CS062

DATA STRUCTURES AND ADVANCED PROGRAMMING

16: Quicksort, Binary Trees and Heaps

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Lecture 16: Quicksort, Binary Trees and Heaps

- Quicksort

Algorithm sketch:

- Shuffle the array.
- Partition so that, for some pivot `j`:
  - Entry `a[j]` is in place.
  - There is no larger entry to the left of `j`.
  - No smaller entry to the right of `j`.
- Sort each subarray recursively.

input	Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
shuffle	K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
partition	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
sort left	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
sort right	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

partitioning element

not greater

not less

Quicksort overview

```
private static void sort(Comparable[] a, int lo, int hi) {
    if (hi <= lo) return;
    int j = partition(a, lo, hi);
    sort(a, lo, j-1);
    sort(a, j+1, hi);
}
```

Quicksort Trace

	lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
initial values				Q	U	I	C	K	S	O	R	T	E	X	A	M	P	L	E
random shuffle				K	R	A	T	E	L	E	P	U	I	M	Q	C	X	O	S
	0	5	15	E	C	A	I	E	K	L	P	U	T	M	Q	R	X	O	S
	0	3	4	E	C	A	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	2	2	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	0	0	1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	1		1	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	4		4	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	6	6	15	A	C	E	E	I	K	L	P	U	T	M	Q	R	X	O	S
	7	9	15	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	7	7	8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	8		8	A	C	E	E	I	K	L	M	O	P	T	Q	R	X	U	S
	10	13	15	A	C	E	E	I	K	L	M	O	P	S	Q	R	T	U	X
	10	12	12	A	C	E	E	I	K	L	M	O	P	R	Q	S	T	U	X
	10	11	11	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	10		10	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	14	14	15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
	15		15	A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X
result				A	C	E	E	I	K	L	M	O	P	Q	R	S	T	U	X

no partition for subarrays of size 1

Great algorithms are better than good ones

- Your laptop executes 10^8 comparisons per second
- A supercomputer executes 10^{12} comparisons per second

	Insertion sort			Mergesort			Quicksort		
Computer	**Thousand inputs**	**Million inputs**	**Billion inputs**	**Thousand inputs**	**Million inputs**	**Billion inputs**	**Thousand inputs**	**Million inputs**	**Billion inputs**
Home	Instant	2 hours	300 years	instant	1 sec	15 min	Instant	0.5 sec	10 min
Supercomputer	Instant	1 second	1 week	instant	instant	instant	instant	instant	Instant

Quicksort analysis: best case

- Quicksort divides everything exactly in half.
- Similar to merge sort.
- Number of compares is $\sim n \log n$.

Quicksort analysis: worst case

- Data are already sorted or we pick the smallest or largest key as pivot.
- Number of compares is ~n^2 - quadratic!
- Extremely unlikely (less likely than the probably that your computer is struck by lightning) if we shuffle and our shuffling is not broken.

Things to remember about quick sort

- $O(n \log n)$ average, $O(n^2)$ worst, in practice faster than mergesort.

- 39% more compares than merge sort but in practice it is faster because it does not move data much.
  - Compare and increment pointer
  - Mergesort moves items into and out of aux array

- Random shuffle = probabilistic guarantee against worst case

- In-place sorting.
- **Not** stable.

Quicksort practical improvements

- Use insertion sort for small subarrays.
- Best choice of pivot is the median of a small sample.

- For years, Java used quicksort for collections of primitives and mergesort for collections of objects due to stability.
  - Has moved to dual-pivot quick sort (Yaroslavskiy, Bentley, and Bloch, 2009) and timsort (Peters, 1993), respectively.

Sorting: the story so far

Which Sort	In place	Stable	Best	Average	Worst	Remarks
Selection	X		$O(n^2)$	$O(n^2)$	$O(n^2)$	n exchanges
Insertion	X	X	$O(n)$	$O(n^2)$	$O(n^2)$	Use for small arrays or partially ordered
Merge		X	$O(n \log n)$	$O(n \log n)$	$O(n \log n)$	Guaranteed performance; stable
Quick	X		$O(n \log n)$	$O(n \log n)$	$O(n^2)$	$n \log n$ probabilistic guarantee; fastest in practice

Sorting based on comparisons

- All sorting algorithms we have seen so far and we will see in this class are compare-based.
- No compare-based sorting algorithm can sort n elements in less than $O(n \log n)$ time in the worst case.
  - Proof and proper notation in CS140.

Readings:

- Textbook:
  - Chapter 2.3 (Pages 288-296)
- Website:
  - Quicksort: https://algs4.cs.princeton.edu/23quicksort/
  - Code: https://algs4.cs.princeton.edu/23quicksort/Quick.java.html

Practice Problems:

- 2.3.1-2.3.4

Basic data structures

- Arrays,
- Resizing arrays or arraylists,
- Linked Lists,
- Queues, and
- Stacks.
- Runtime and memory analysis for each one.

Sorting

- Selection sort,
- Insertion sort,
- Mergesort, and
- Quicksort.
- Runtime (comparisons and exchanges), stability, in-place for each one.
- Comparators: How to sort a data structure with objects of any class.
- Iterators: How to traverse a data structure.

Lecture 16: Binary Trees and Heaps

- Binary Trees
- Tree traversals
- Binary Heaps

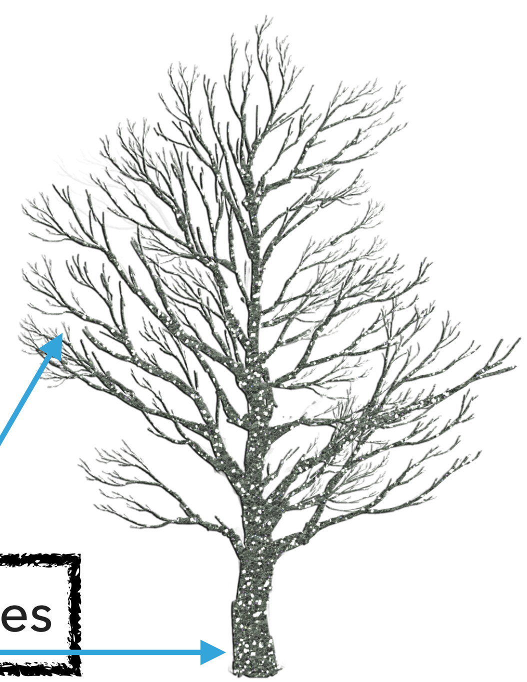
Trees in Computer Science

- Abstract data types that store elements hierarchically rather than linearly.
- Examples of hierarchical structures:
  - Organization charts for
    - Companies (CEO at the top followed by CFO, CMO, COO, CTO, etc).
    - Universities (Board of Trustees at the top, followed by President, then by VPs, etc).
  - Sitemaps (home page links to About, Products, etc. They link to other pages).
  - Computer file systems (user at top followed by Documents, Downloads, Music, etc. Each folder can hold more folders.).

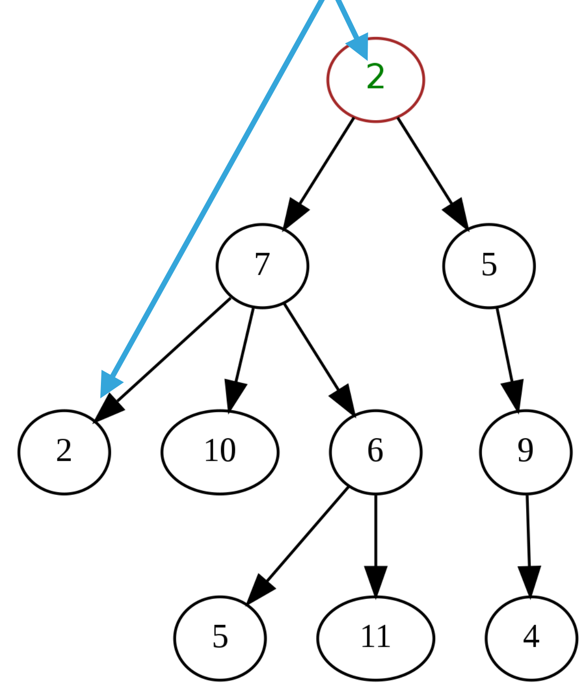
Trees in Computer Science

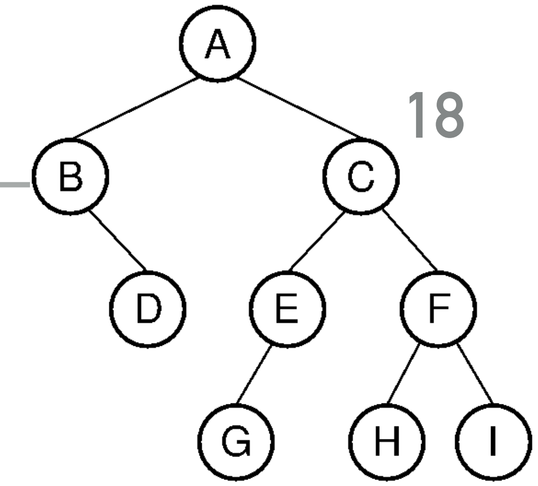
- Hierarchical: Each element in a tree has a **single** parent (immediate ancestor) and zero or more children (immediate descendants).

CS tree root and leaves



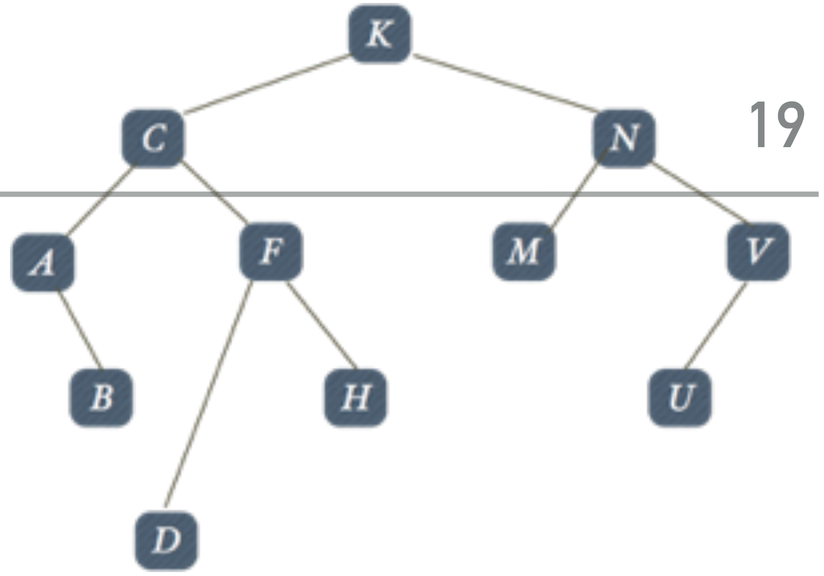
Real tree root and leaves





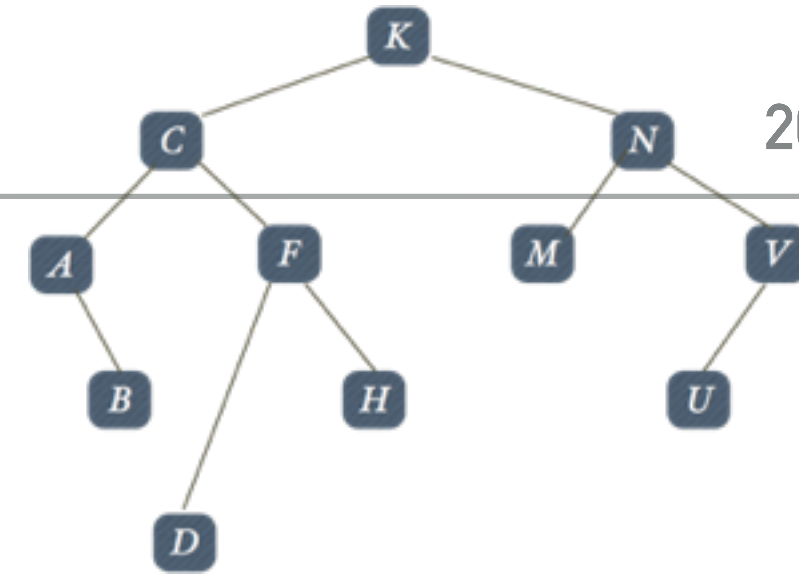
Definition of a tree

- A tree T is a set of nodes that store elements based on a parent-child relationship:
  - If T is non-empty, it has a node called the root of T, that has no parent.
    - Here, the root is A.
  - Each node v, other than the root, has a unique parent node u. Every node with parent u is a child of u.
    - E.g., E's parent is C and F has two children, H and I.



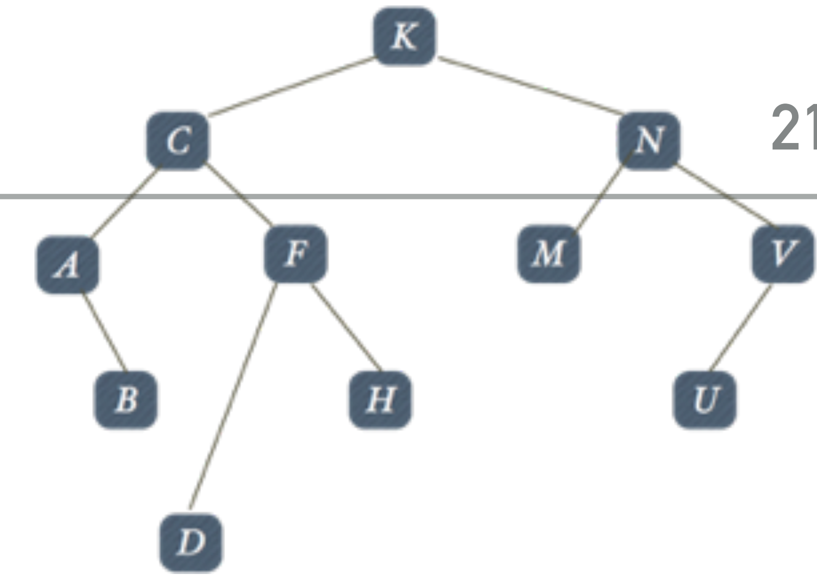
Tree Terminology

- Edge: a pair of nodes s.t. one is the parent of the other, e.g., (K,C).
- Parent node is directly above child node, e.g., K is parent of C and N.
- Sibling nodes have same parent, e.g., A and F.
- K is ancestor of B.
- B is descendant of K.
- Node plus all descendants gives subtree.
- Nodes without descendants are called leaves or external. The rest are called internal.
- A set of trees is called a forest.



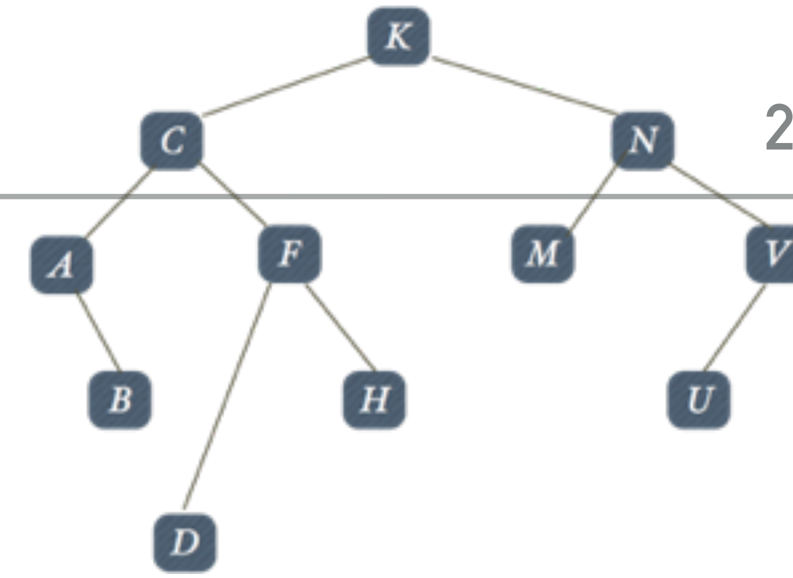
More Terminology

- Simple path: a series of distinct nodes s.t. there are edges between successive nodes, e.g., K-N-V-U.
- Path length: number of edges in path, e.g., path K-C-A has length 2.
- Height of node: length of longest path from the node to a leaf.
- Height of tree: length of longest path from the root to a leaf.
- Degree of node: number of its children.
- Degree of tree (arity): max degree of any of its nodes.
- Binary tree: a tree with arity of 2.



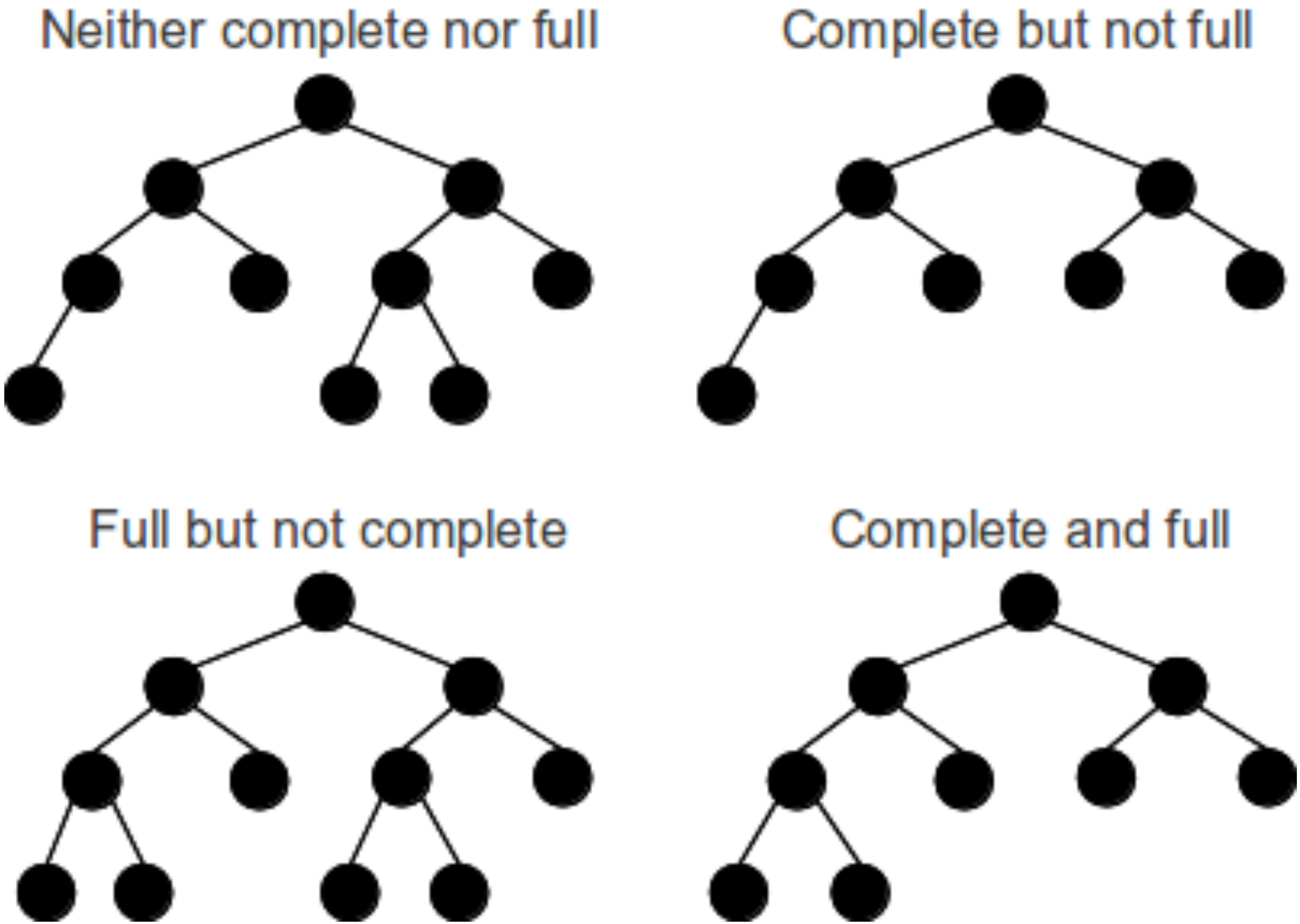
Even More Terminology

- Level/depth of node defined recursively:
  - Root is at level 0.
  - Level of any other node is equal to level of parent + 1.
  - It is also known as the length of path from root or number of ancestors excluding itself.
- Height of node defined recursively:
  - If leaf, height is 0.
  - Else, height is max height of child + 1.



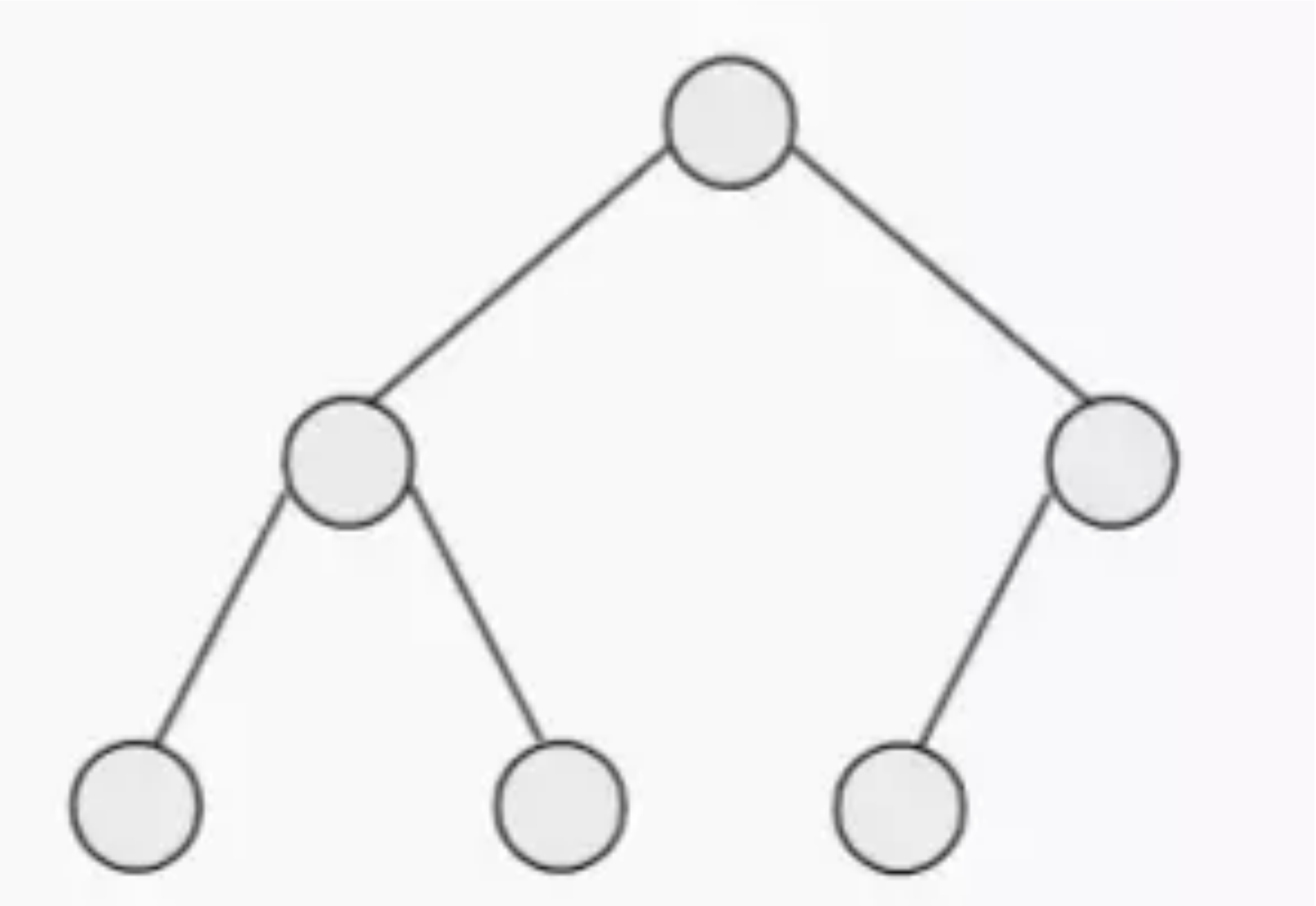
But wait there's more!

- Full (or proper): a binary tree whose every node has 0 or 2 children.
- Complete: a binary tree with minimal height. Any holes in tree would appear at last level to right, i.e., all nodes of last level are as left as possible.



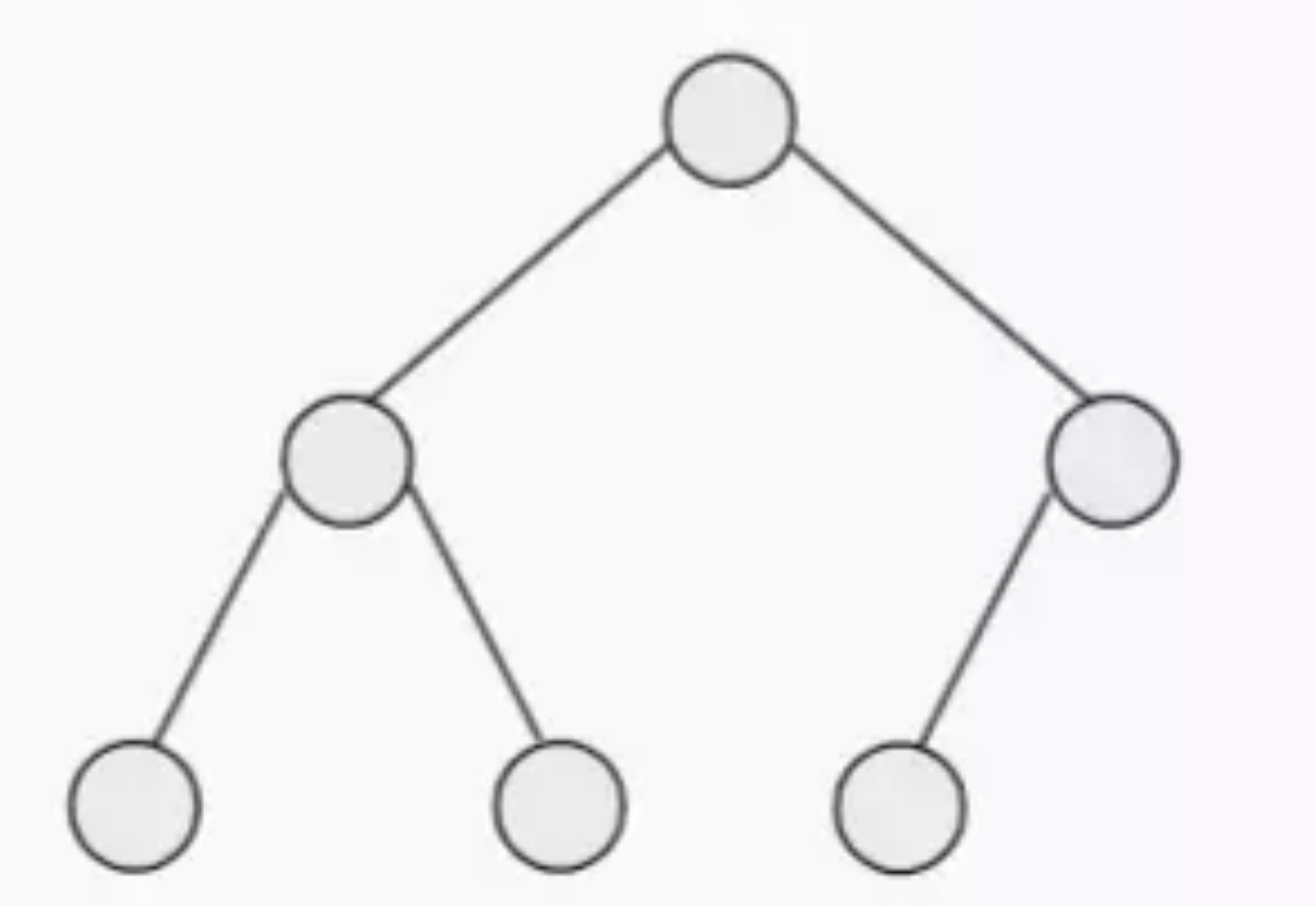
http://code.cloudkaksha.org/binary-tree/types-binary-tree

Practice Time: This tree is



- A: Full
- B: Complete
- C: Full and Complete
- D: Neither Full nor Complete

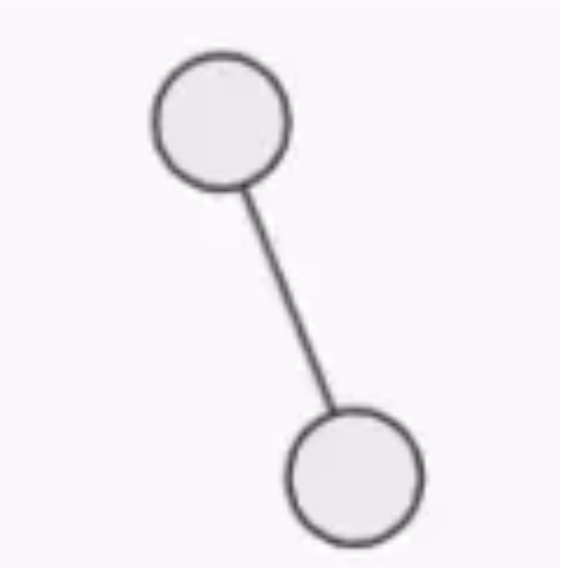
Answer



- A: Full
- **B: Complete**
- C: Full and Complete
- D: Neither Full nor Complete

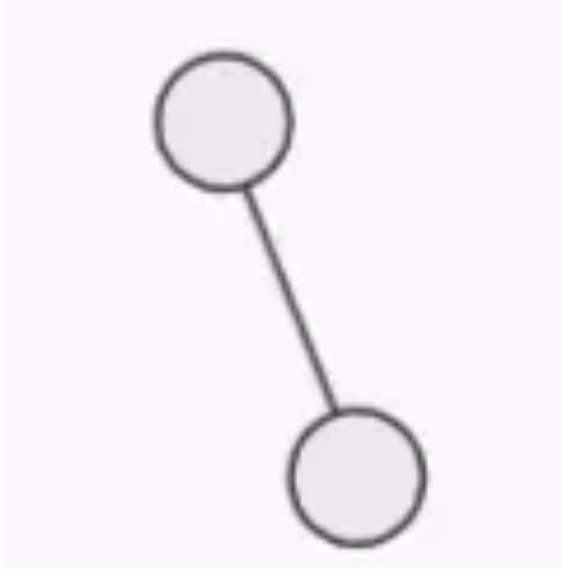
Practice Time: This tree is

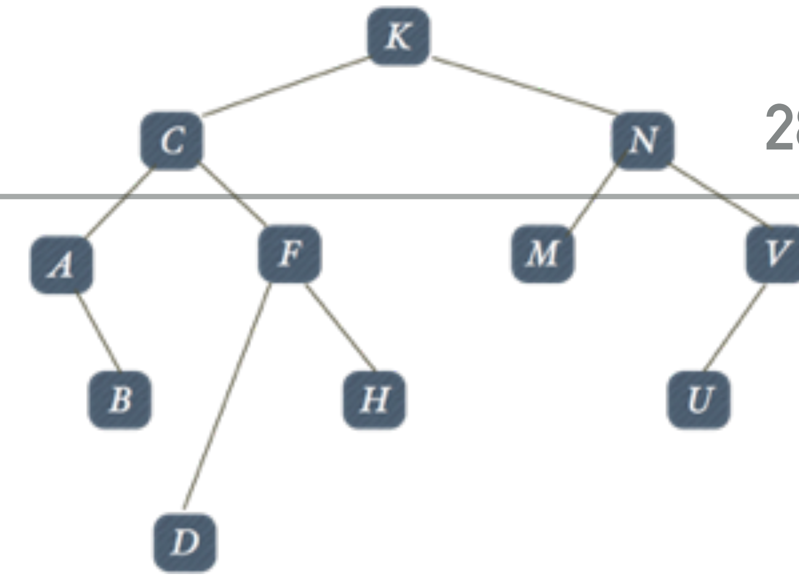
- A: Full
- B: Complete
- C: Full and Complete
- D: Neither Full nor Complete



Answer

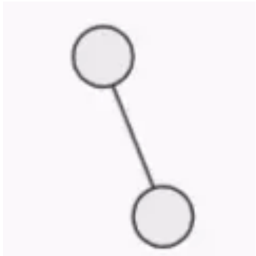
- A: Full
- B: Complete
- C: Full and Complete
- **D: Neither Full nor Complete**



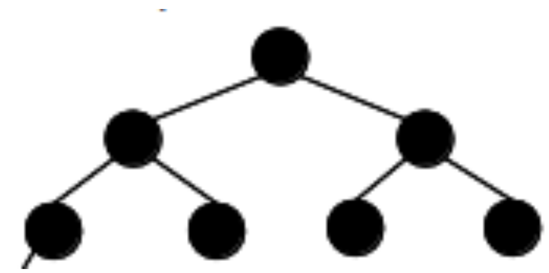


Counting in binary trees

- Lemma: if T is a binary tree, then at level k, T has $\leq 2^k$ nodes.
    - E.g., at level 2, at most 4 nodes (A, F, M, V)
- Theorem: If T has height h, then # of nodes n in T satisfy: $h + 1 \leq n \leq 2^{h+1} - 1$.



- Equivalently, if T has n nodes, then $log(n + 1) - 1 \leq h \leq n - 1$.
    - Worst case: When $h = n - 1$ or $O(n)$, the tree looks like a left or right-leaning "stick".
    - Best case: When a tree is as compact as possible (e.g., complete) it has $O(\log n)$ height.



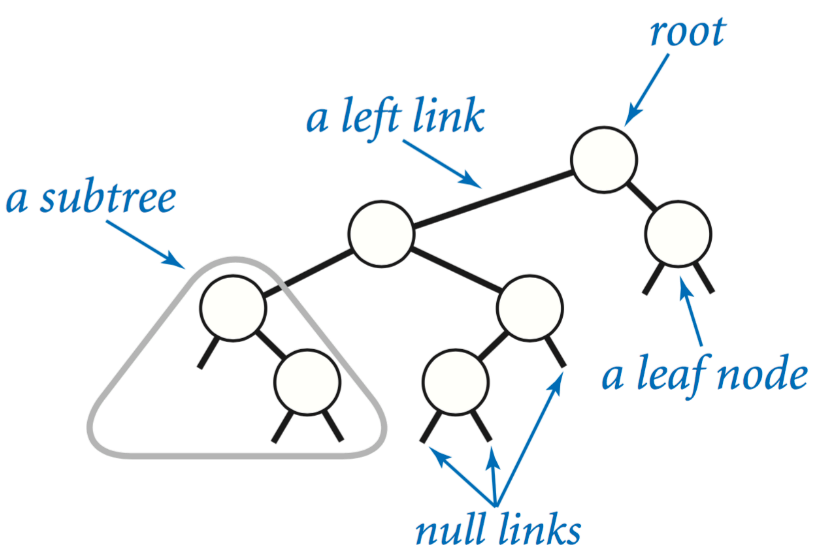
Basic idea behind a simple implementation

```
public class BinaryTree<Item> {
    private Node root;

    /**
     * A node subclass which contains various recursive methods
     *
     * @param <Item>  The type of the contents of nodes
     */
    private class Node {
        private Item item;

        private Node left;
        private Node right;

        /**
         * Node constructor with subtrees
         *
         * @param left   the left node child
         * @param right      the right node child
         * @param item   the item contained in the node
         */
        public Node(Node left, Node right, Item item) {
            this.left = left;
            this.right = right;
            this.item = item;
        }
```

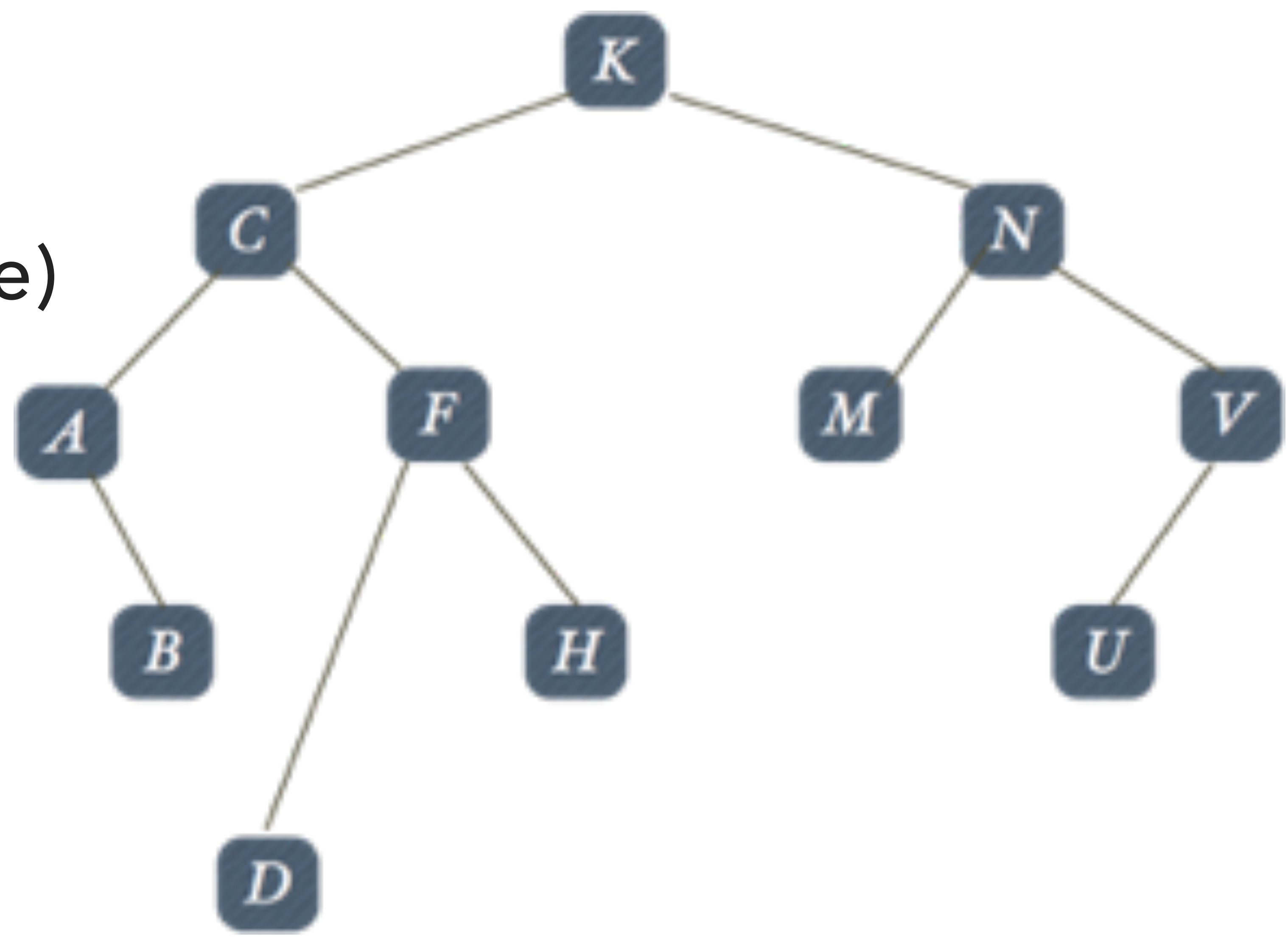


Lecture 16: Binary Trees and Heaps

- Binary Trees
- Tree traversals
- Binary Heaps

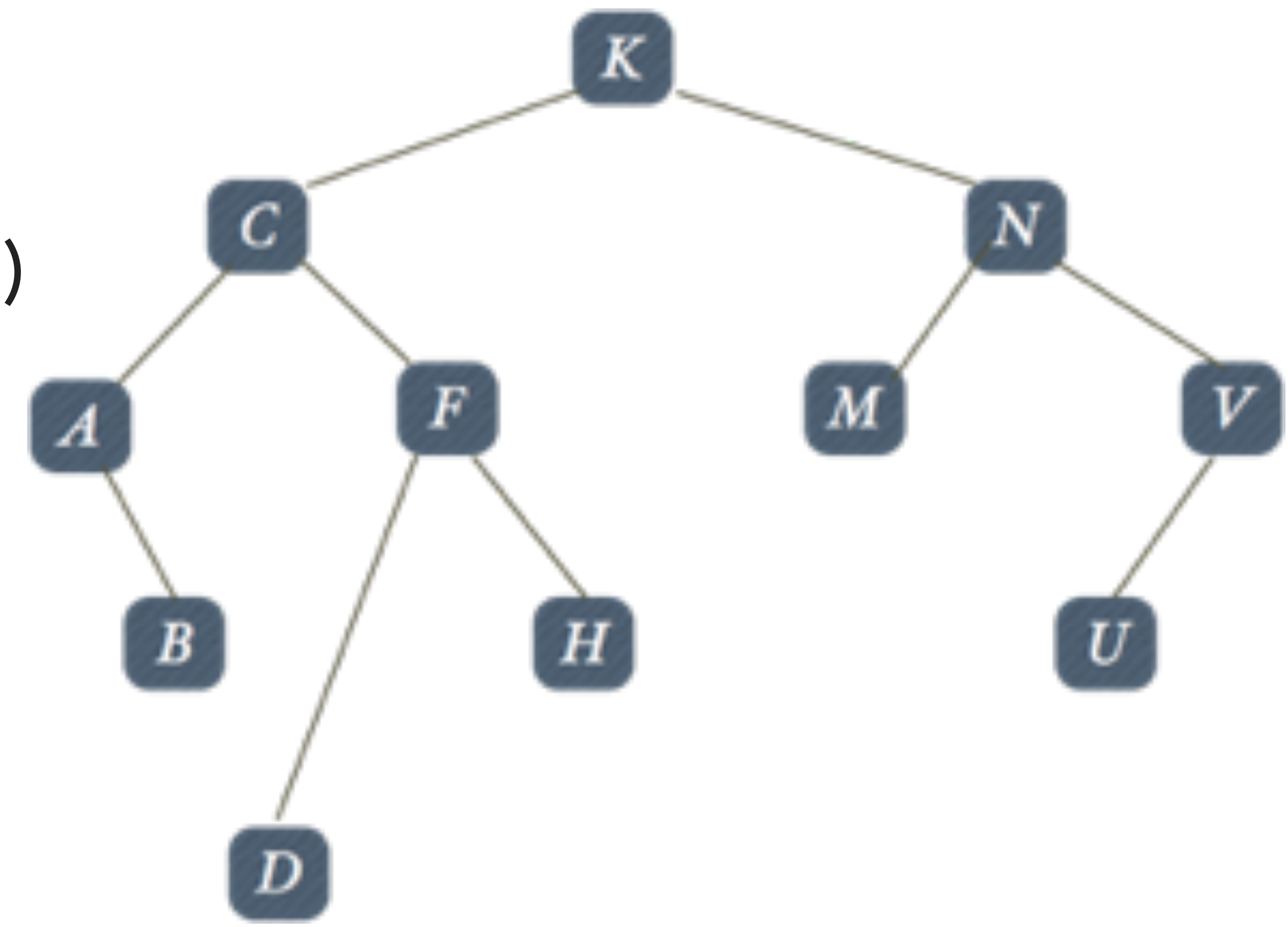
Pre-order traversal

- Preorder(Tree)
  - Mark root as visited
  - Preorder(Left Subtree)
  - Preorder(Right Subtree)
- K C A B F D H N M V U



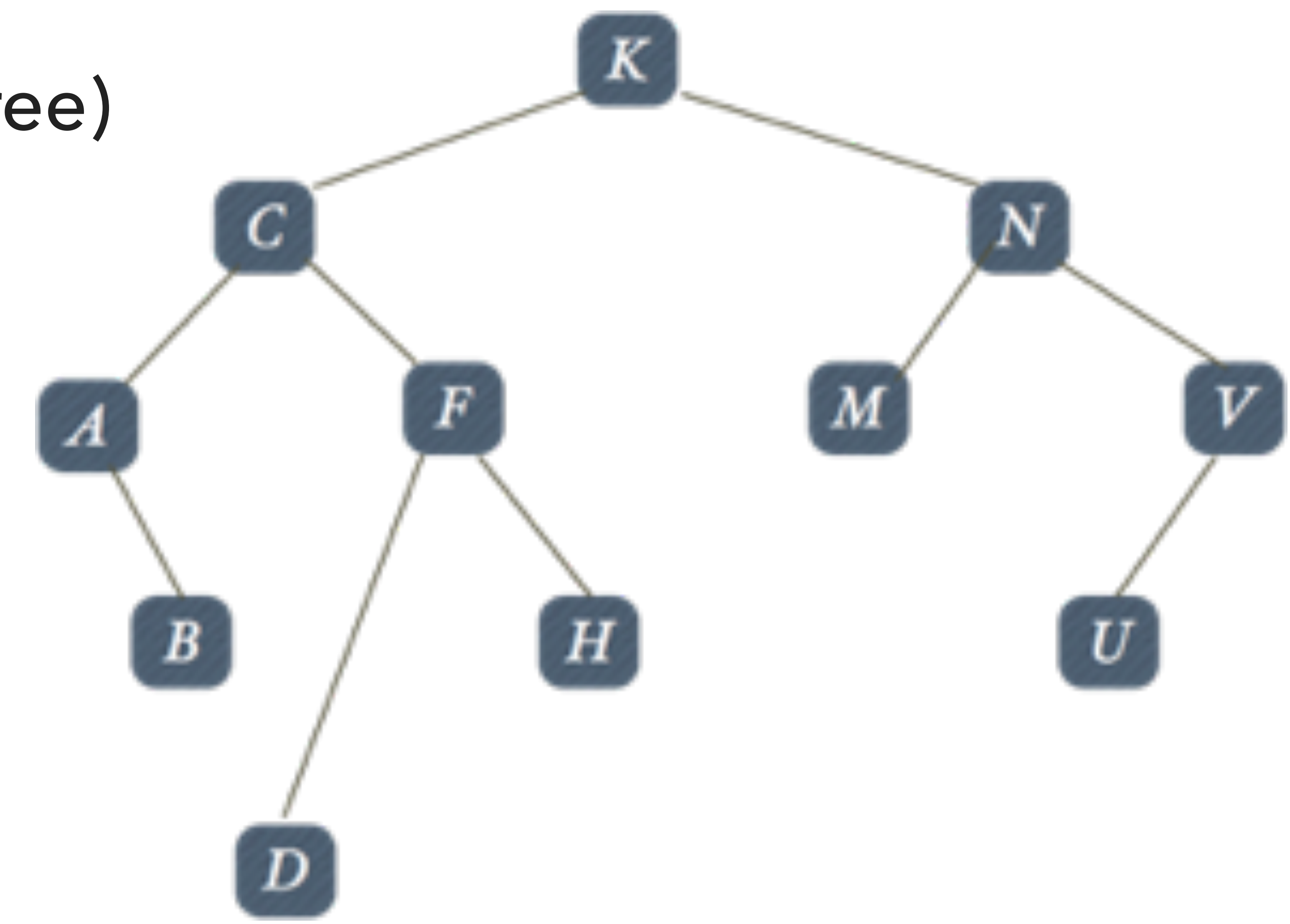
In-order traversal

- Inorder(Tree)
  - Inorder(Left Subtree)
  - Mark root as visited
  - Inorder(Right Subtree)
- A B C D F H K M N U V



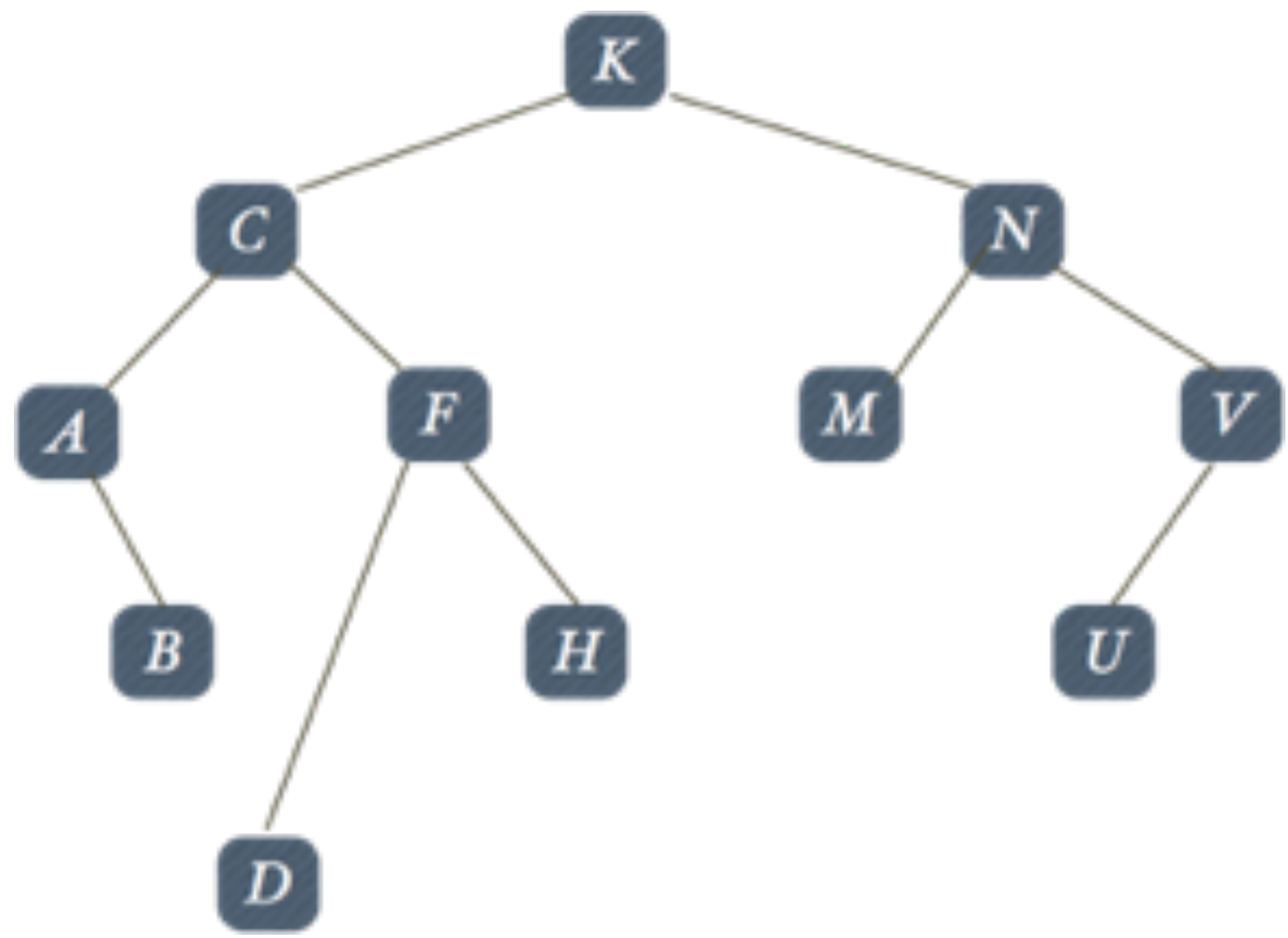
Post-order traversal

- Postorder(Tree)
  - Postorder(Left Subtree)
  - Postorder(Right Subtree)
  - Mark root as visited
- B A D H F C M U V N K



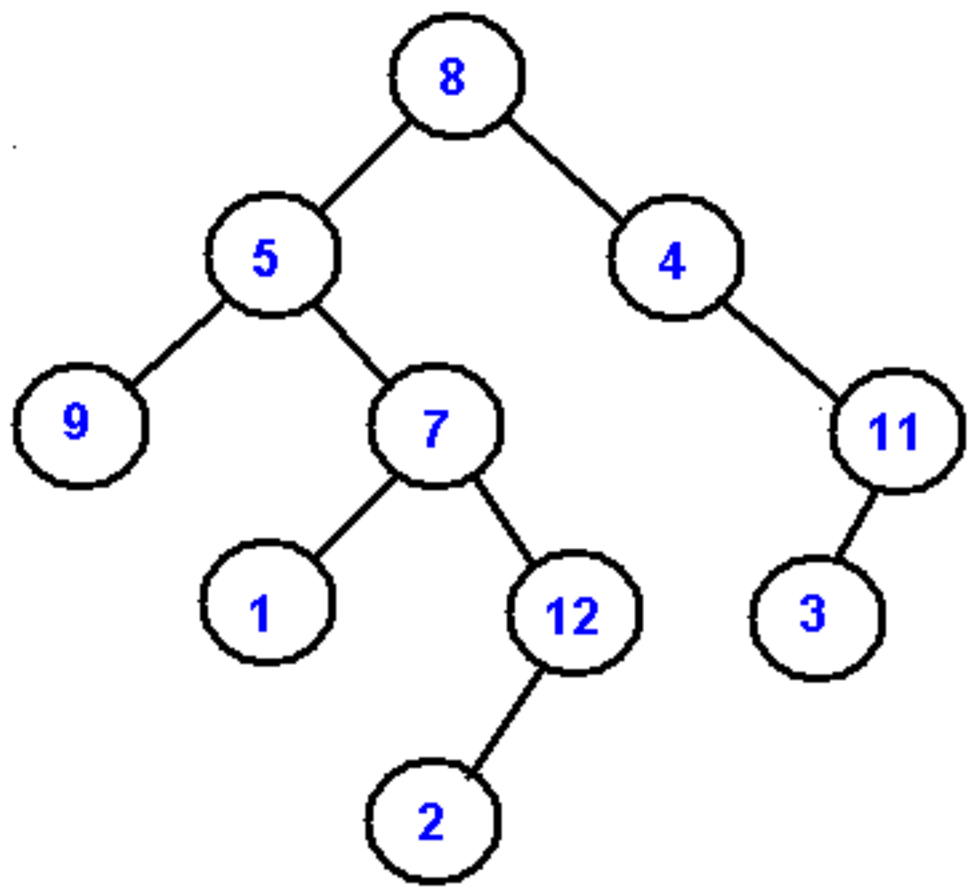
Level-order traversal

- From left to right, mark nodes of level i as visited before nodes in level $i+1$. Start at level 0.
- K C N A F M V B D H U



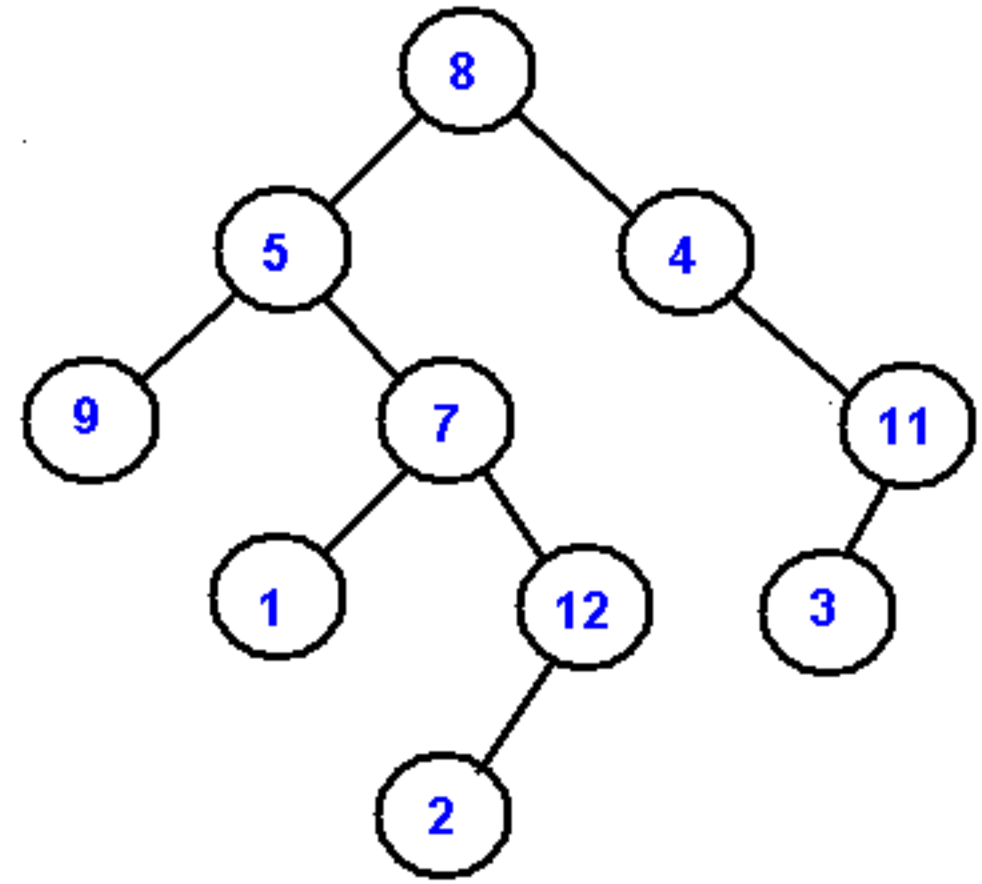
Practice Time

- List the nodes in pre-order, in-order, post-order, and level order:



Answer

- Pre-order: 8, 5, 9, 7, 1, 12, 2, 4, 11, 3
- In-order: 9, 5, 1, 7, 2, 12, 8, 4, 3, 11
- Post-order: 9, 1, 2, 12, 7, 5, 3, 11, 4, 8
- Level-order: 8, 5, 4, 9, 7, 11, 1, 12, 3, 2



Lecture 16: Binary Trees and Heaps

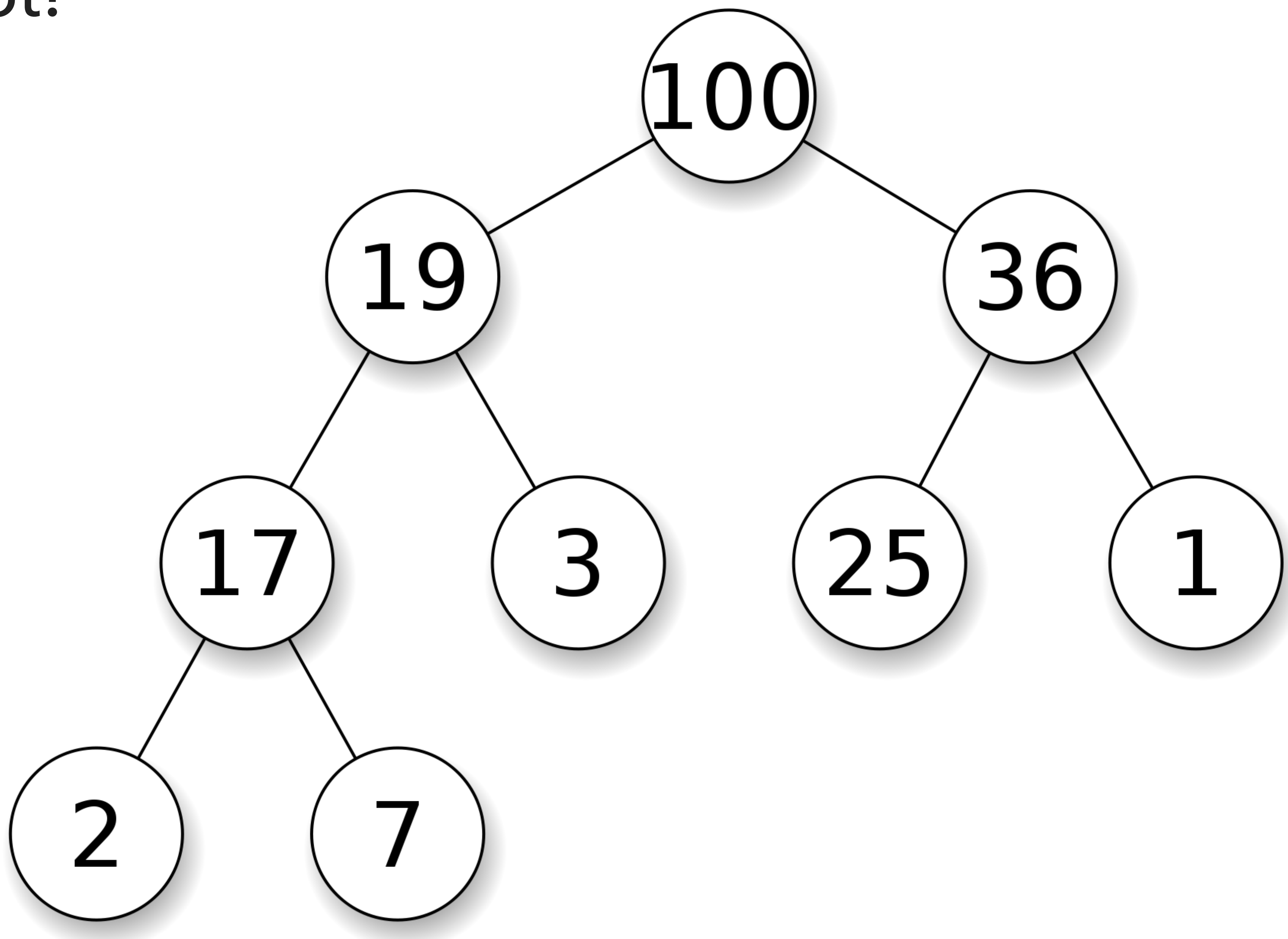
- Binary Trees
- Tree traversals
- Binary Heaps

Heap-ordered binary trees

- A binary tree is heap-ordered if the key in each node is larger than or equal to the keys in that node's two children (if any).
- Equivalently, the key in each node of a heap-ordered binary tree is smaller than or equal to the key in that node's parent (if any).
- No assumption of which child is smaller.
- Moving up from any node, we get a non-decreasing sequence of keys.
- Moving down from any node we get a non-increasing sequence of keys.

Heap-ordered binary trees

- The largest key in a heap-ordered binary tree is found at the root!



Binary heap representation

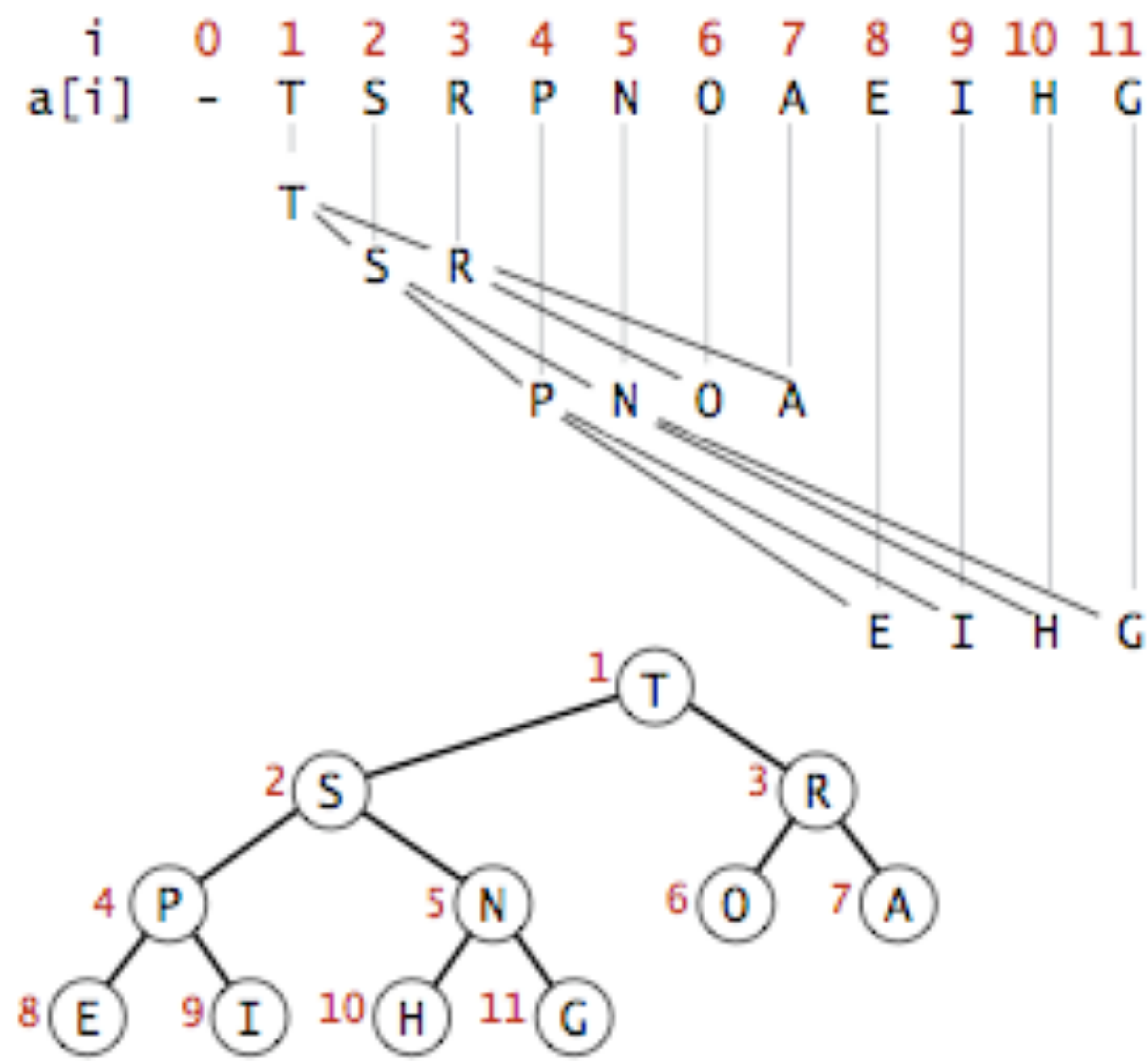
- We could use a linked representation but we would need three links for every node (one for parent, one for left subtree, one for right subtree).
- If we use complete binary trees, we can use instead an array.
  - Compact arrays vs explicit links means memory savings!

Binary heaps

- Binary heap: the array representation of a complete heap-ordered binary tree.
  - Items are stored in an array such that each key is guaranteed to be larger (or equal to) than the keys at two other specific positions (children).
- Max-heap but there are min-heaps, too.

Array representation of heaps

- Nothing is placed at index 0.
- Root is placed at index 1.
- Rest of nodes are placed in level order.
- No unnecessary indices and no wasted space because it’s complete.



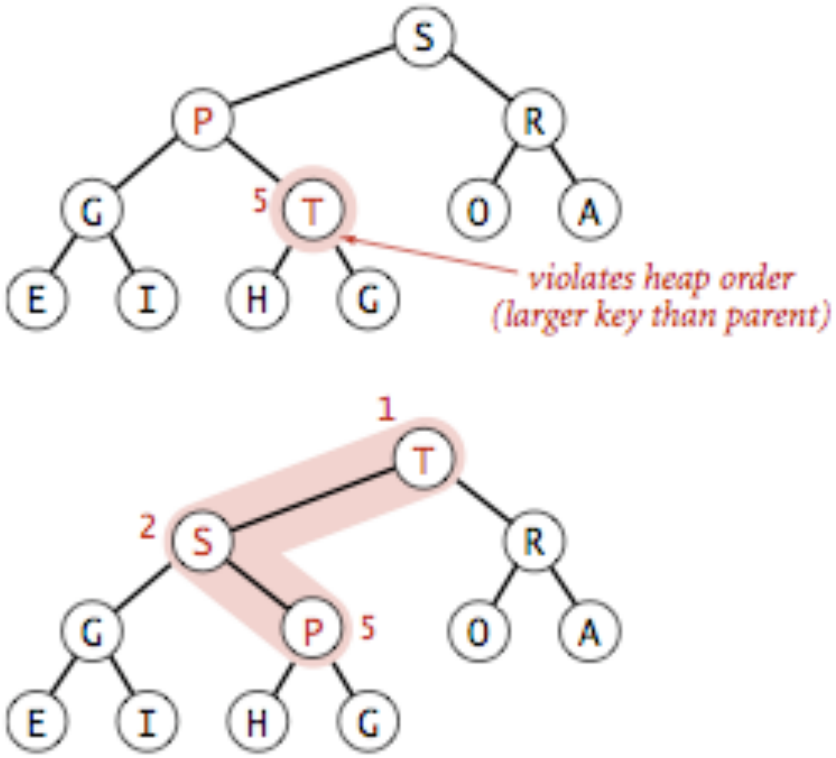
Heap representations

Reuniting immediate family members.

- For every node at index k, its parent is at index $\lfloor k/2 \rfloor$.
- Its two children are at indices $2k$ and $2k + 1$.
- We can travel up and down the heap by using this simple arithmetic on array indices.

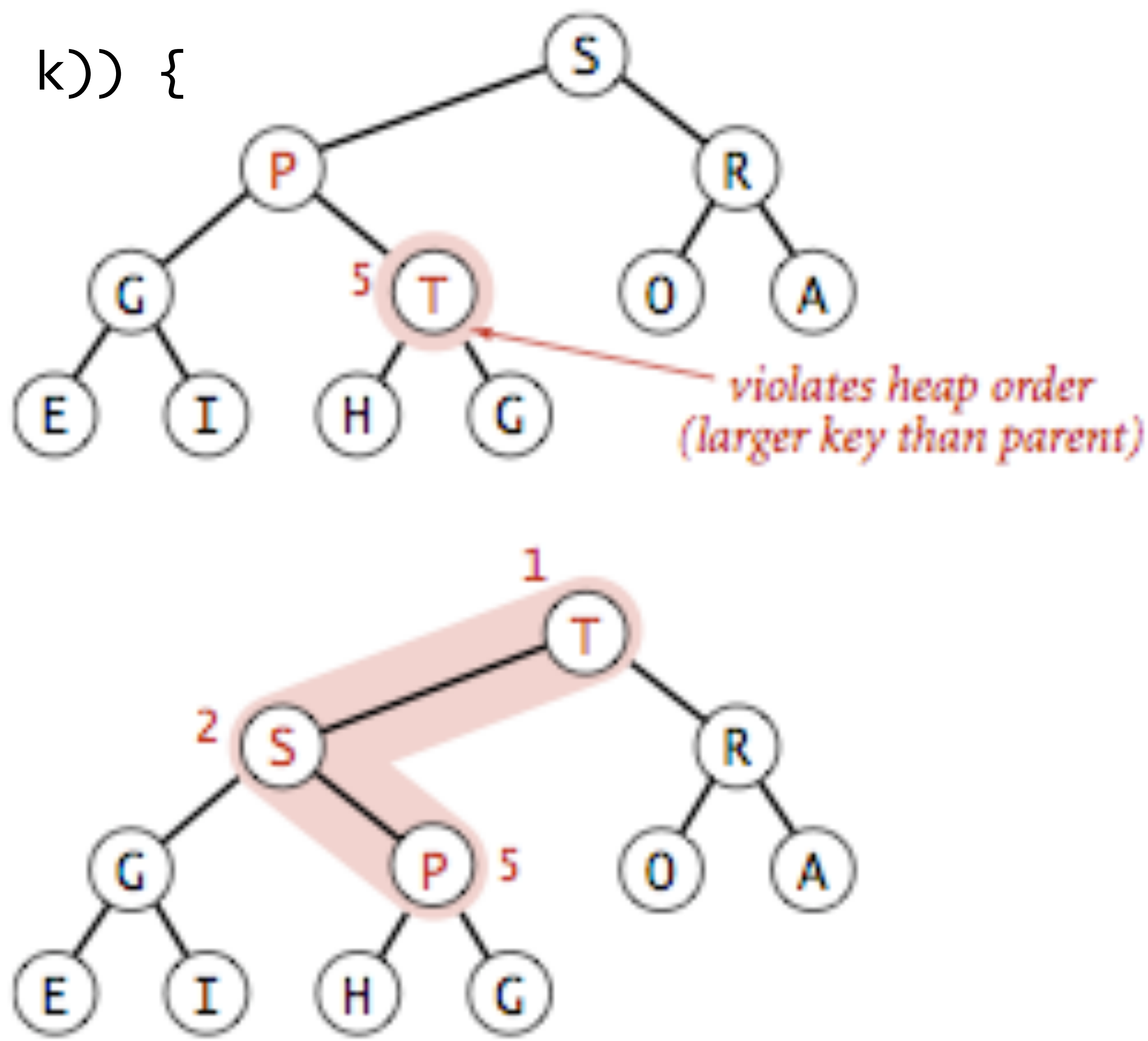
Swim/promote/percolate up/bottom up reheapify

- Scenario: a key becomes larger than its parent therefore it violates the heap-ordered property.
- To eliminate the violation:
  - Exchange key in child with key in parent.
  - Repeat until heap order restored.



Swim/promote/percolate up

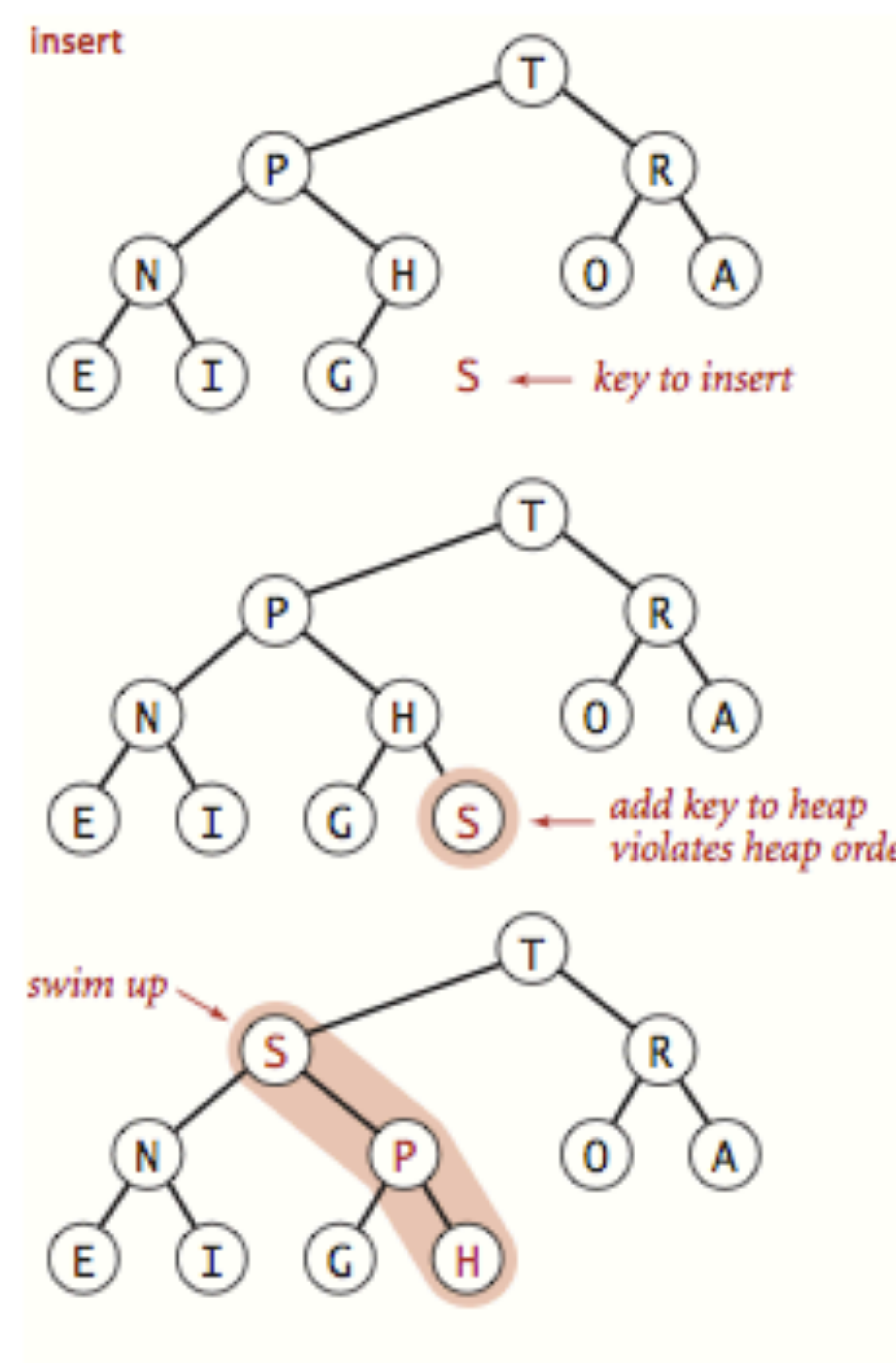
```
private void swim(int k) {
   while (k > 1 && less(k/2, k)) {
      exch(k, k/2);
      k = k/2;
   }
}
```



Binary heap: insertion

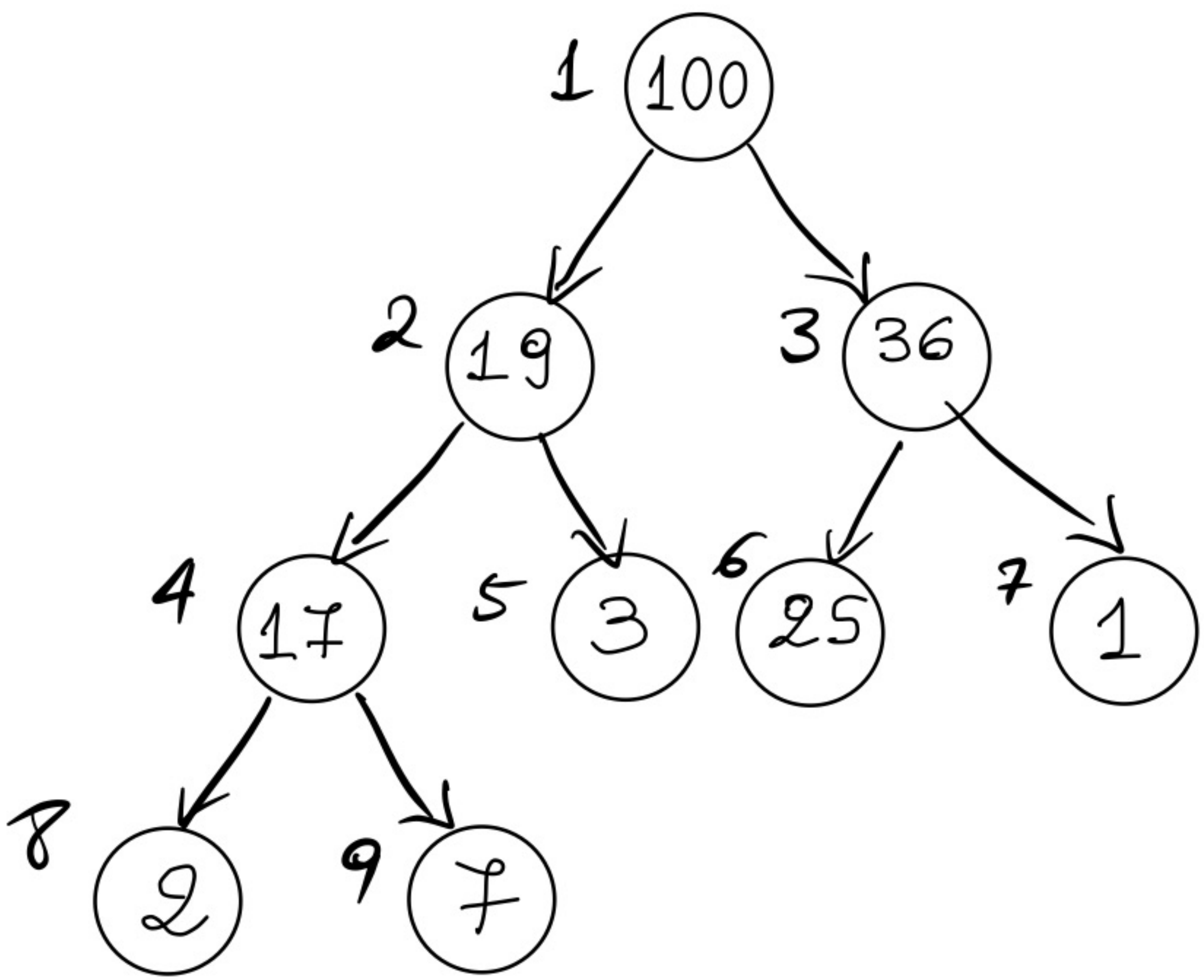
- Insert: Add node at end in bottom level, then swim it up.
- Cost: At most $\log n + 1$ compares.

```
public void insert(Key x) {
    pq[++n] = x;
    swim(n);
}
```

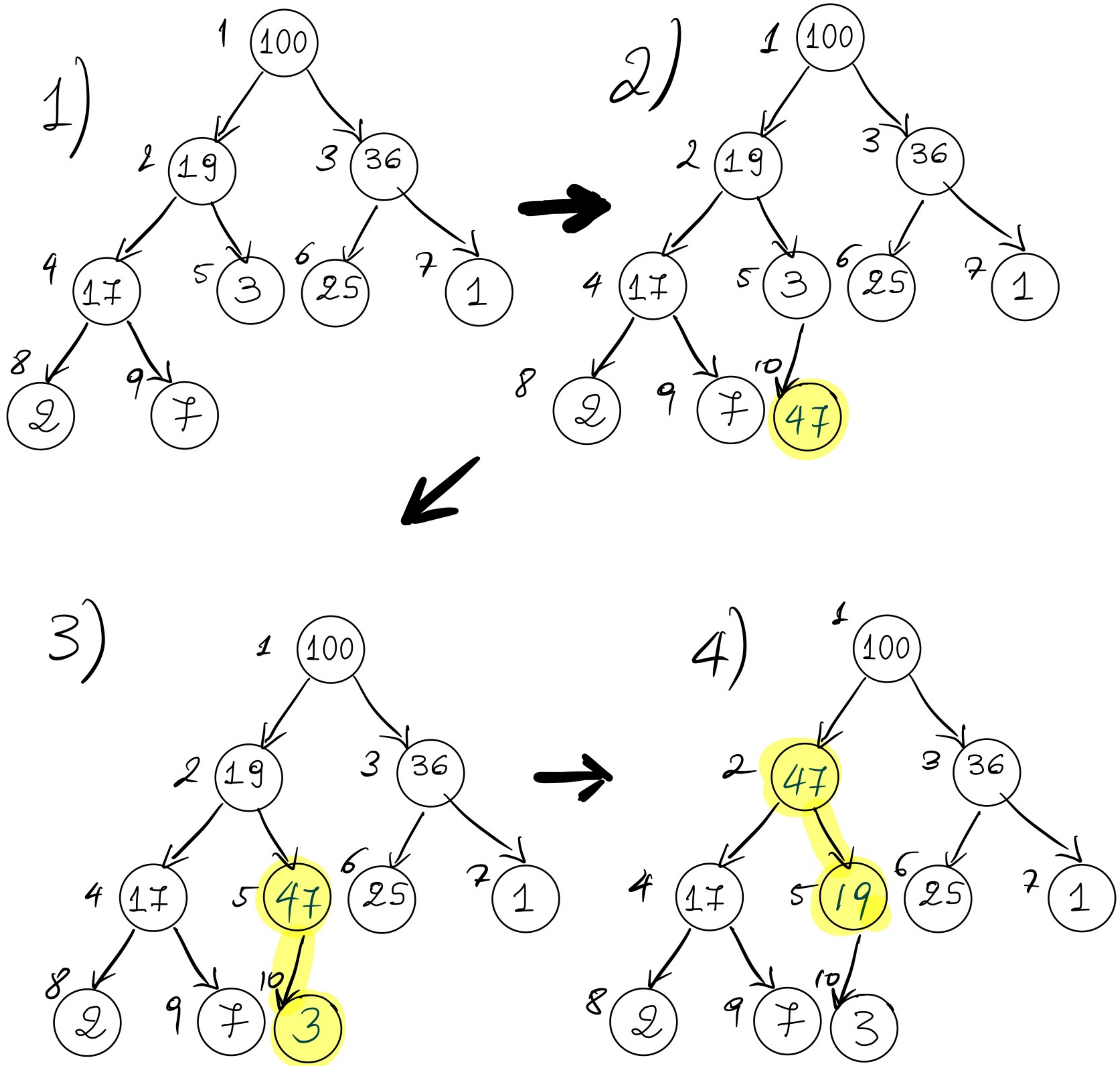


Practice Time

- Insert 47 in this binary heap.

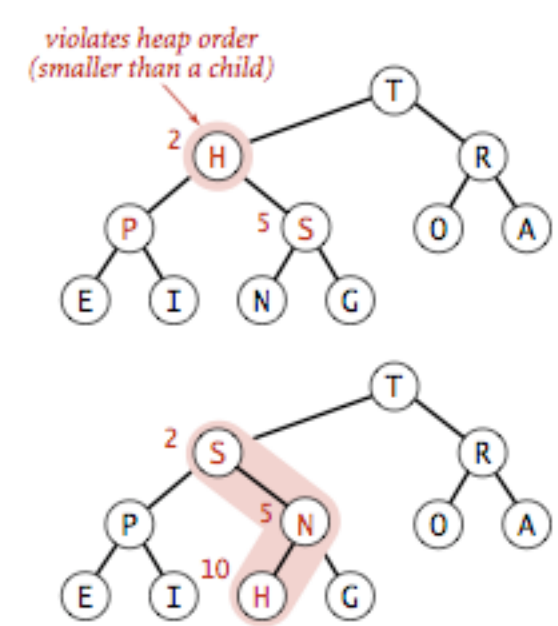


Answer


1)
100
19
36
17
3
25
1
2
7
2)
100
19
36
17
3
25
1
2
7
47
3)
100
19
36
17
47
25
1
2
7
3
4)
100
47
36
17
19
25
1
2
7
3

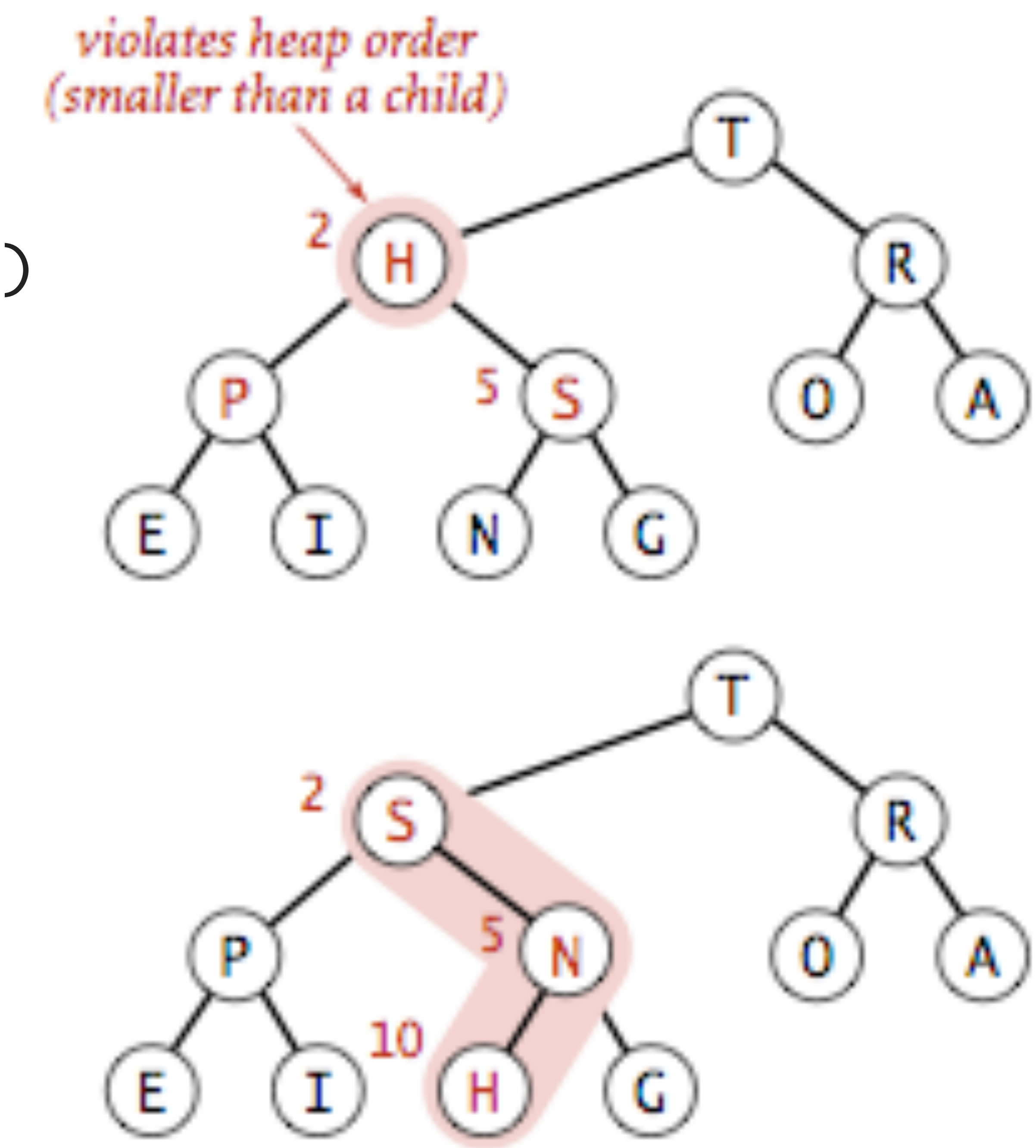
Sink/demote/top down heapify

- Scenario: a key becomes smaller than one (or both) of its children's keys.
- To eliminate the violation:
  - Exchange key in parent with key in **larger** child.
  - Repeat until heap order is restored.



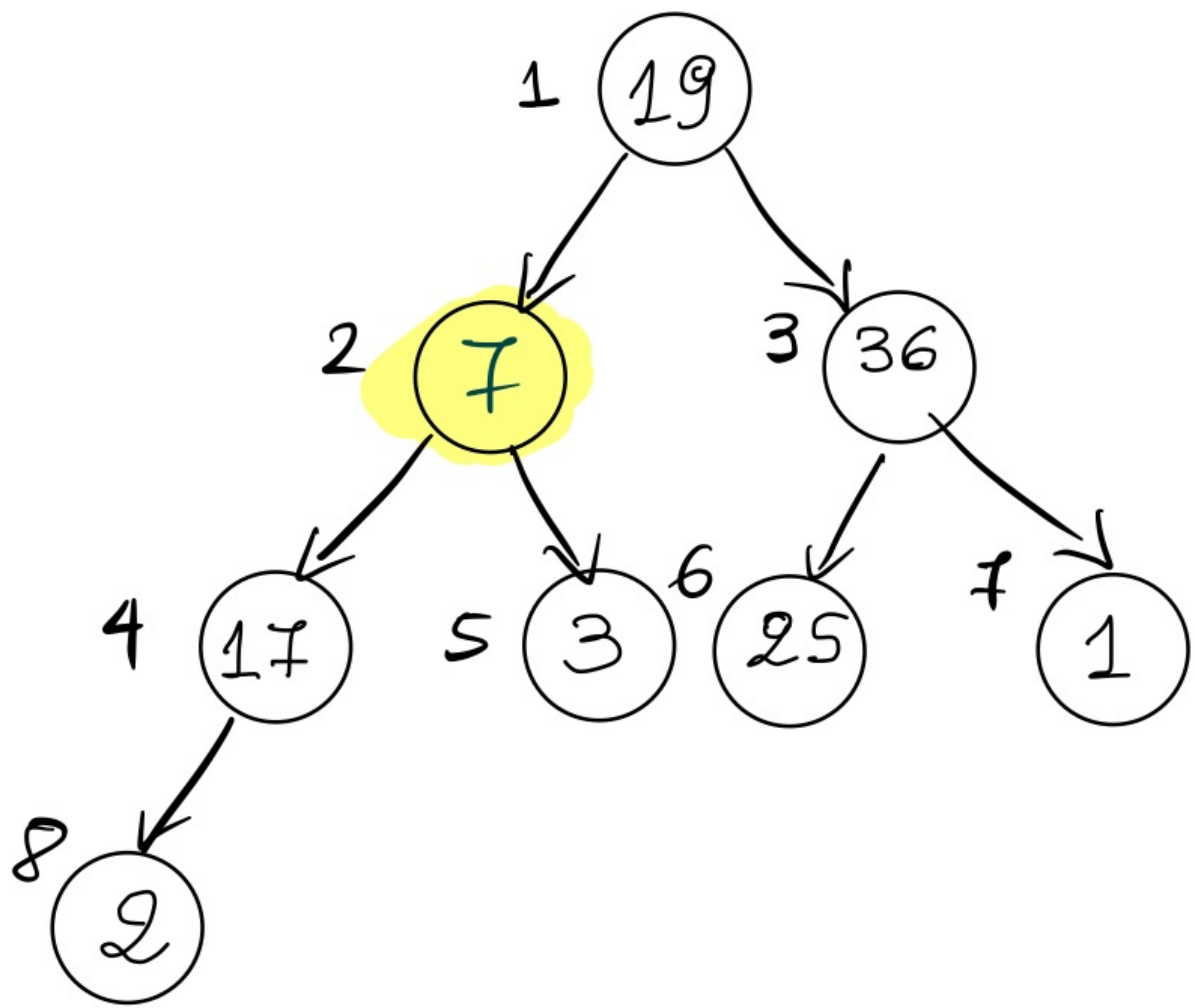
Sink/demote/top down heapify

```
private void sink(int k) {
    while (2*k <= n) {
        int j = 2*k;
        if (j < n && less(j, j+1))
            j++;
        if (!less(k, j))
            break;
        exch(k, j);
        k = j;
    }
}
```

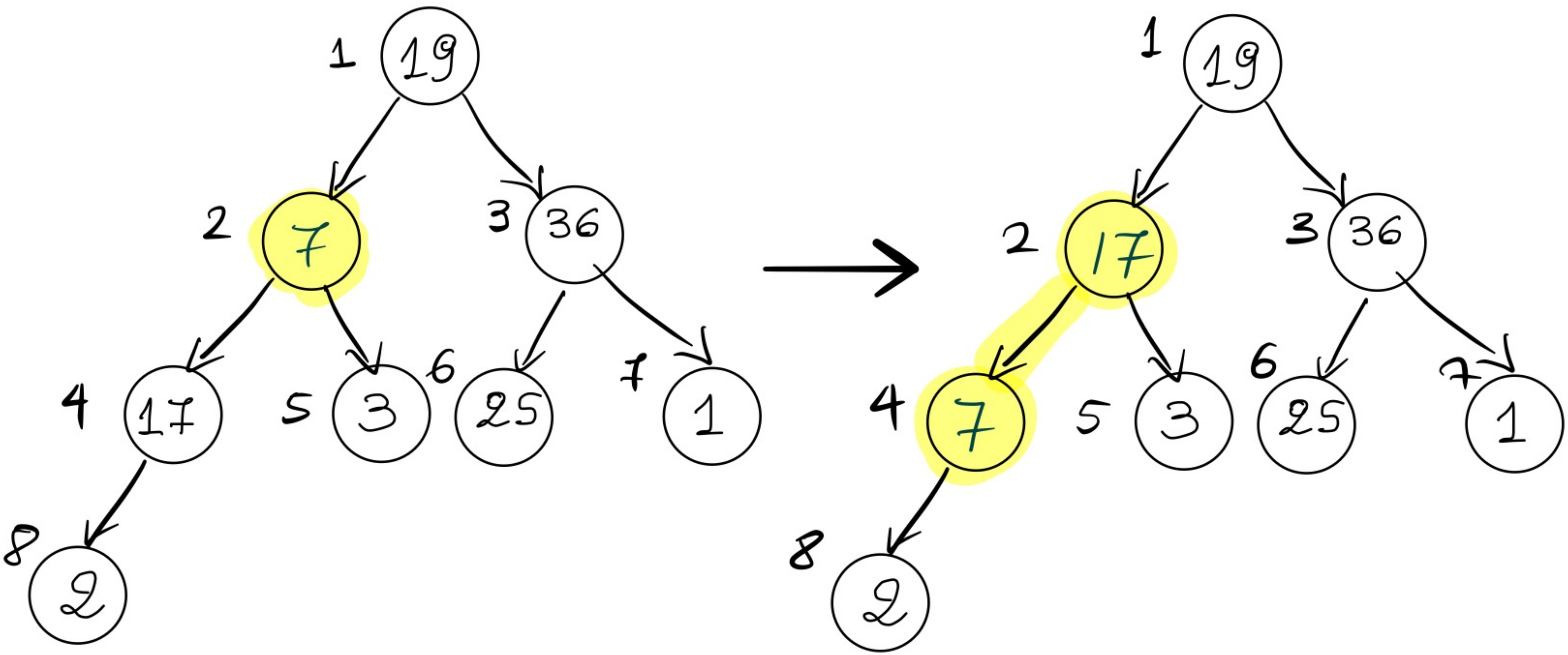


Practice Time

- Sink 7 to its appropriate place in this binary heap.



Answer

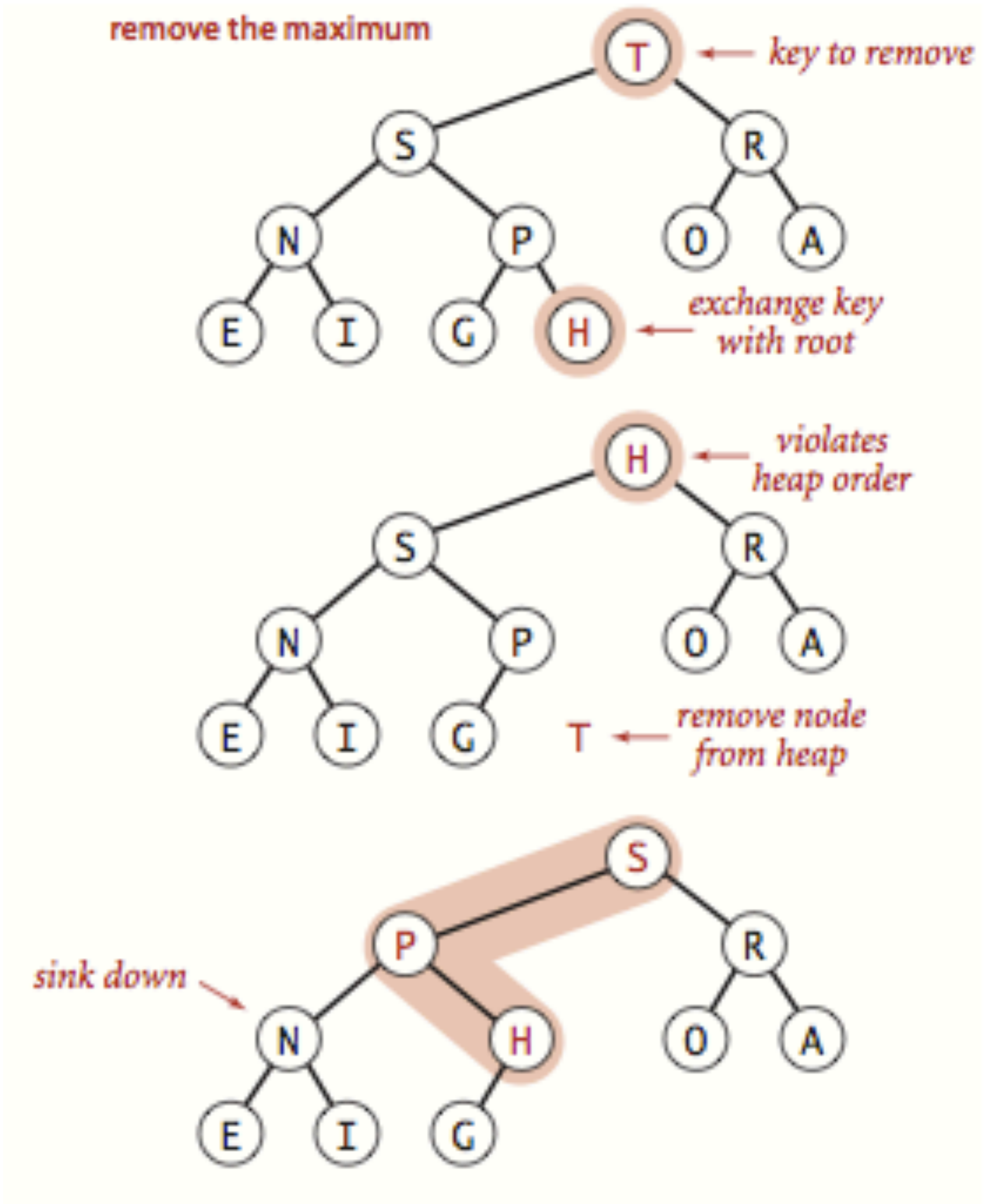


Binary heap: return (and delete) the maximum

- Delete max: Exchange root with node at end. Return it and delete it. Sink the new root down.
- Cost: At most $2 \log n$ compares.

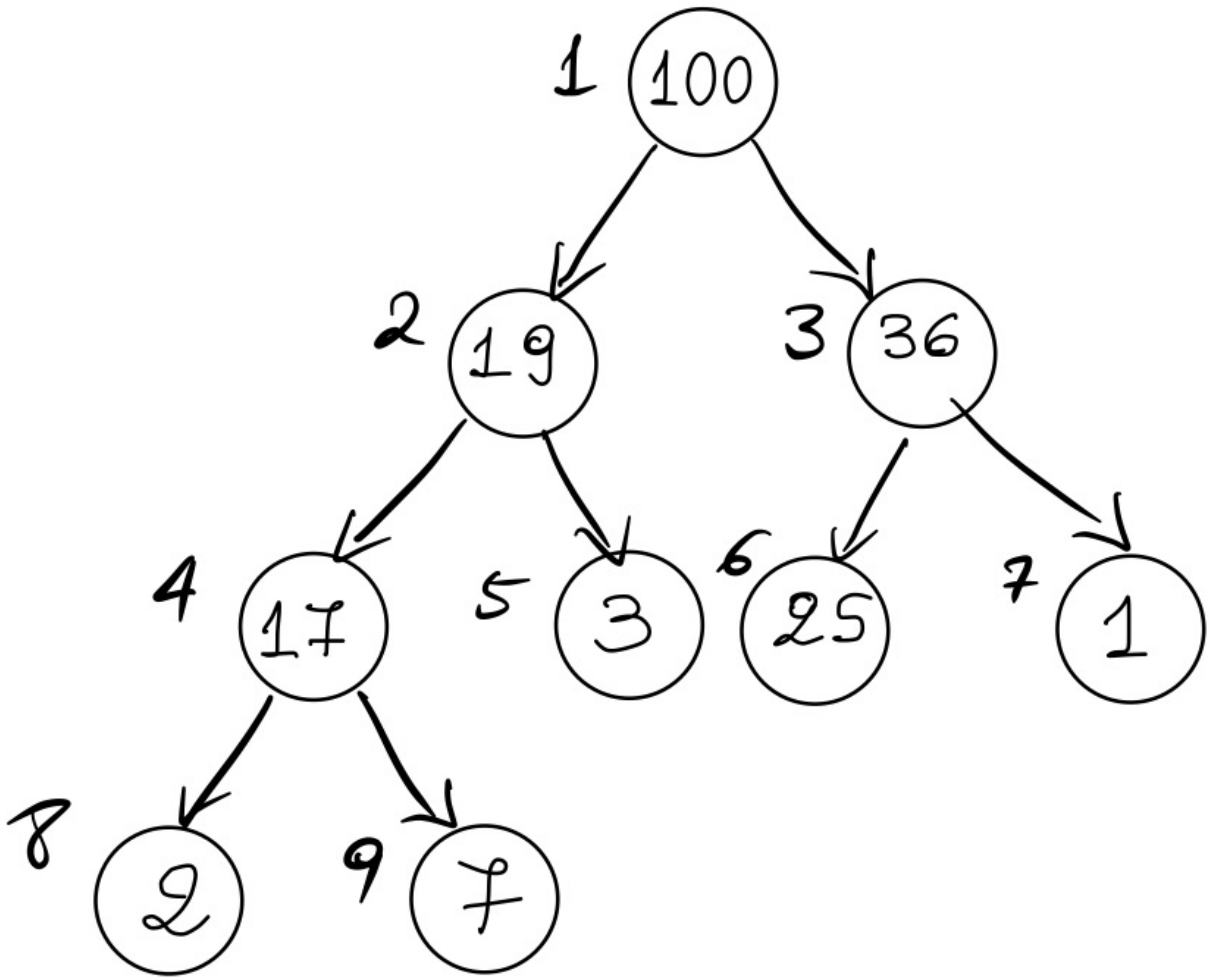
```
public Key delMax() {
    Key max = pq[1];
    exch(1, n--);
    sink(1);
    pq[n+1] = null;
    return max;
}
```

Binary heap: delete and return maximum

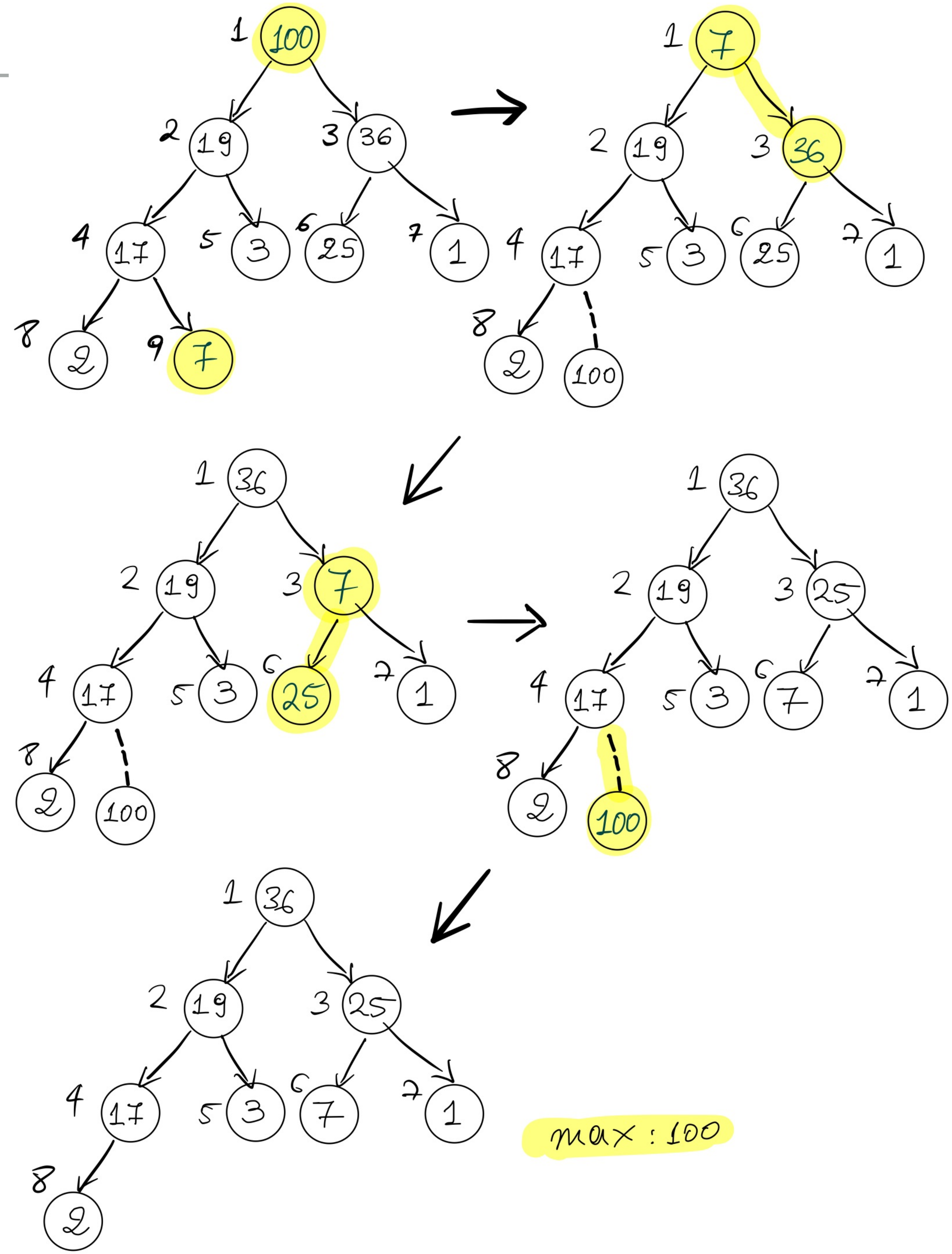


Practice Time

- Delete max (and return it!)



Answer



Things to remember about runtime complexity of heaps

- Insertion is $O(\log n)$.
- Delete max is $O(\log n)$.
- Space efficiency is $O(n)$.

http://algs4.cs.princeton.edu

2.4 BINARY HEAP DEMO

Lecture 16: Binary Trees and Heaps

- Binary Trees
- Tree traversals
- Binary Heaps

Readings:

- Textbook:
    - Chapter 2.4 (Pages 308-327)
- Website:
    - Priority Queues: https://algs4.cs.princeton.edu/24pq/
- Visualization:
    - Insert and ExtractMax: https://visualgo.net/en/heap

Practice Problems:

- Practice with traversals of trees and insertions and deletions in binary heaps