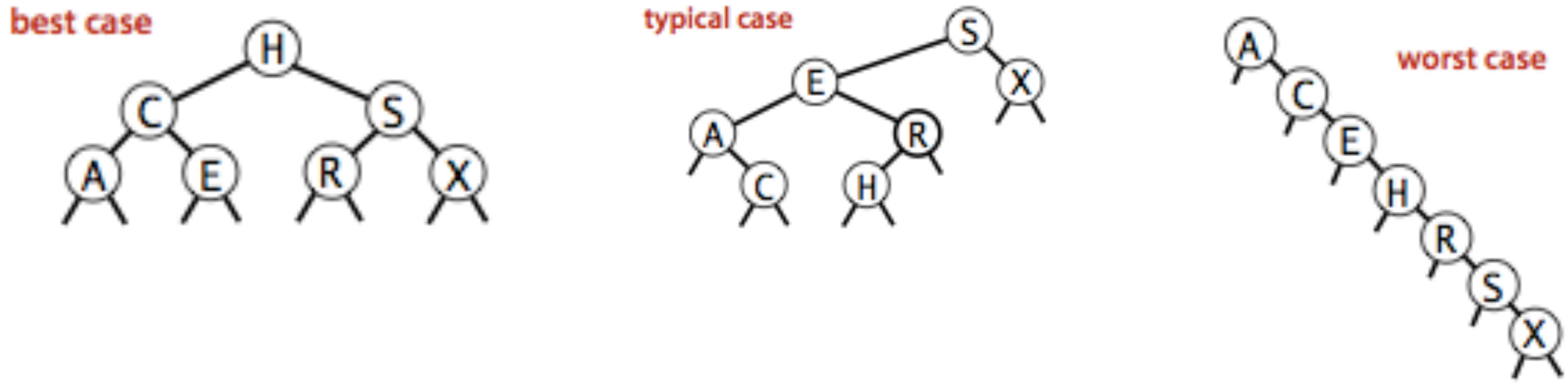


http://algs4.cs.princeton.edu

3.2 BINARY SEARCH TREE DEMO

Tree shape

- The same set of keys can result to different BSTs based on their order of insertion.
- Number of compares for search/insert is equal to depth of node +1.

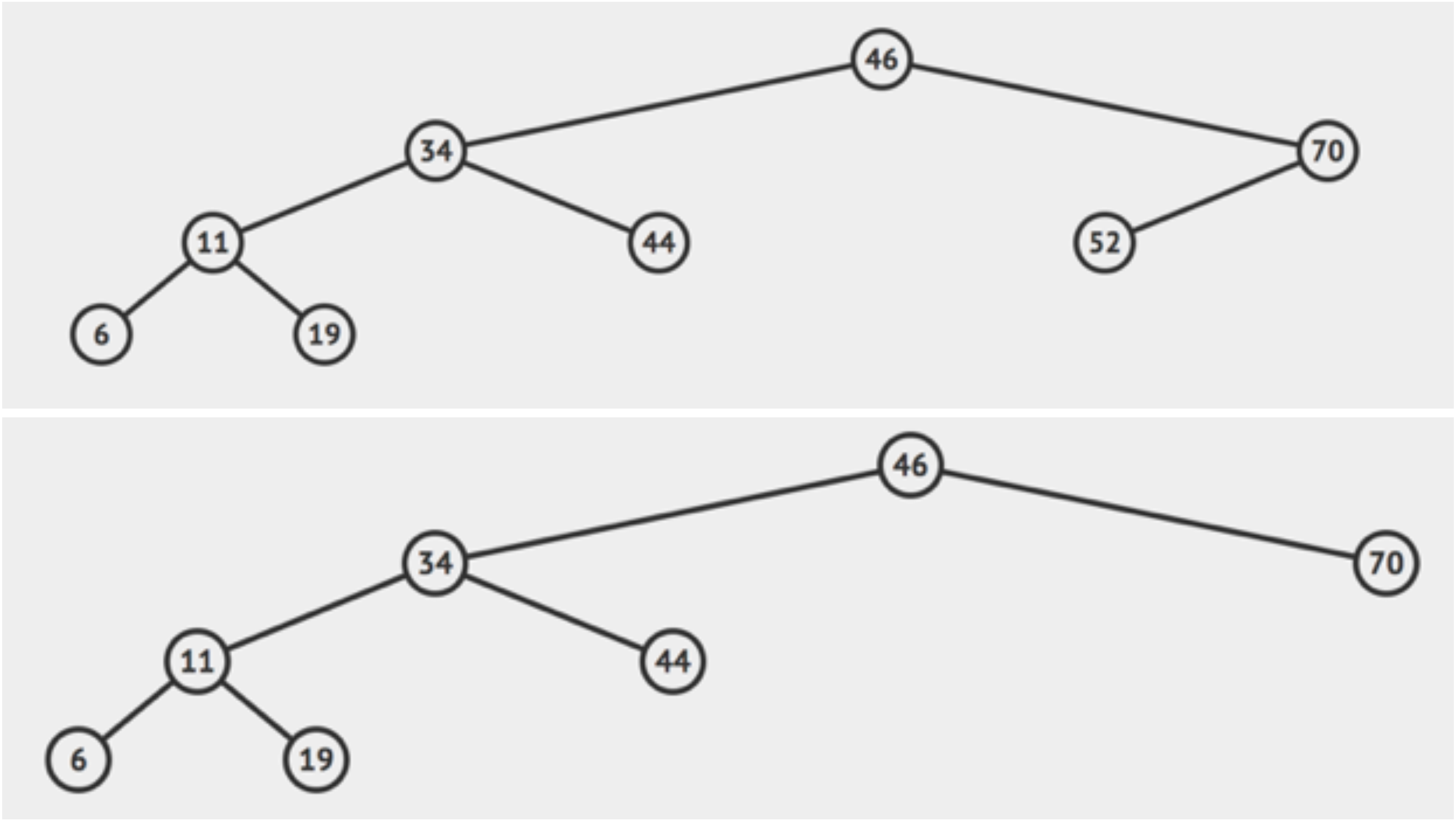


BSTs mathematical analysis

- If n distinct keys are inserted into a BST in random order, the expected number of compares of search/insert is $O(\log n)$.
  - If n distinct keys are inserted into a BST in random order, the expected height of tree is $O(\log n)$. [Reed, 2003].
- Worst case height is n but highly unlikely.
  - Keys would have to come (reversely) sorted!
- All ordered operations in a dictionary implemented with a BST depend on the height of the BST.

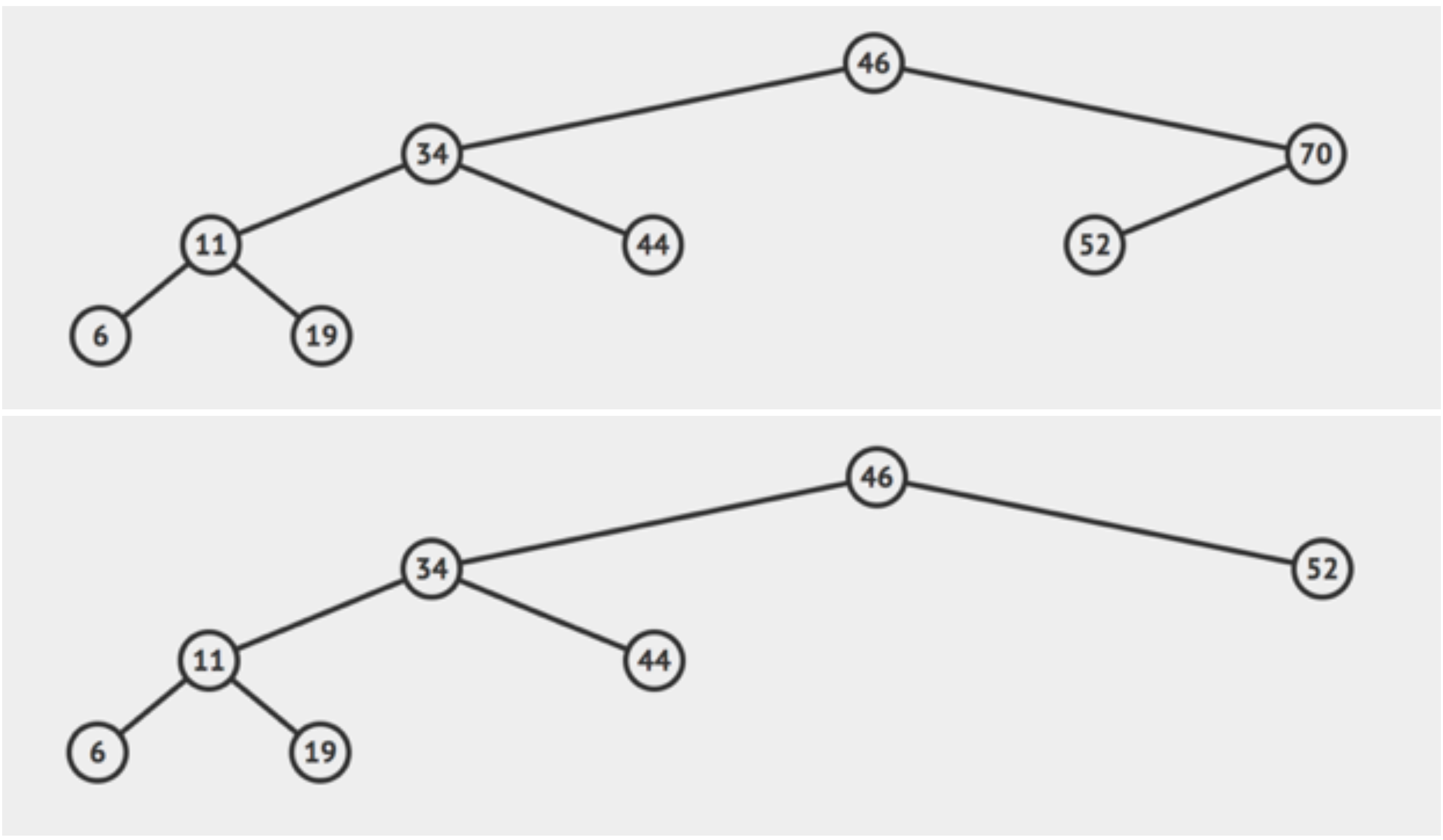
Hibbard deletion: Delete node which is a leaf

- Simply delete node.
- Example: delete 52 locates a node which is a leaf and removes it.



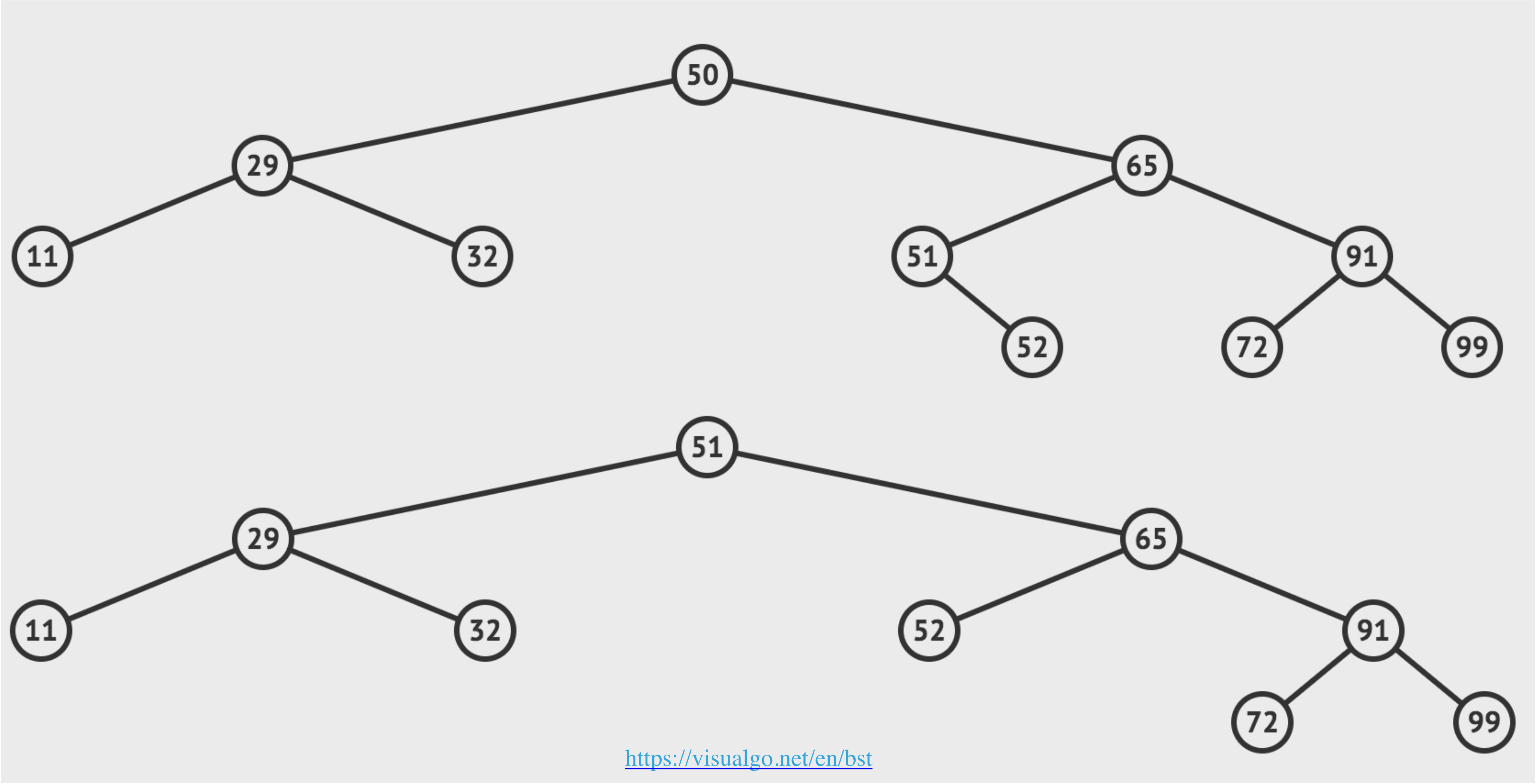
Hibbard deletion: Delete node with one child

- Delete node and replace it with its child.
- Example: delete 70 locates a node which has one child and replaces it with the child.



Hibbard deletion: Delete node with two children

- Delete node and replace it with successor (node with smallest of the larger keys). Move successor's child (if any) where successor was.
- Example: delete 50 locates a node which has two children. Successor is 51.



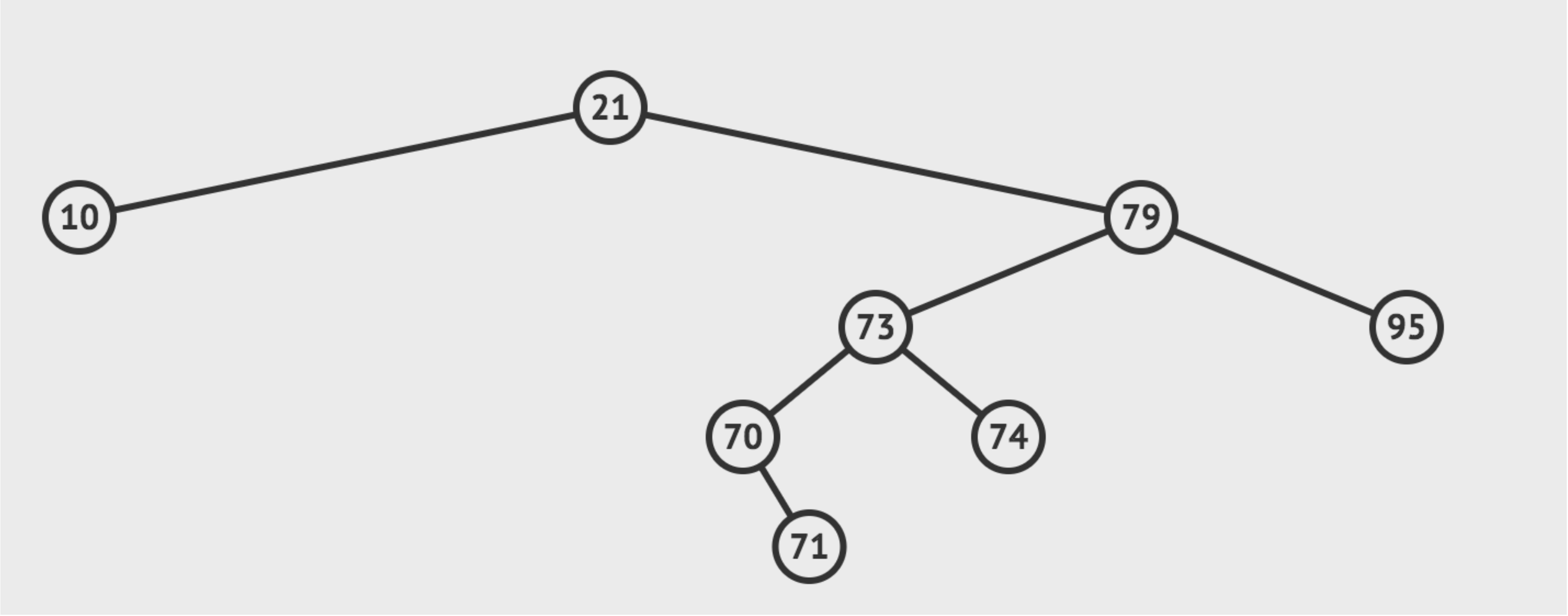
```
public void delete(Key key) {
    root = delete(root, key);
}

 private Node delete(Node x, Key key) {
     if (x == null) return null;

     int cmp = key.compareTo(x.key);
     if (cmp < 0)
         x.left  = delete(x.left,  key);
     else if (cmp > 0)
         x.right = delete(x.right, key);
     else {
         if (x.right == null)
             return x.left;
         if (x.left  == null)
             return x.right;
         Node t = x; //replace with successor
         x = min(t.right);
         x.right = deleteMin(t.right);
         x.left = t.left;
     }
     x.size = size(x.left) + size(x.right) + 1;
     return x;
 }
```

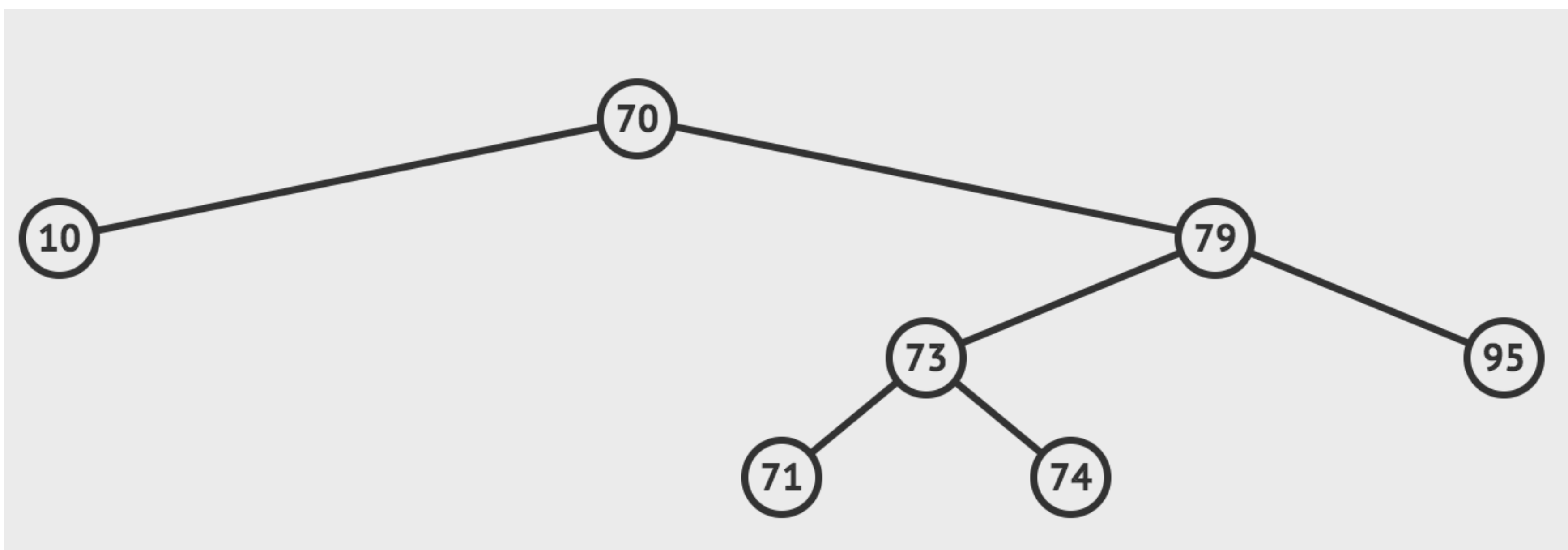
Practice Time

- Delete the node 21 following Hibbard's deletion



Answer

- Delete the node 21 following Hibbard's deletion



Hibbard's deletion

- Unsatisfactory solution. If we were to perform many insertions and deletions the BST ends up being not symmetric and skewed to the left.
  - Extremely complicated analysis, but average cost of deletion ends up being $\sqrt{n}$. Let's simplify things by saying it stays $O(\log n)$.
  - No one has proven that alternating between the predecessor and successor will fix this.
- Hibbard devised the algorithm in 1962. Still no algorithm for efficient deletion in Binary Search Trees!
- Overall, BSTs can have $O(n)$ worst-case for search, insert, and delete. We want to do better (see future lectures).

Lecture 23: Binary Search Trees

- Dictionaries
- Binary Search Trees

Readings:

- Textbook: Chapters 3.1 (Pages 362–386) and 3.2 (Pages 396–414)
- Website:
  - https://algs4.cs.princeton.edu/31elementary/
  - https://algs4.cs.princeton.edu/32bst/
- Visualization:
  - https://visualgo.net/en/bst

Practice Problems:

- 3.1.1-3.1.6, 3.2.1-3.2.13