SORTING

CS062

DATA STRUCTURES AND ADVANCED PROGRAMMING

12: Insertion Sort & Mergesort

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she/her/hers

Tom Yeh
he/him/his

Lecture 12: Insertion Sort & Mergesort

- Insertion Sort
- Mergesort

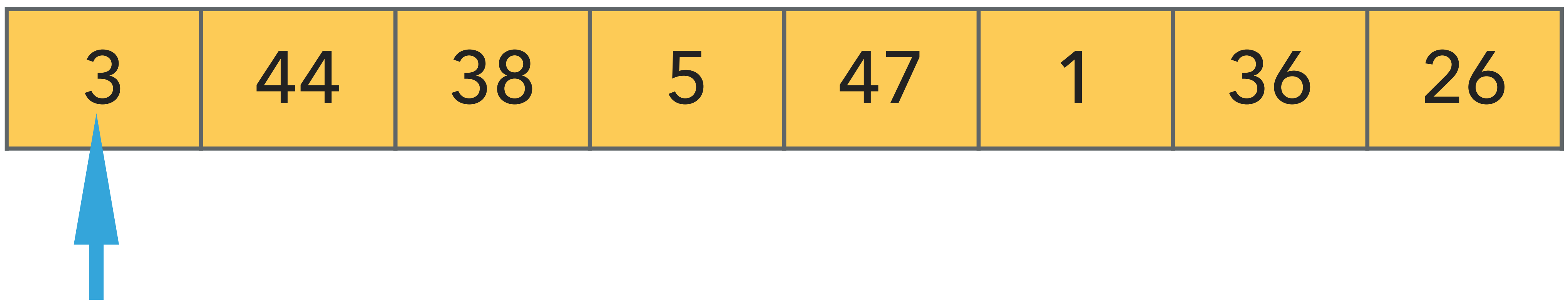
Some slides adopted from Algorithms 4th Edition or COS226

Insertion sort

3	44	38	5	47	1	36	26

- Keep a *partially* sorted subarray on the left and an unsorted subarray on the right
- Repeat:
  - Examine the next element in the unsorted subarray.
  - Insert this element by exchanging with every entry to the left that is greater.
  - Move subarray boundaries one element to the right.

Insertion sort



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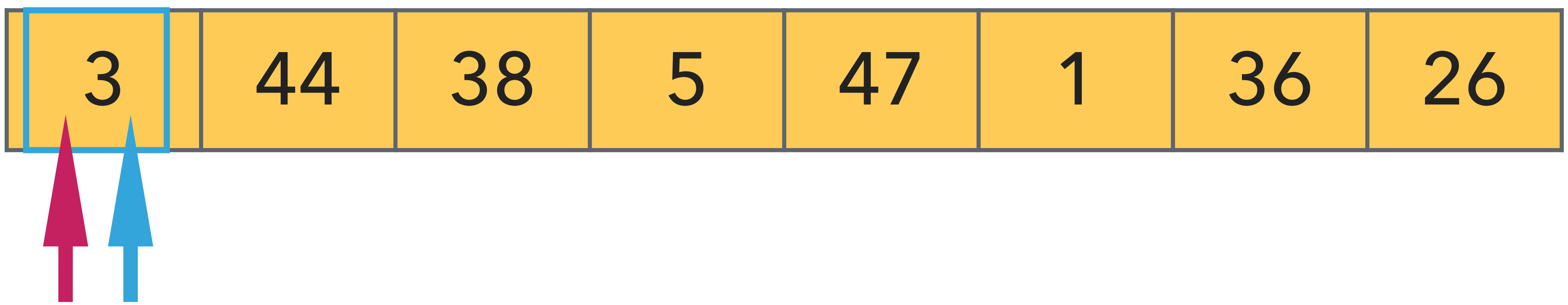

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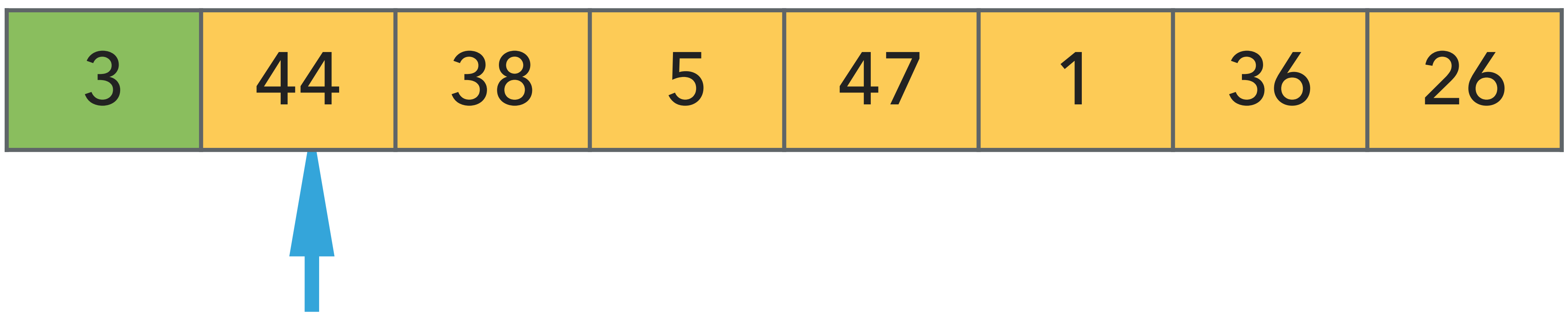

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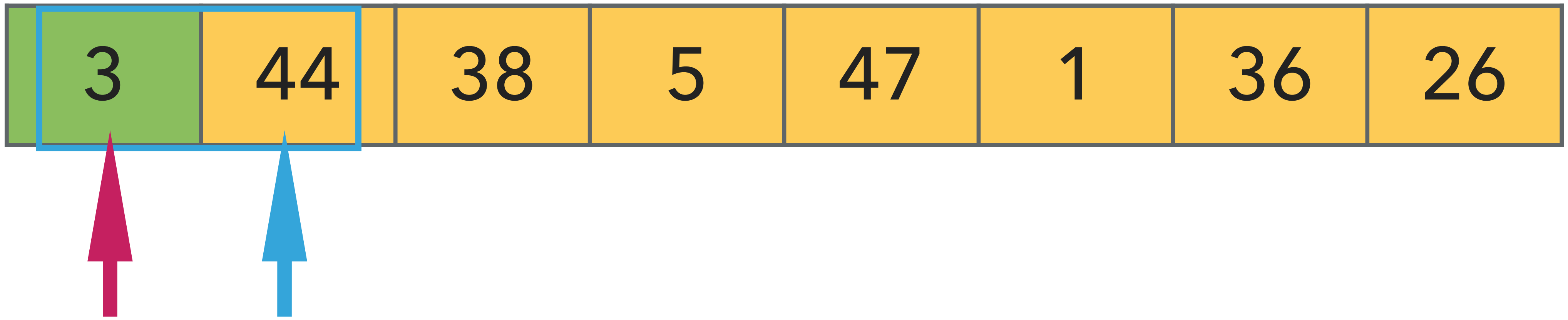

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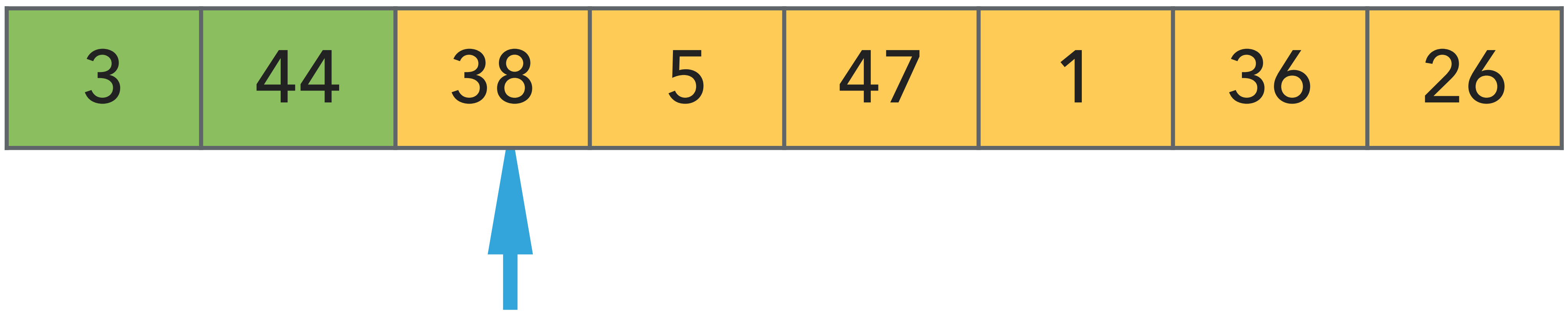

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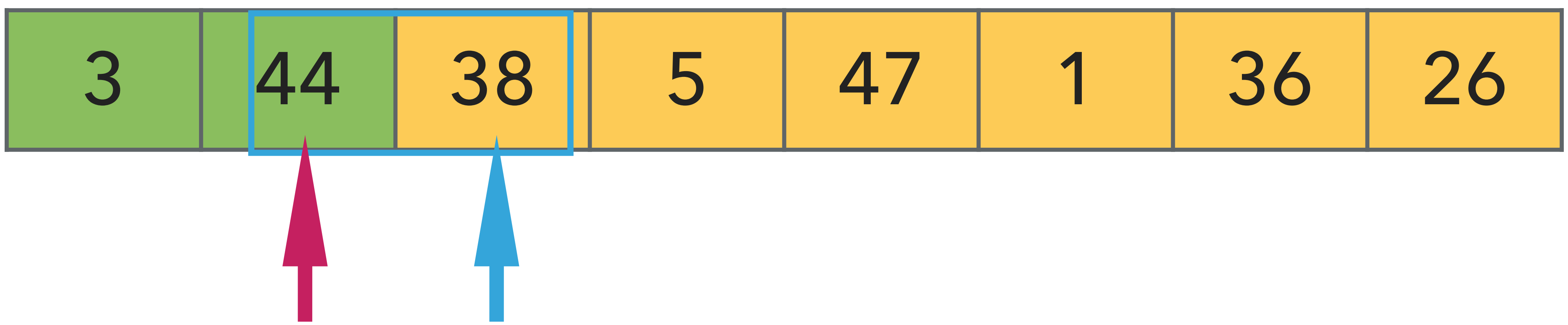

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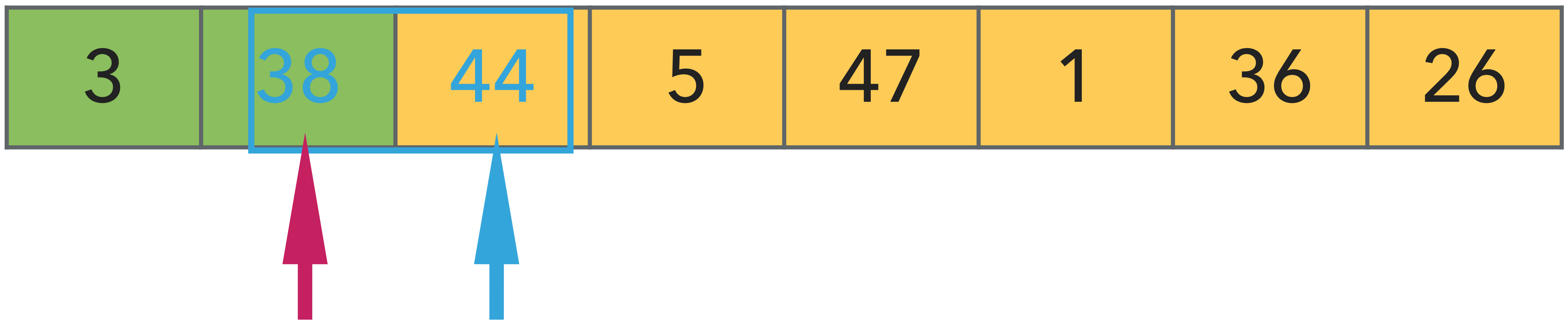

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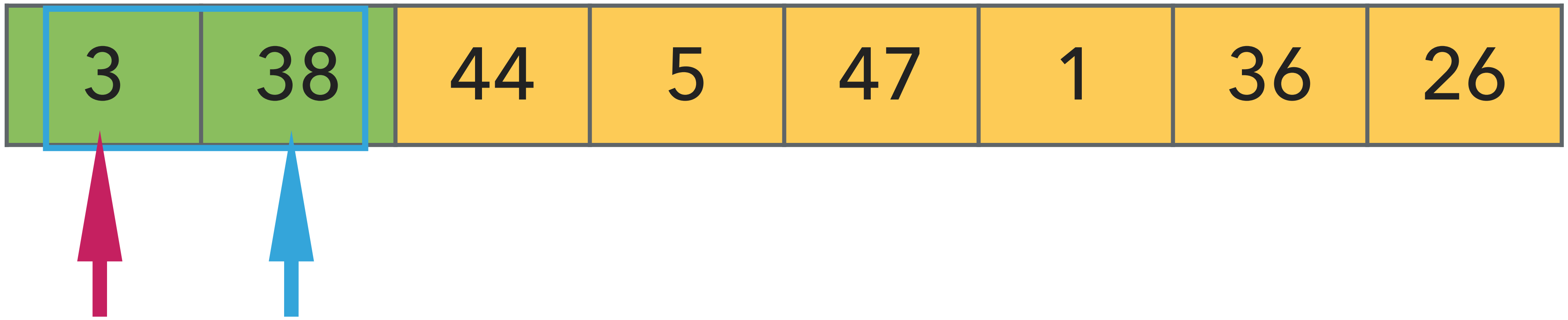

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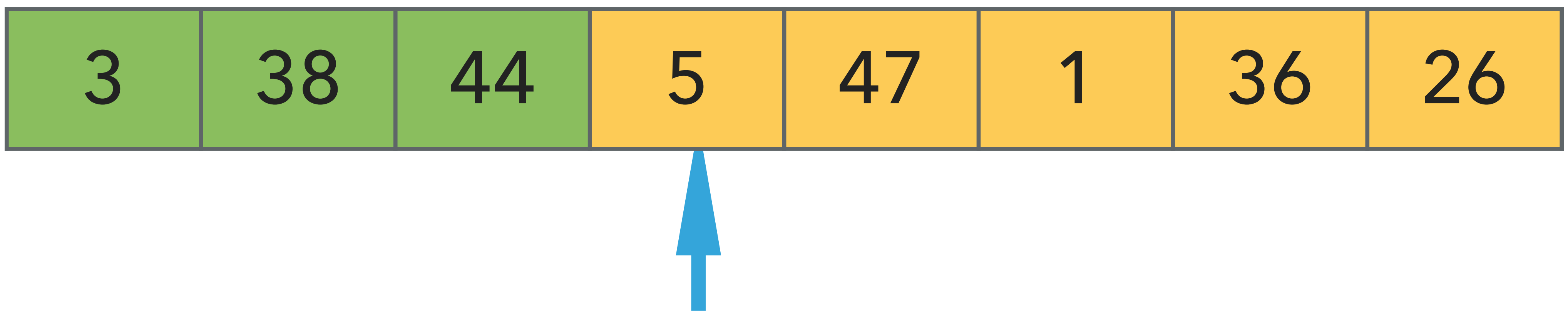

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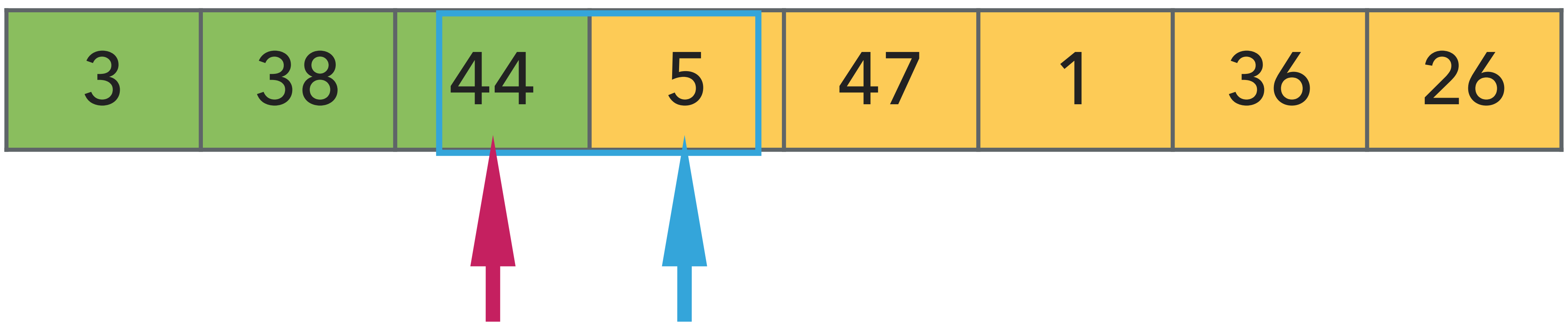

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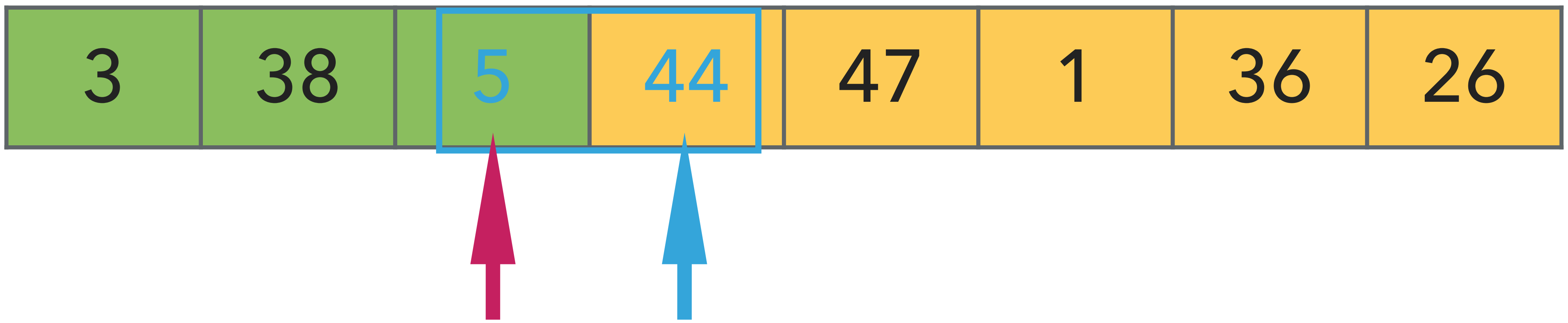

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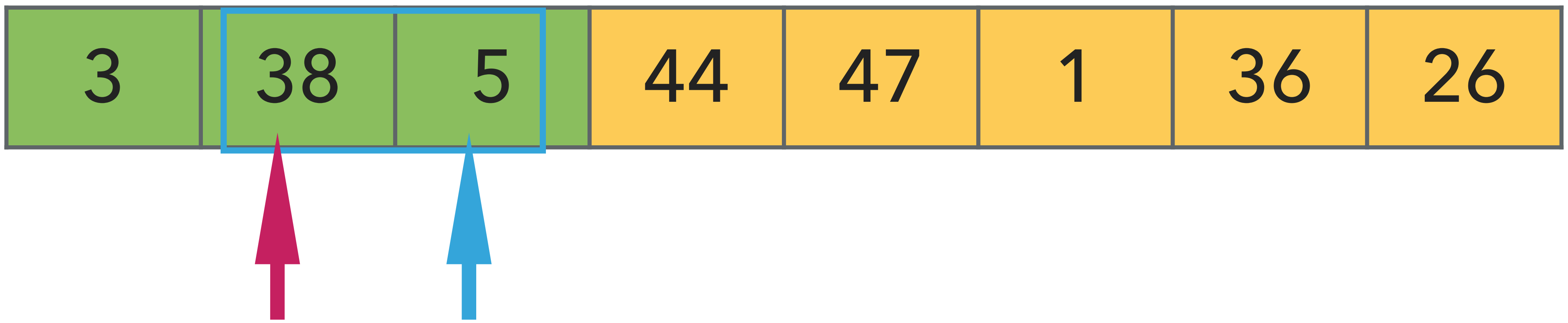

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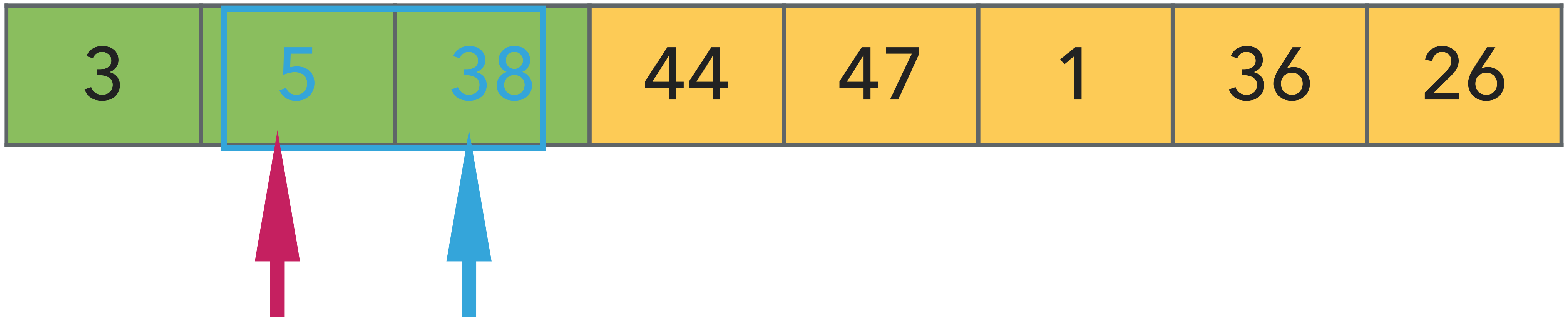

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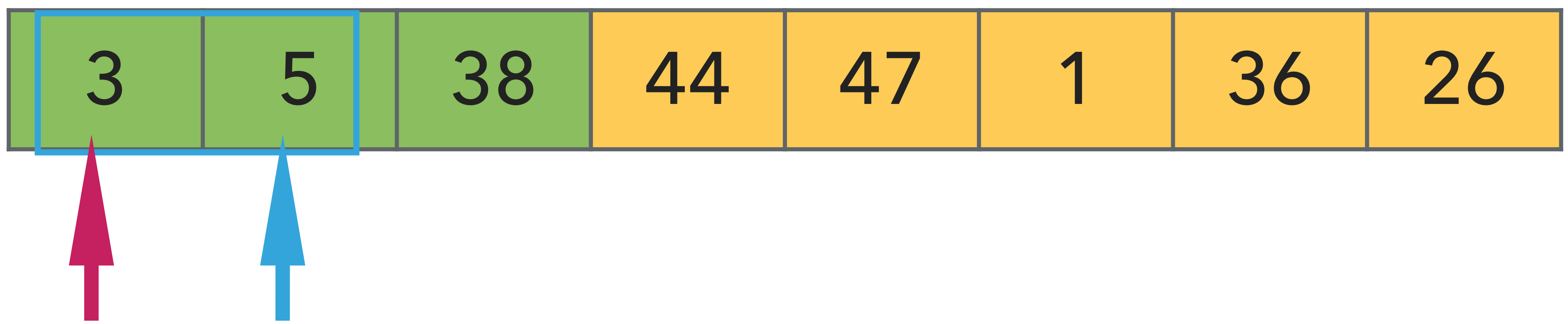

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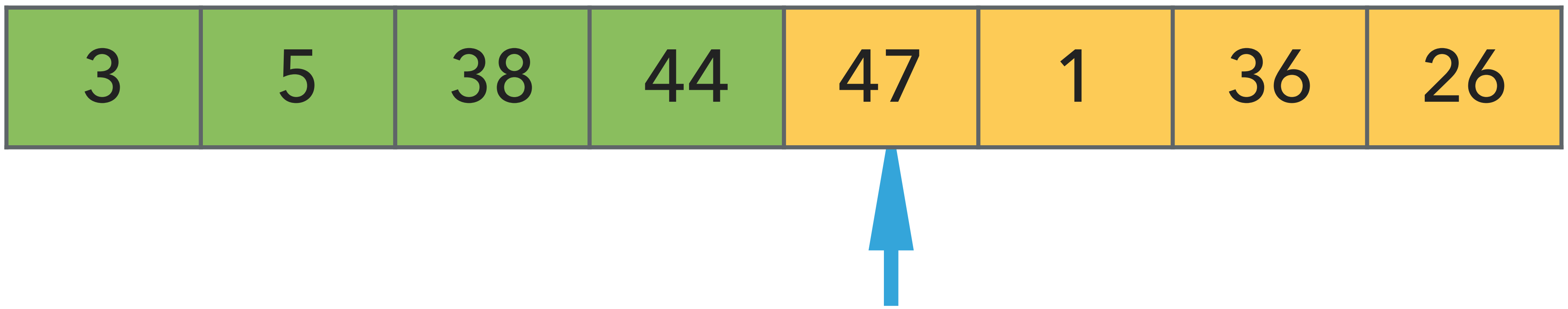

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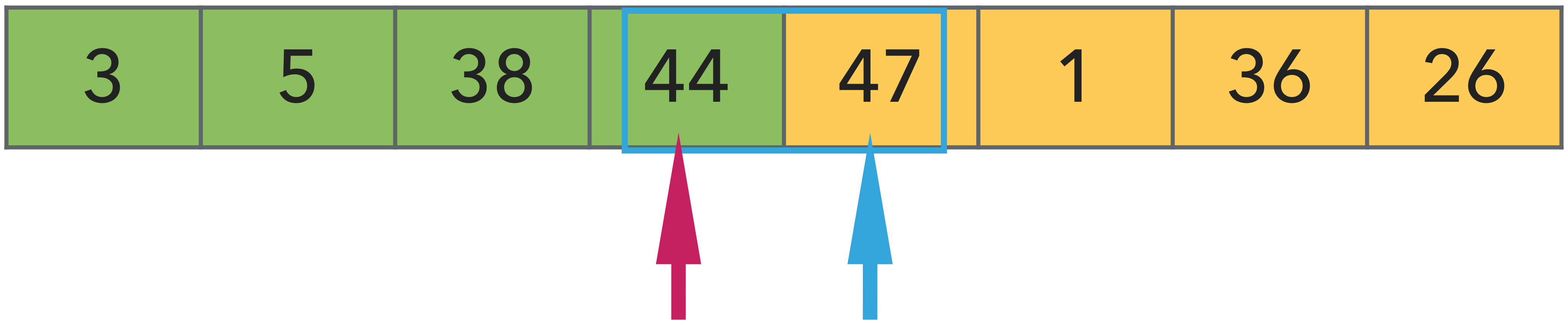

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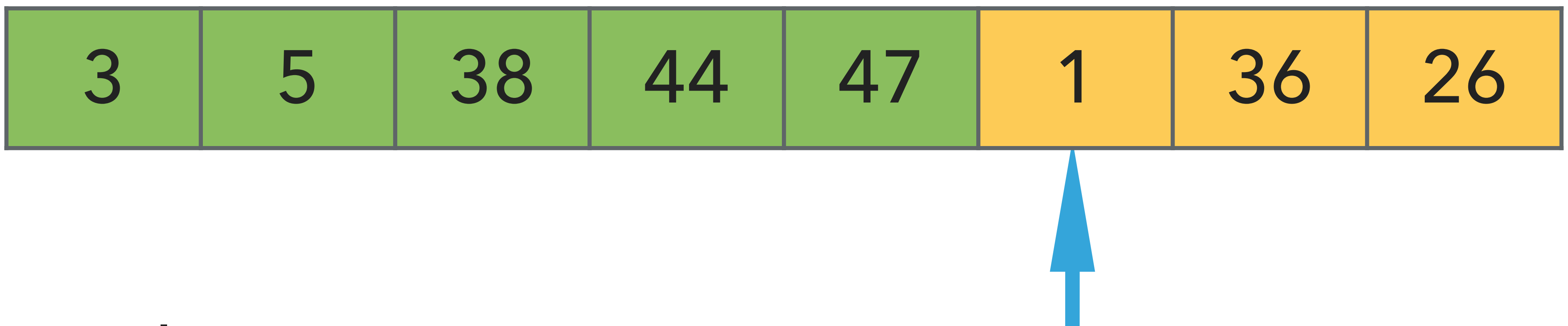

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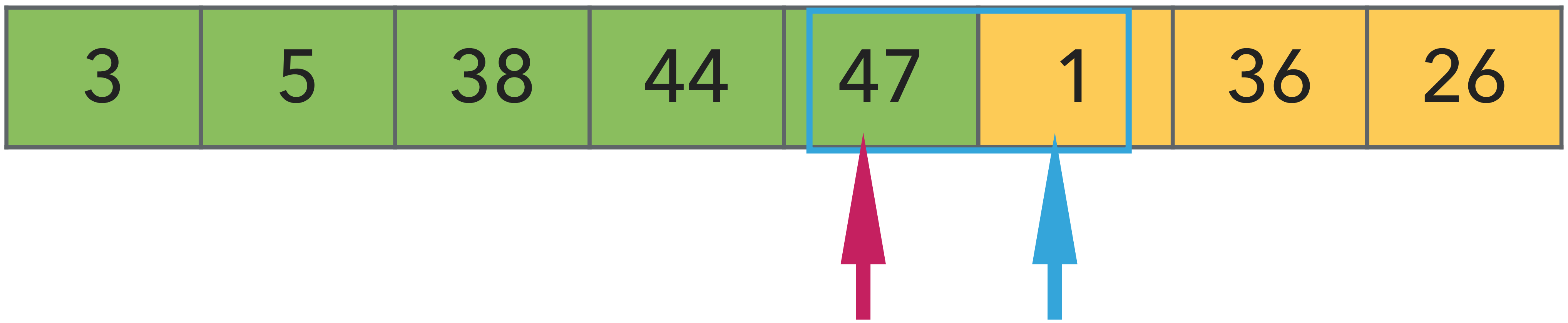

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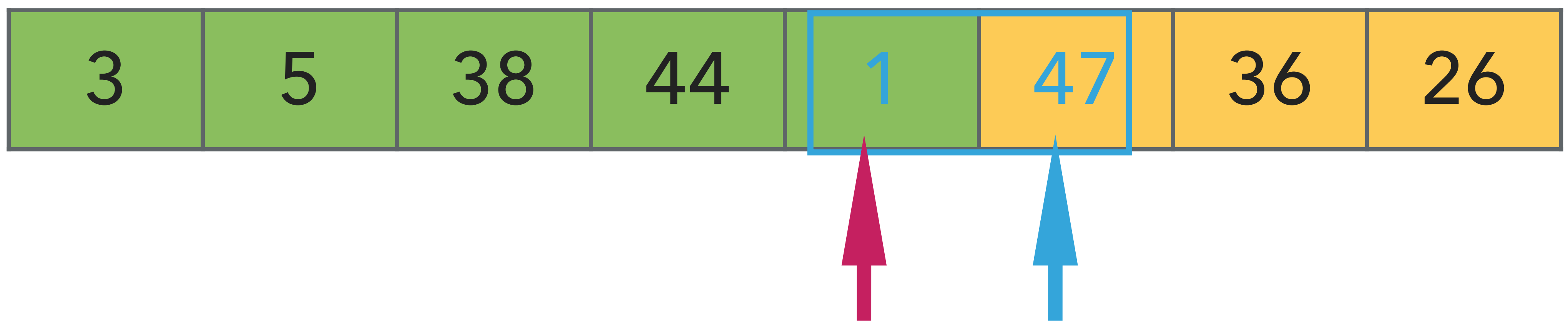

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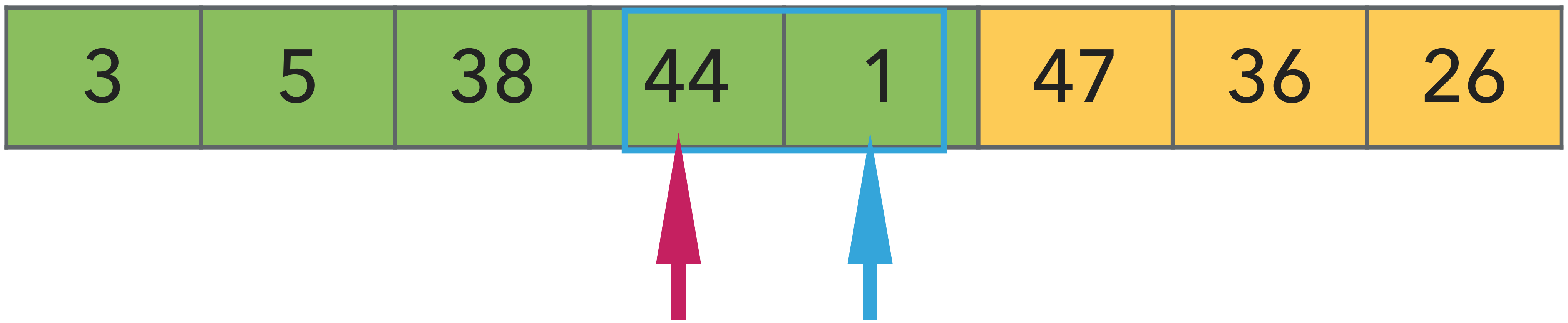

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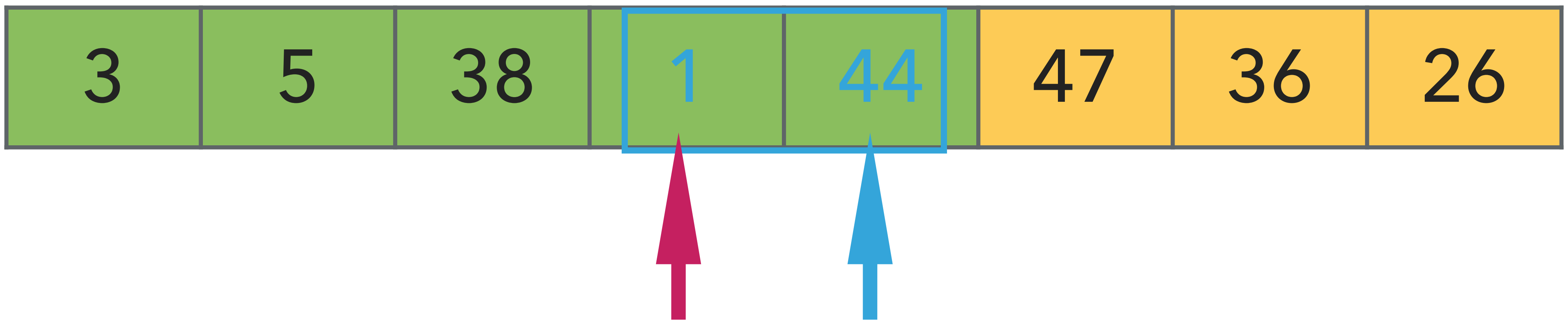

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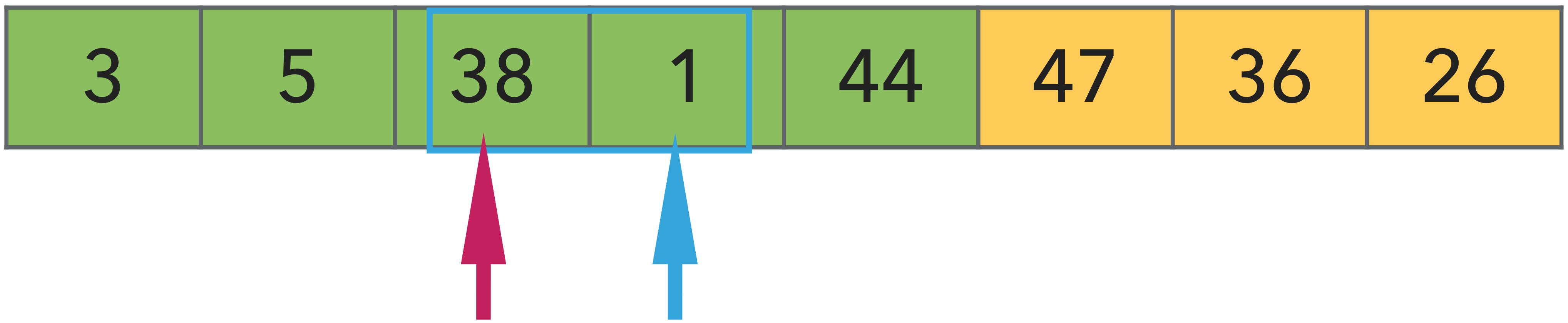

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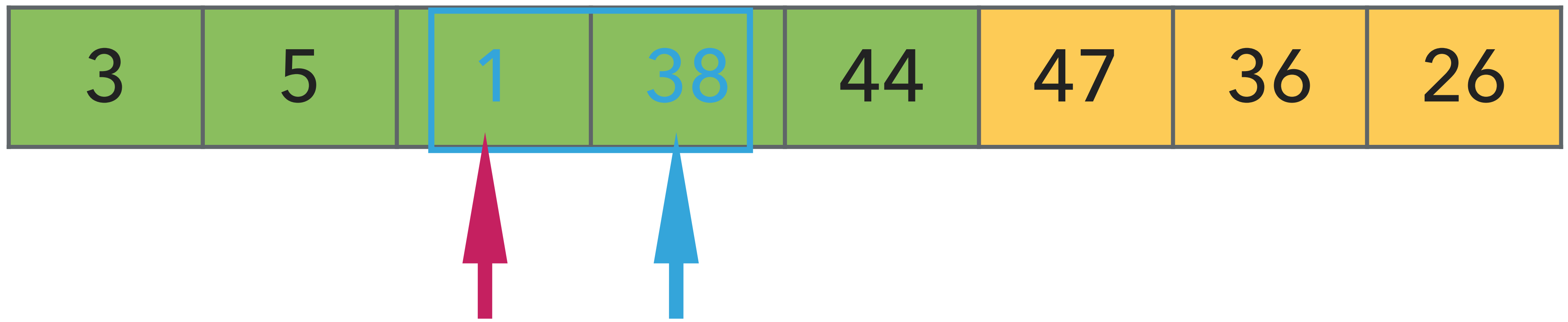

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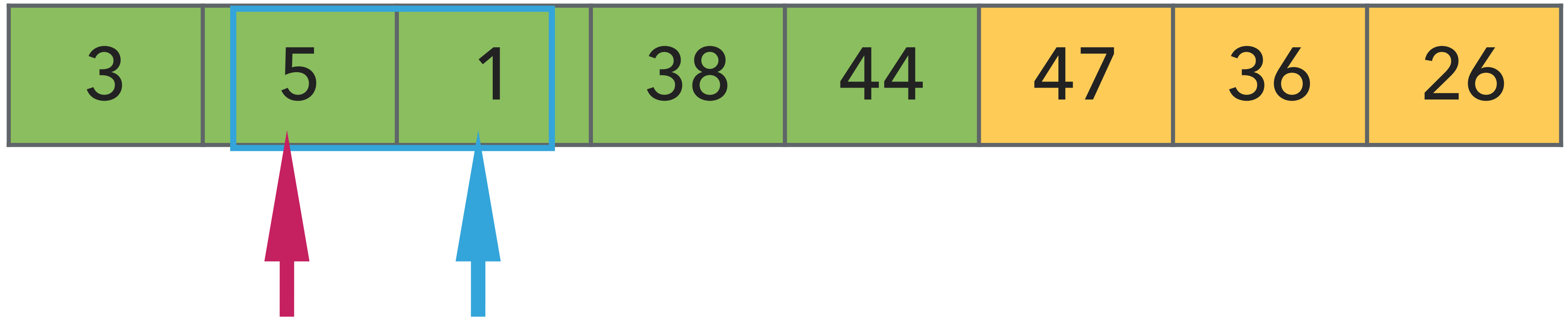

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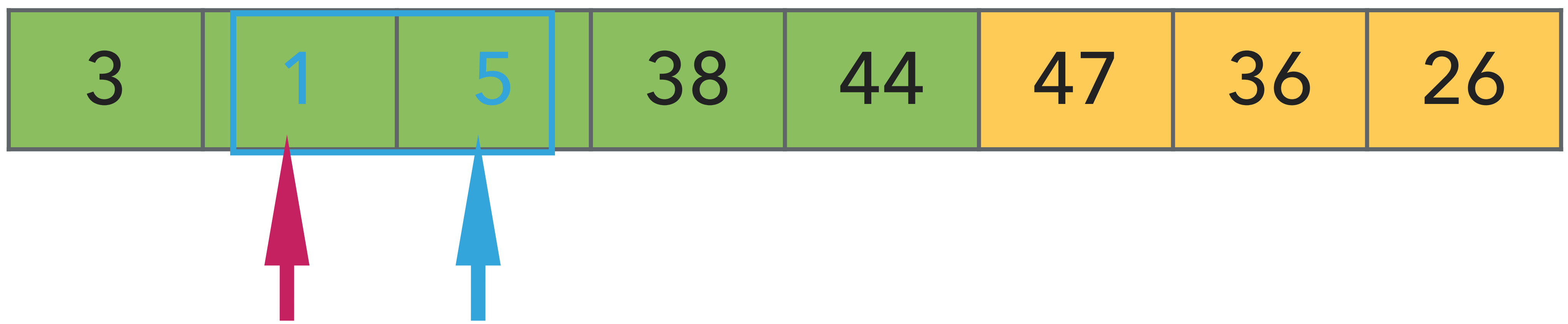

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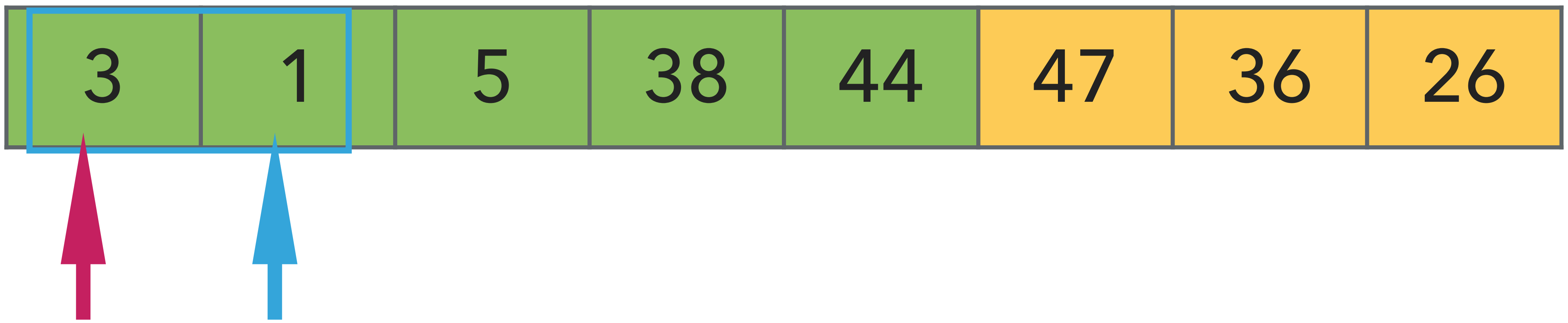

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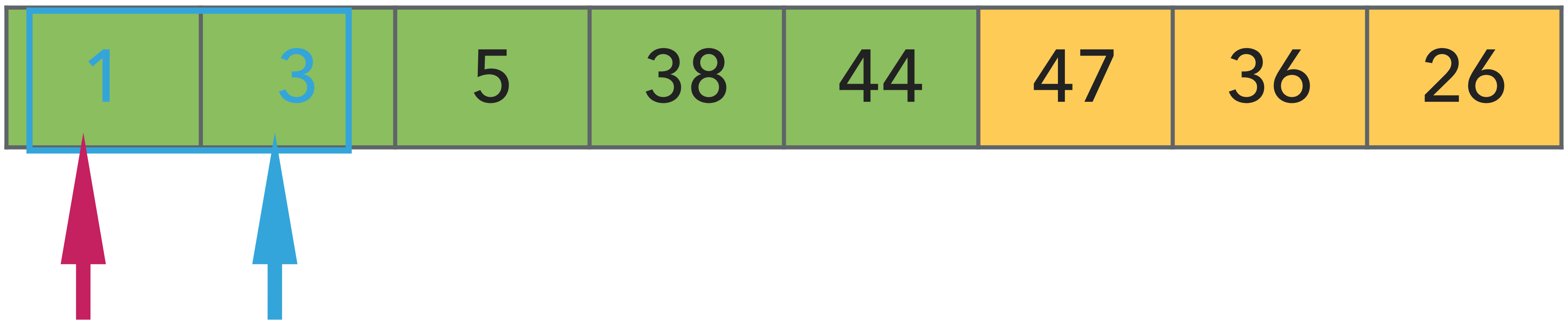

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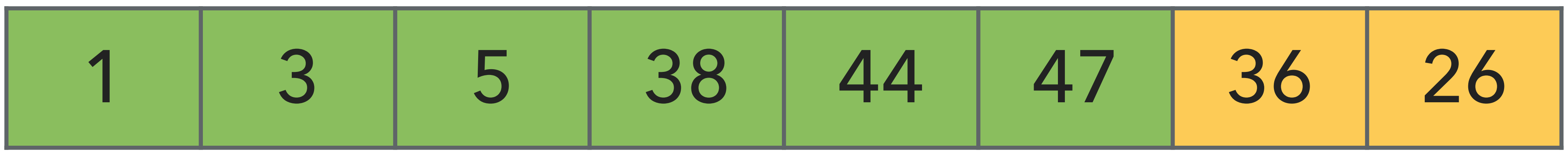

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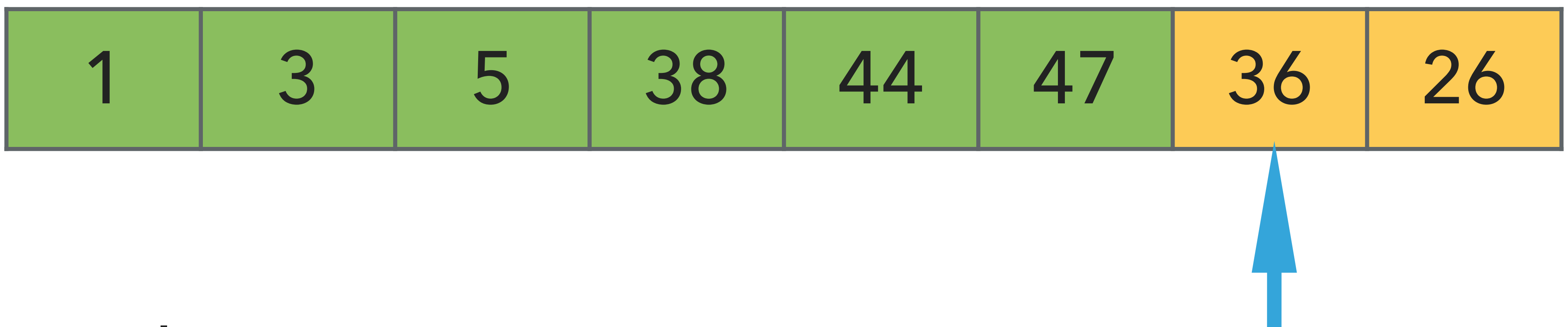

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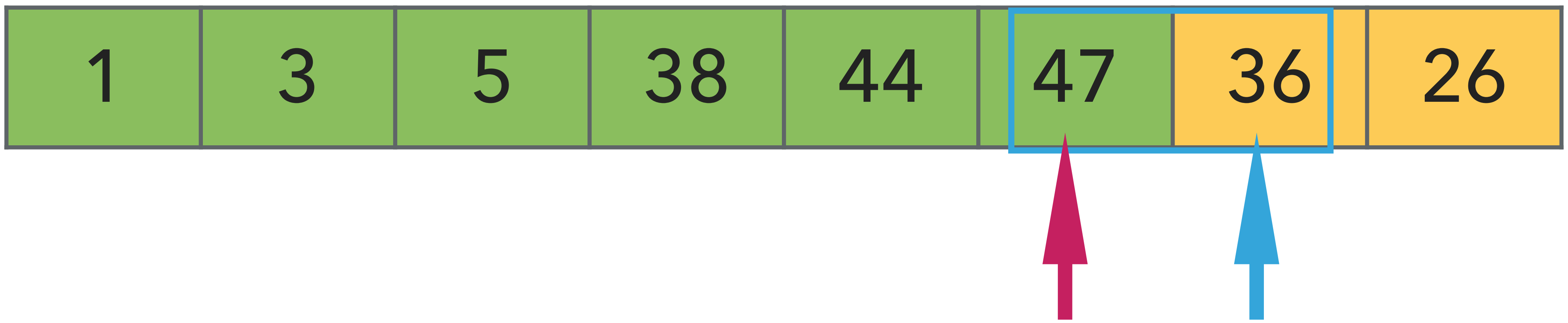

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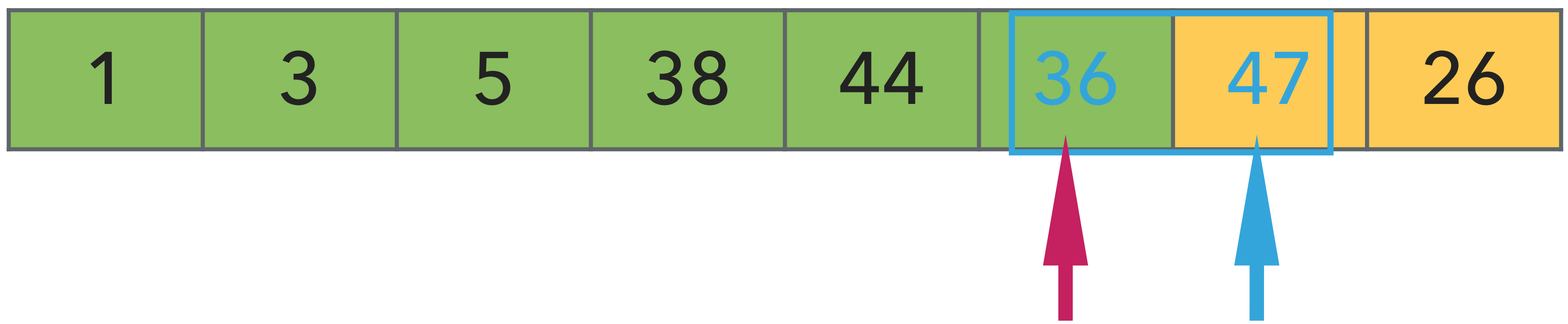

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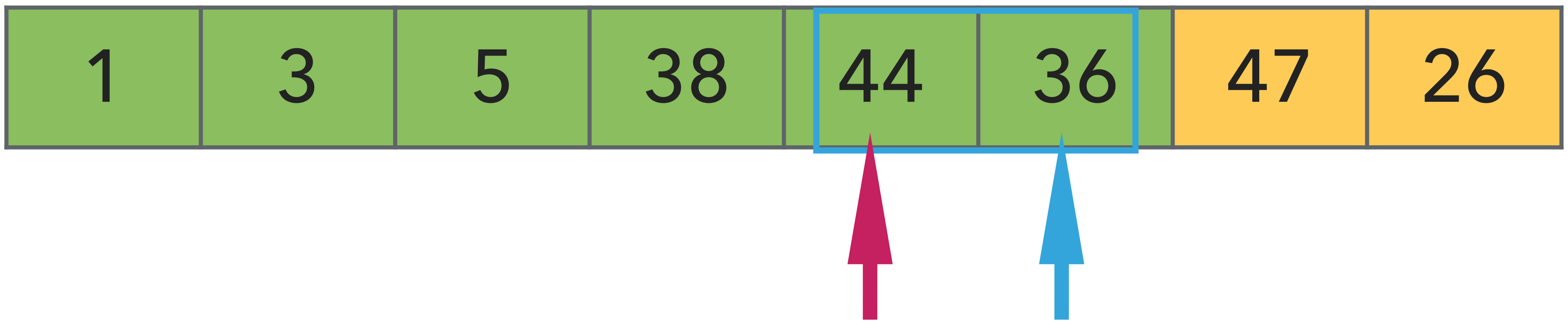

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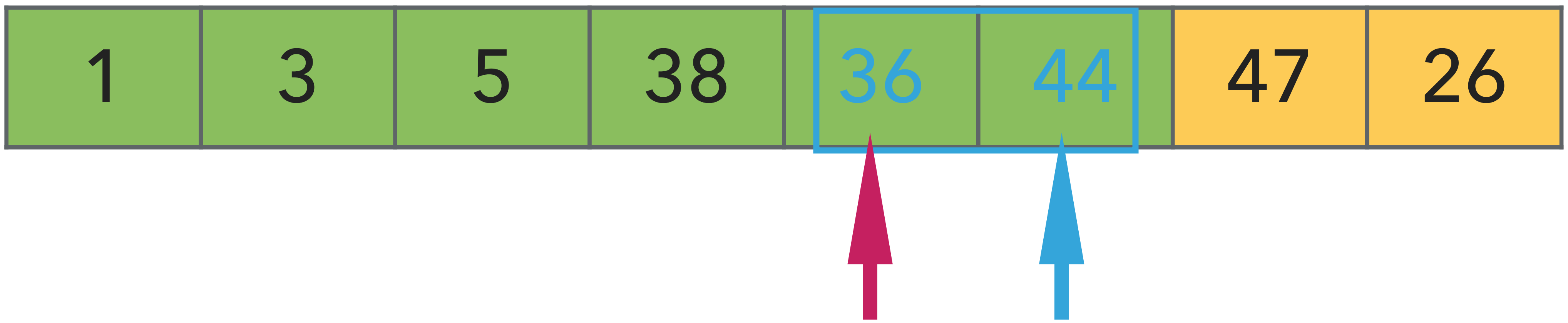

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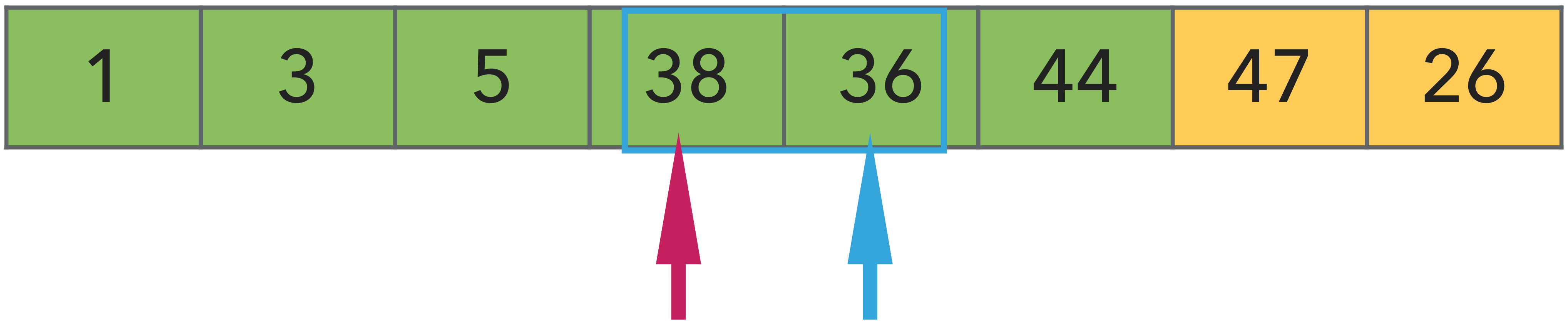

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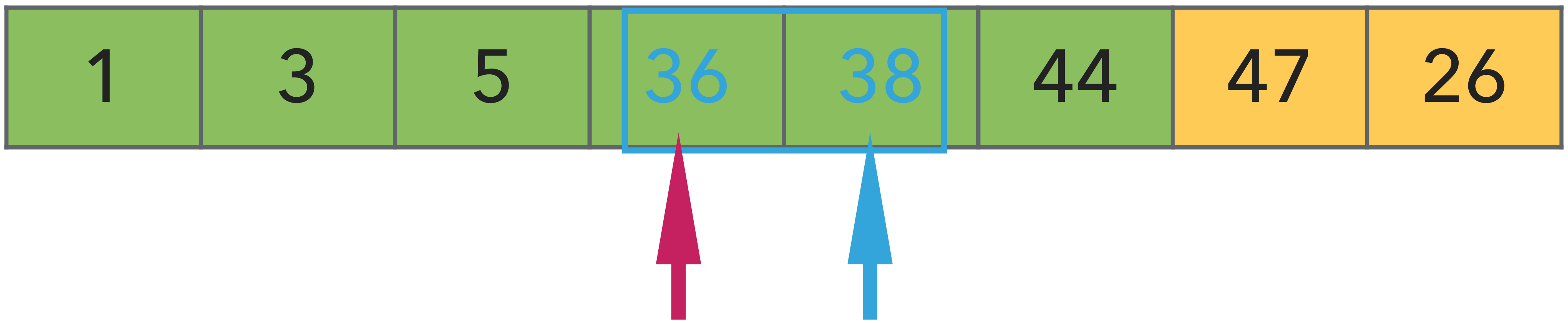

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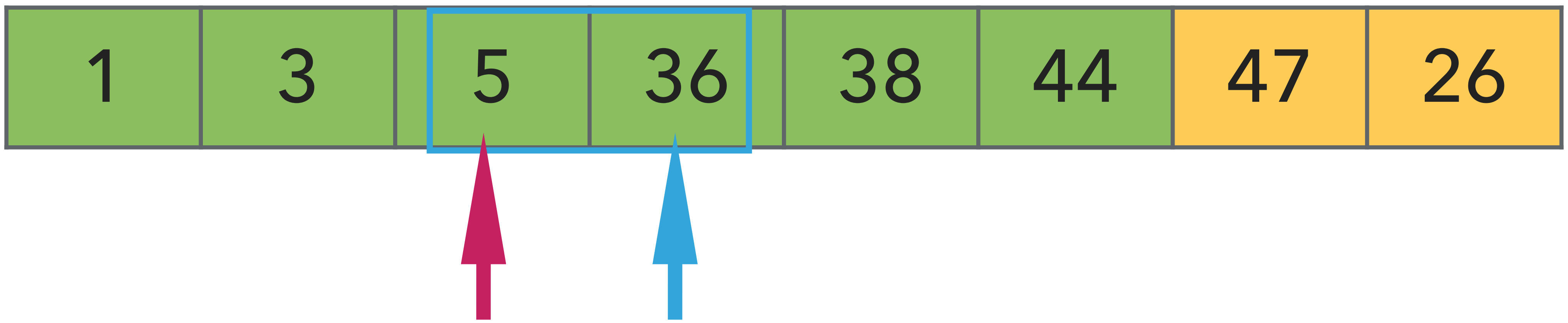

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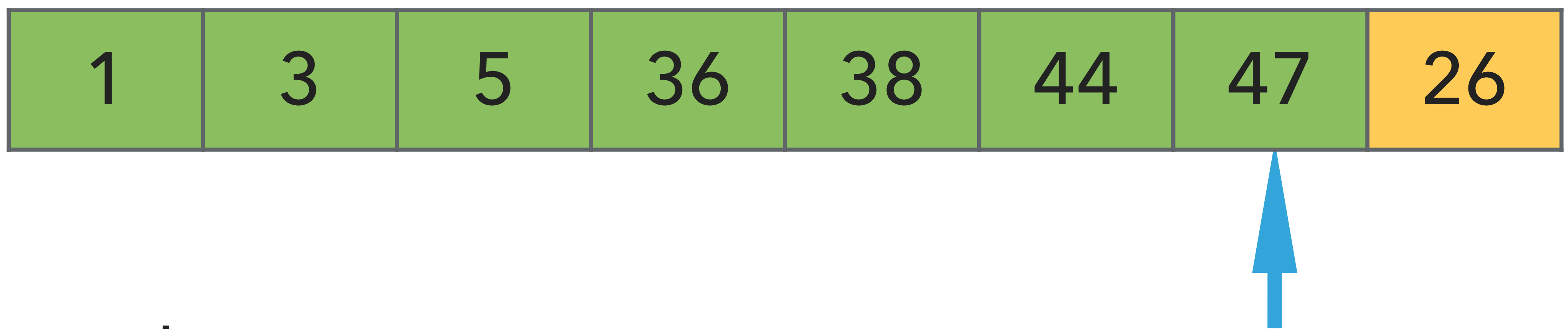

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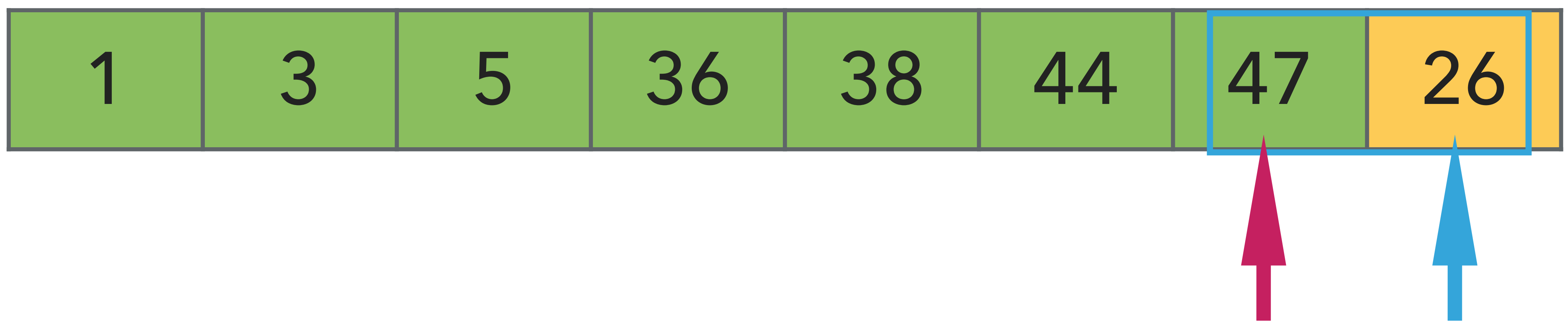

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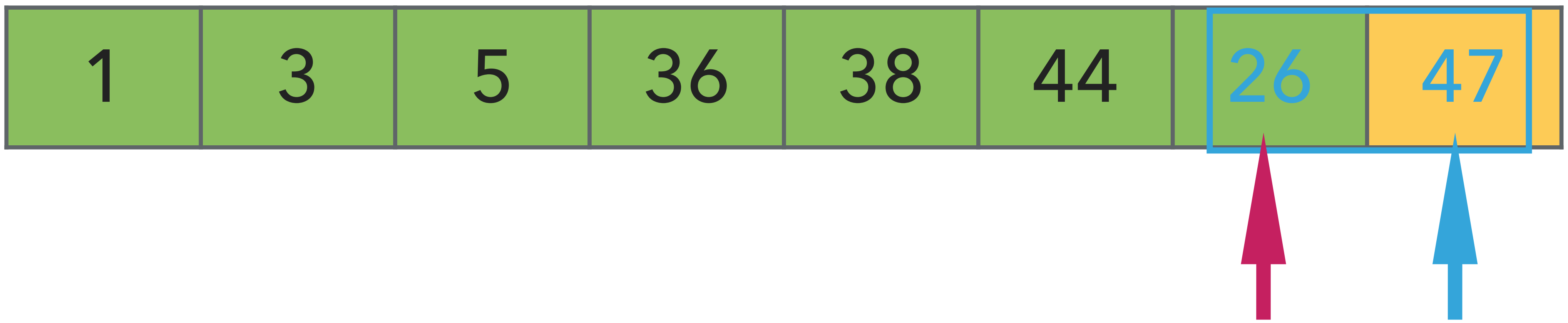

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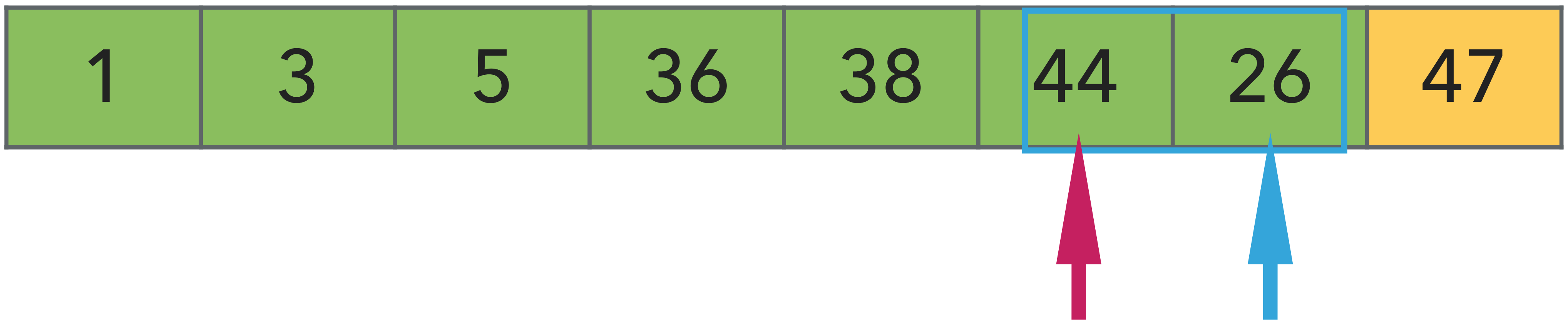

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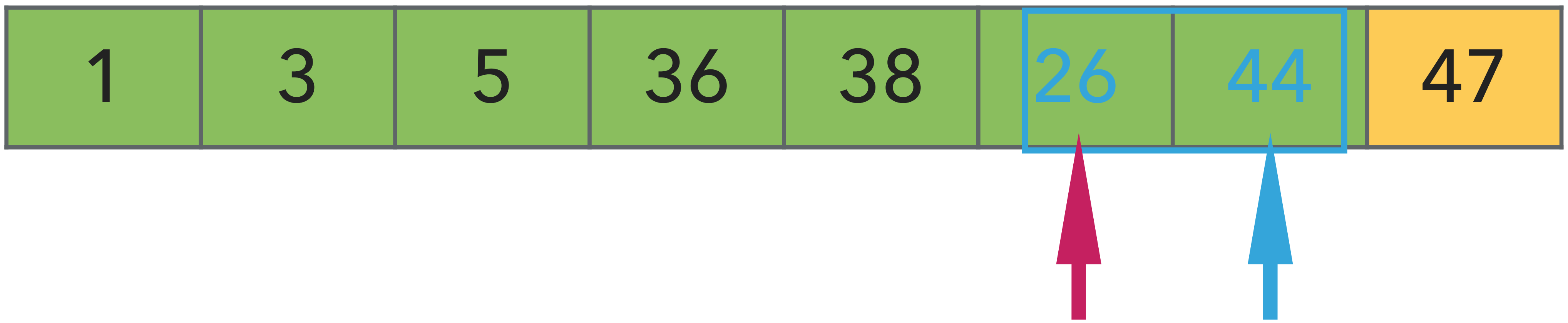

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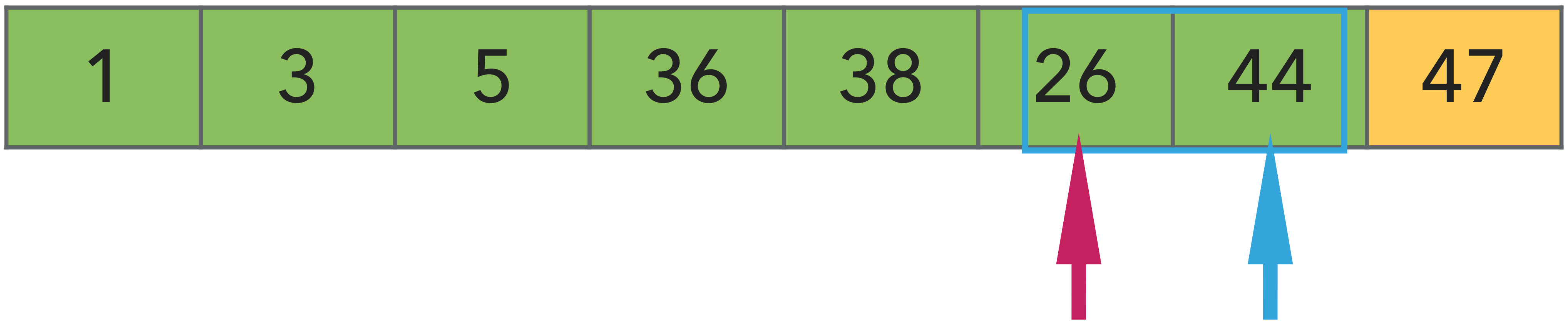

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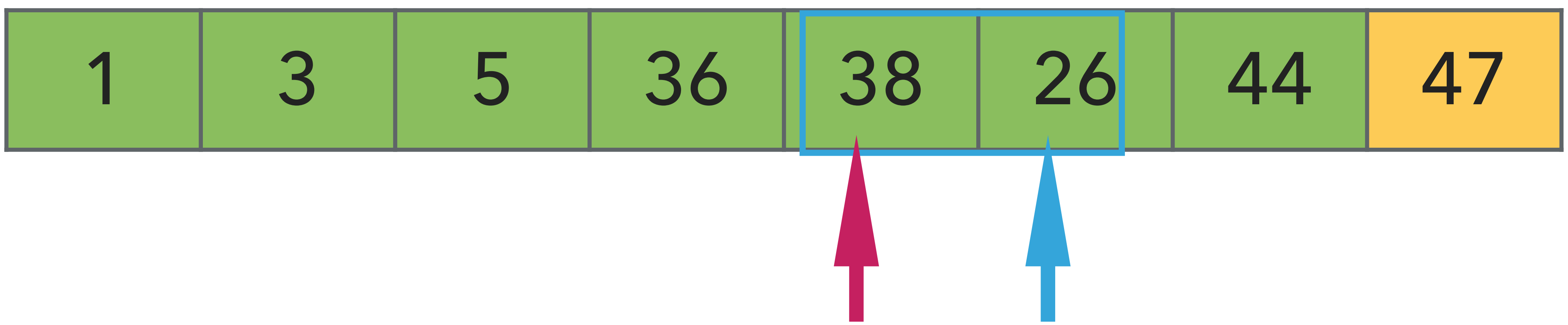

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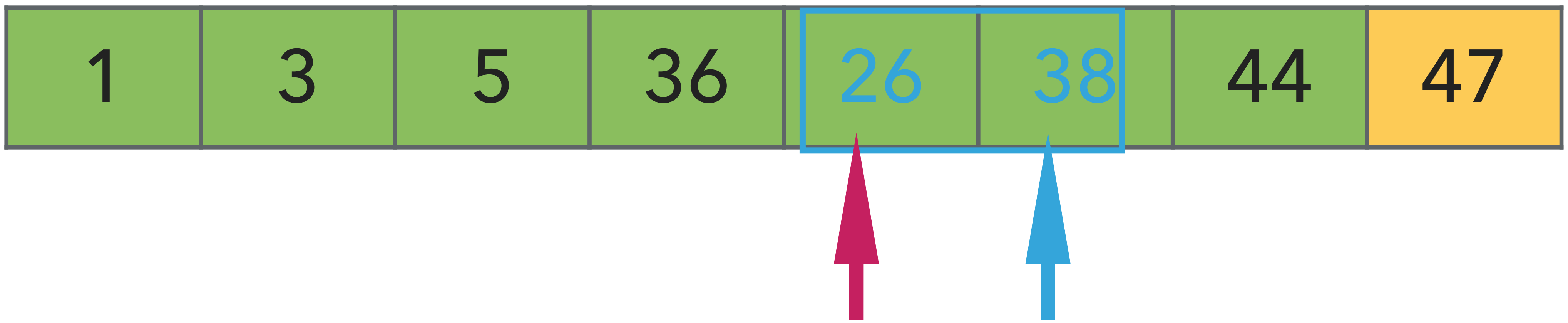

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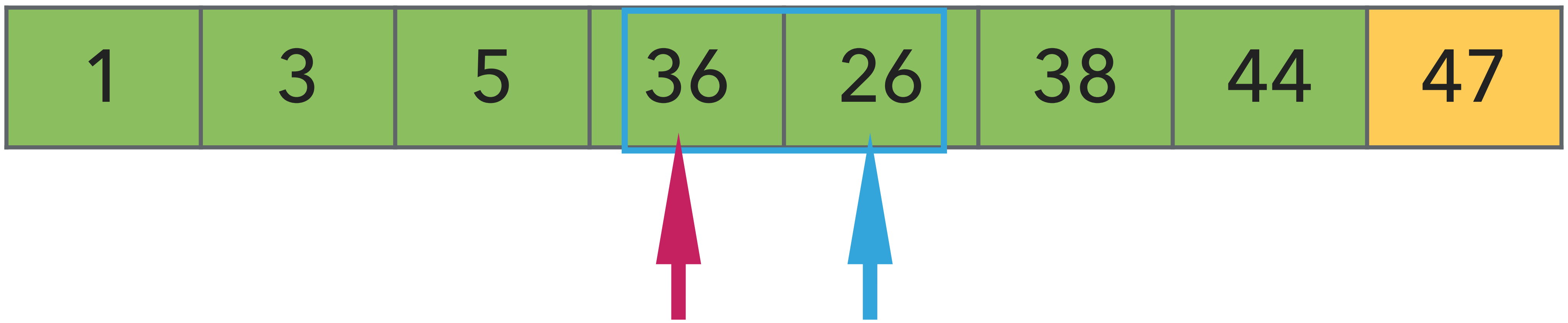

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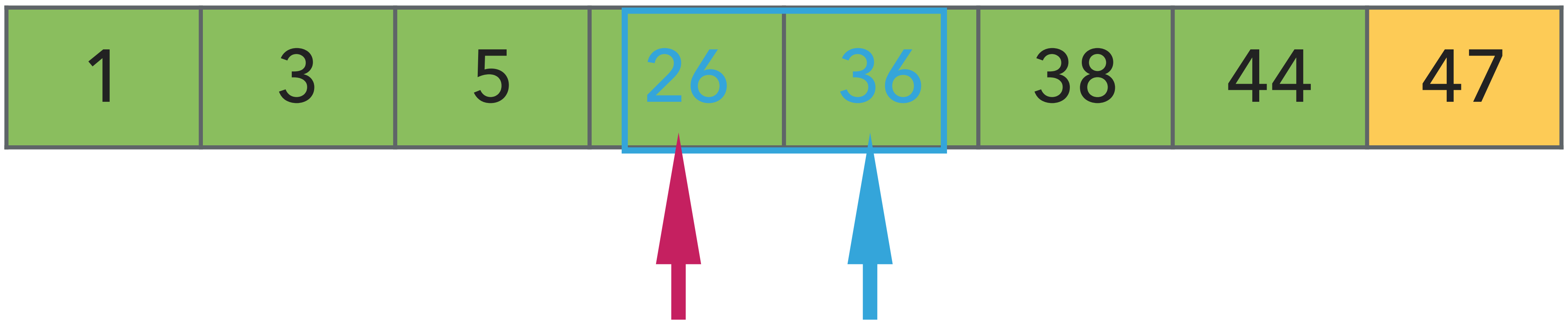

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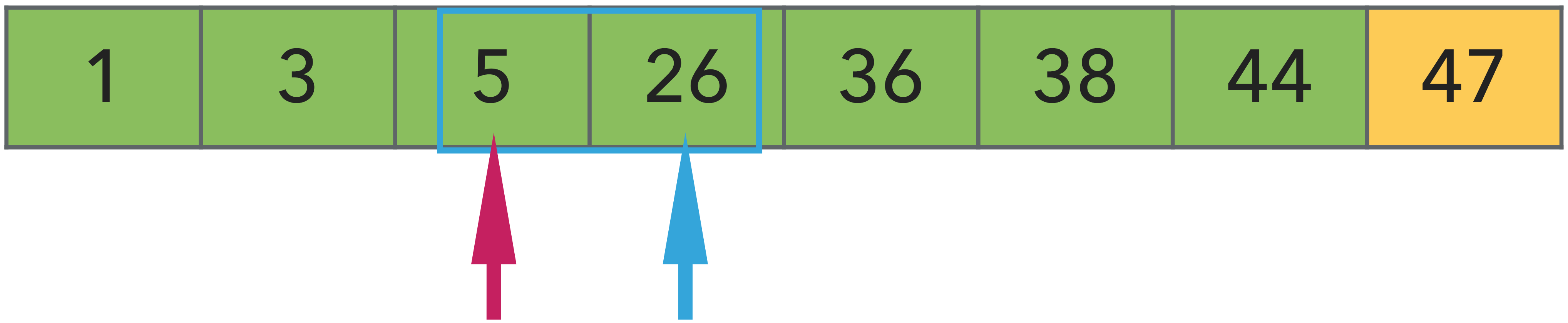

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  - Examine the next element in the unsorted subarray.
  - Exchange this element with every entry to the left that is greater.
  - Move subarray boundaries one element to the right.

Insertion sort

1	3	5	26	36	38	44	47

- Repeat:
  - Examine the next element in the unsorted subarray.
  - Exchange this element with every entry to the left that is greater.
  - Move subarray boundaries one element to the right.

http://algs4.cs.princeton.edu

2.1 INSERTION SORT DEMO

Demo with Cards

In case you didn't get this…

- https://www.youtube.com/watch?v=ROalU379l3U

Insertion sort

```
public static void sort(Comparable[] a) {

         // for loop to iterate through each element of the array



         // Moving right to left, exchange a[i] with every larger
         // entry to its left











 }
```

Insertion sort

```
public static void sort(Comparable[] a) {
        int n = a.length;
        for (int i = 0; i < n; i++) {
            for (int j = i; j > 0; j--) {
                if(less(a[j], a[j-1]))
                   exch(a, j, j-1);
                else
                   break;
            }
        }
    }
```

← In iteration i

← Move from right to left, exchange a[i] with entry to the left, if it's larger

- Invariants: At the end of each iteration `i`:
  - the array a is sorted in ascending order for the first `i+1` elements `a[0…i]`

Insertion sort: mathematical analysis for **worst-case**

```
public static void sort(Comparable[] a) {
        int n = a.length;
        for (int i = 0; i < n; i++) {
           for (int j = i; j > 0; j--) {
               if(less(a[j], a[j-1]))
                  exch(a, j, j-1);
               else
                  break;
           }
       }
   }
```

- Comparisons:
- Exchanges: ?
- In-place?
- Stable?

Insertion sort: mathematical analysis for **worst-case**

```
public static void sort(Comparable[] a) {
        int n = a.length;
        for (int i = 0; i < n; i++) {
           for (int j = i; j > 0; j--) {
               if(less(a[j], a[j-1]))
                  exch(a, j, j-1);
               else
                  break;
           }
       }
   }
```

- Comparisons: $0 + 1 + 2 + \ldots + (n-2) + (n-1) \sim n^2/2$, that is $O(n^2)$.
- Exchanges: ?
- In-place?
- Stable?

Insertion sort: mathematical analysis for **worst-case**

```
public static void sort(Comparable[] a) {
        int n = a.length;
        for (int i = 0; i < n; i++) {
           for (int j = i; j > 0; j--) {
               if(less(a[j], a[j-1]))
                  exch(a, j, j-1);
               else
                  break;
           }
       }
   }
```

- Comparisons: $0 + 1 + 2 + \ldots + (n-2) + (n-1) \sim n^2/2$, that is $O(n^2)$.
- Exchanges: $0 + 1 + 2 + \ldots + (n-2) + (n-1) \sim n^2/2$, that is $O(n^2)$.
- Worst-case running time is quadratic. Worst case = array sorted in reverse order.
- Every element moves all the way to the left.
- In-place, requires almost no additional memory.
- Stable

Insertion sort: average and best case

```
public static void sort(Comparable[] a) {
        int n = a.length;
        for (int i = 0; i < n; i++) {
           for (int j = i; j > 0; j--) {
               if(less(a[j], a[j-1]))
                  exch(a, j, j-1);
               else
                  break;
           }
       }

   }
```

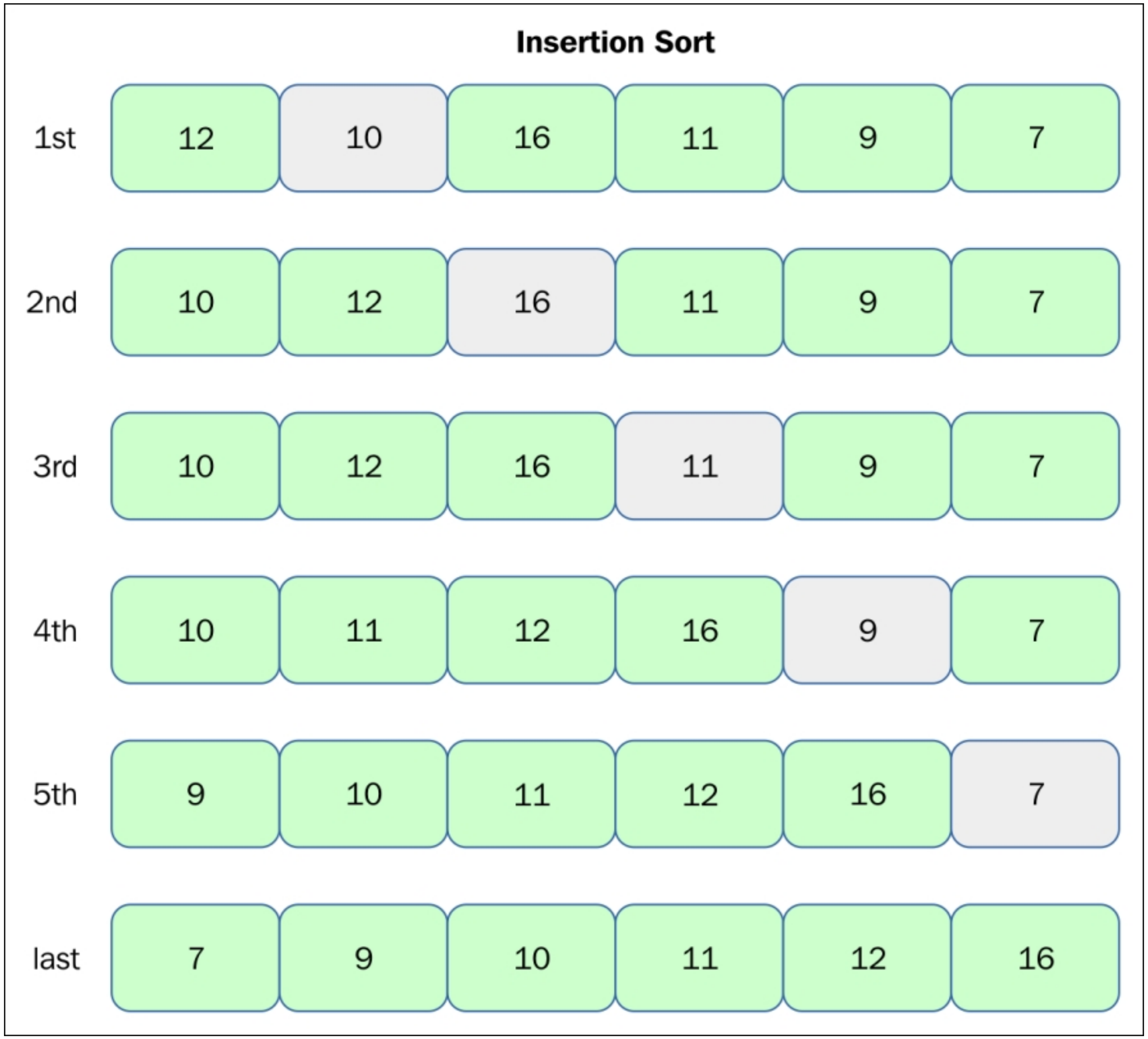
- Average case: quadratic for both comparisons and exchanges $\sim n^2/4$ when sorting a randomly ordered array. (2X faster than selection sort on average)
  - Expect each entry to move halfway back: 0 + 0.5 + 1 +...(n-1)/2 ~ (n/2)*(n/2) ~ n^2/4
- Best case: $n - 1$ comparisons (validate) and 0 exchanges for an already sorted array.

https://www.toptal.com/developers/sorting-algorithms/insertion-sort

Practice Time (Use your cards)

- Using insertion sort, sort the array with elements [12,10,16,11,9,7].
- Visualize your work for every iteration of the algorithm.

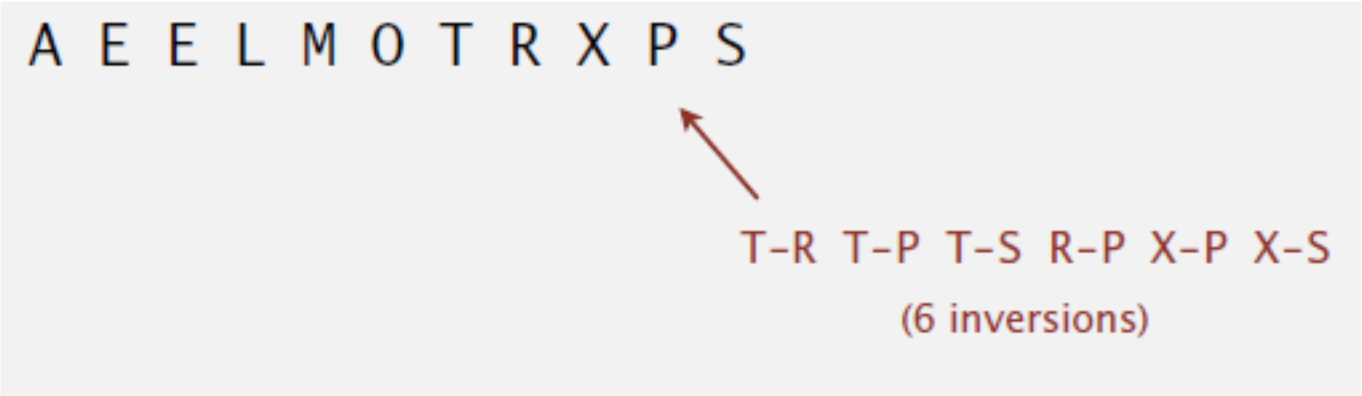
Answer



https://subscription.packtpub.com/book/application_development/9781785888731/13/ch13lvl1sec90/insertion-sort?query=insertion%20sort

Insertion Sort

- For partially-sorted arrays, insertion sort runs in ***linear time***
- Number of exchanges equals number of **inversions**
- **Inversion** = pair of keys that are out of order



- Partially sorted examples
  - 1) Appending a subarray of size 10 to a sorted subarray of size N
  - 2) An array of size N with only 10 entries out of place

Lecture 12: Insertion Sort & Mergesort

- Insertion sort
- Comparators
- Mergesort

`Comparable`

- Interface with a single method that we need to implement: `public int compareTo(T that)`
- Implement it so that `v.compareTo(w)`:
    - Returns >0 if v is greater than `w`.
    - Returns <0 if v is smaller than `w`.
    - Returns 0 if v is equal to `w`.
- Corresponds to natural ordering.

How to make your class T comparable?

1. Implement `Comparable<T>` interface.
2. Implement `compareTo(T that)` method to compare `this` T object to `that` based on natural ordering.

Comparator

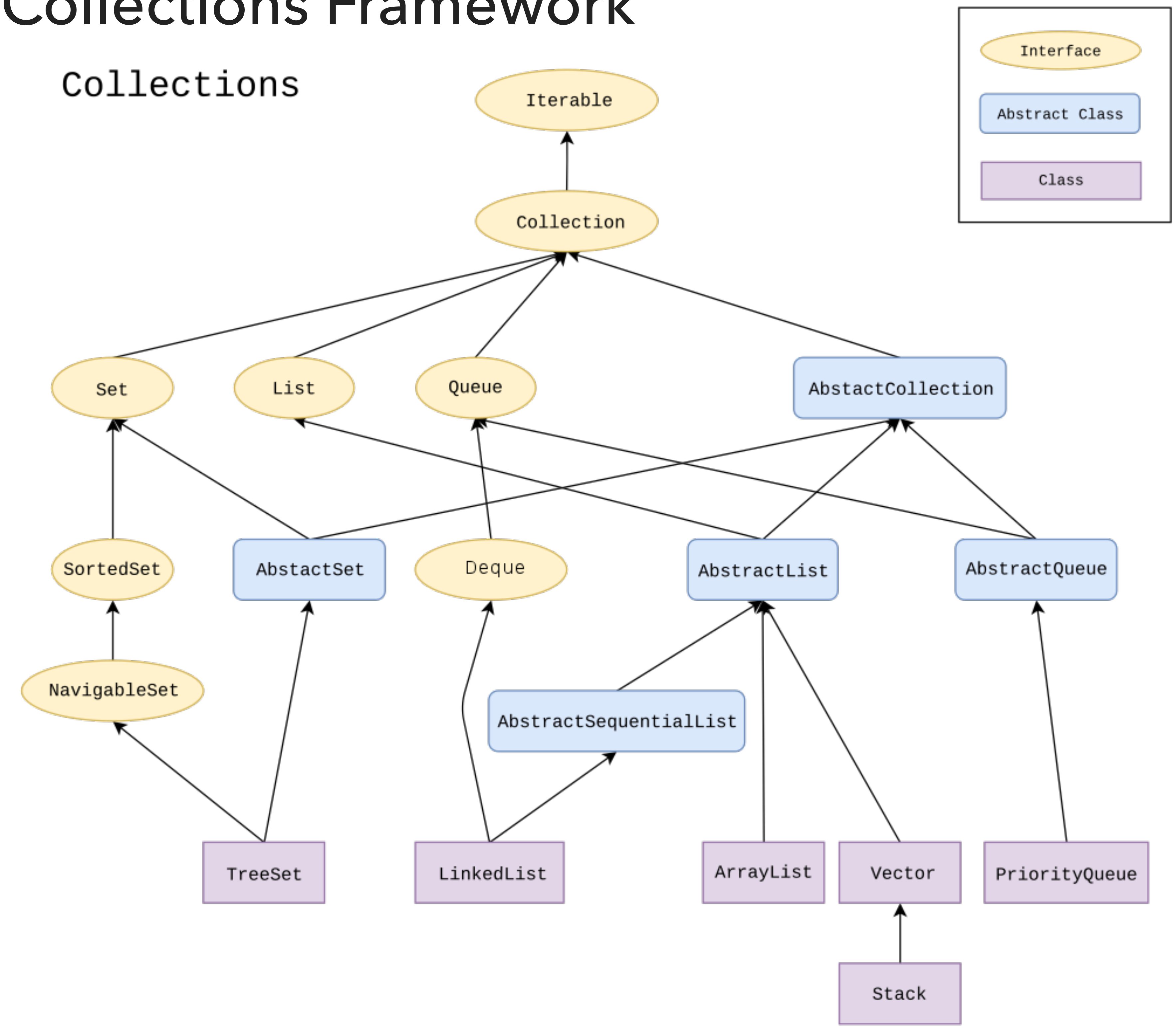
- Sometimes the natural ordering is **not** the type of ordering we want.
- Comparator is an interface which allows us to **dictate what kind of ordering** we want by implementing the method:
  `public int compare(T this, T that)`
- Implement it so that `compare(v, w)`:
  - Returns >0 if v is greater than `w`.
  - Returns <0 if v is smaller than `w`.
  - Returns 0 if v is equal to `w`.

How to define an alternative ordering for your class T?

1. Make a new class that implements `Comparator<T>` interface.
2. Implement `compare(T t1, T t2)` method to compare t1 object to t2 based on an alternative ordering.
3. Alternatively, implement an anonymous inner class:

```
public static Comparator<T> nameOfComparator = new Comparator<T>()
{
    @Override      // indicates method overriding the superclass' method
    public int compare(T t1, T t2) {
     {
          //return something;
    }
};
```

The Java Collections Framework



https://en.wikipedia.org/wiki/Java_collections_framework

Alternative sorting of `Collections`

- Collections class contains:
  - `static <T> void sort(List<T> list, Comparator<? super T> c)`
- `Collections.sort(list, someComparator);`
  - `Collections.sort(list, new ExternalComparatorClass());` or:
  - `Collections.sort(list, T.InnerAnonymousClass);`
  - If list's elements do not implement `Comparable` or cannot be compared with `Comparator`, throw `ClassCastException`.

Example: Natural and alternative sorting for `Employees`

https://github.com/pomonacs622021fa/LectureCode/blob/main/Lecture11/Employee.java

https://stackoverflow.com/questions/2266827/when-to-use-comparable-and-comparator

Lecture 12: Insertion Sort & Mergesort

- Insertion sort
- Comparators
- Mergesort

Lecture 12: Insertion Sort & Mergesort

- Mergesort

Some slides adopted from Algorithms 4th Edition or COS226

input	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sort left half	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
sort right half	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge results	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

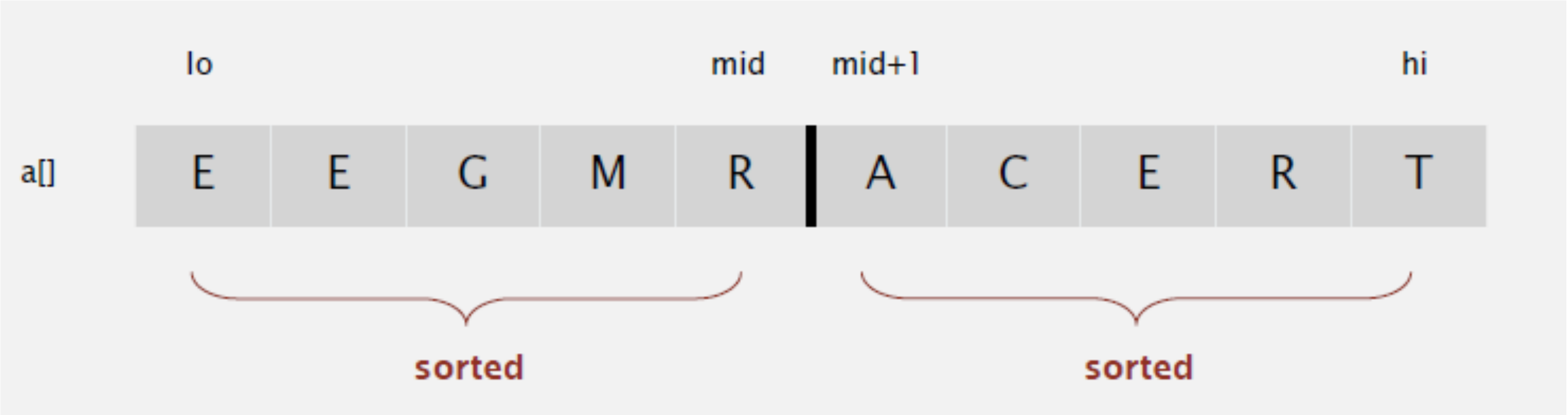
Mergesort overview

Basics

- Invented by John von Neumann in 1945
- Algorithm sketch:
  - Divide array into two halves.
  - Recursively sort each half.
  - Merge the two halves

https://en.wikipedia.org/wiki/File:JohnvonNeumann-LosAlamos.gif

Merging two already sorted halves into one sorted array



Copy to auxiliary array

Merging Example - copying to auxiliary array

Array a

A	G	L	O	R	H	I	M	S	T
0	1	2	3	4	5	6	7	8	9

Array aux

0	1	2	3	4	5	6	7	8	9

Merging Example - copying to auxiliary array

Array a

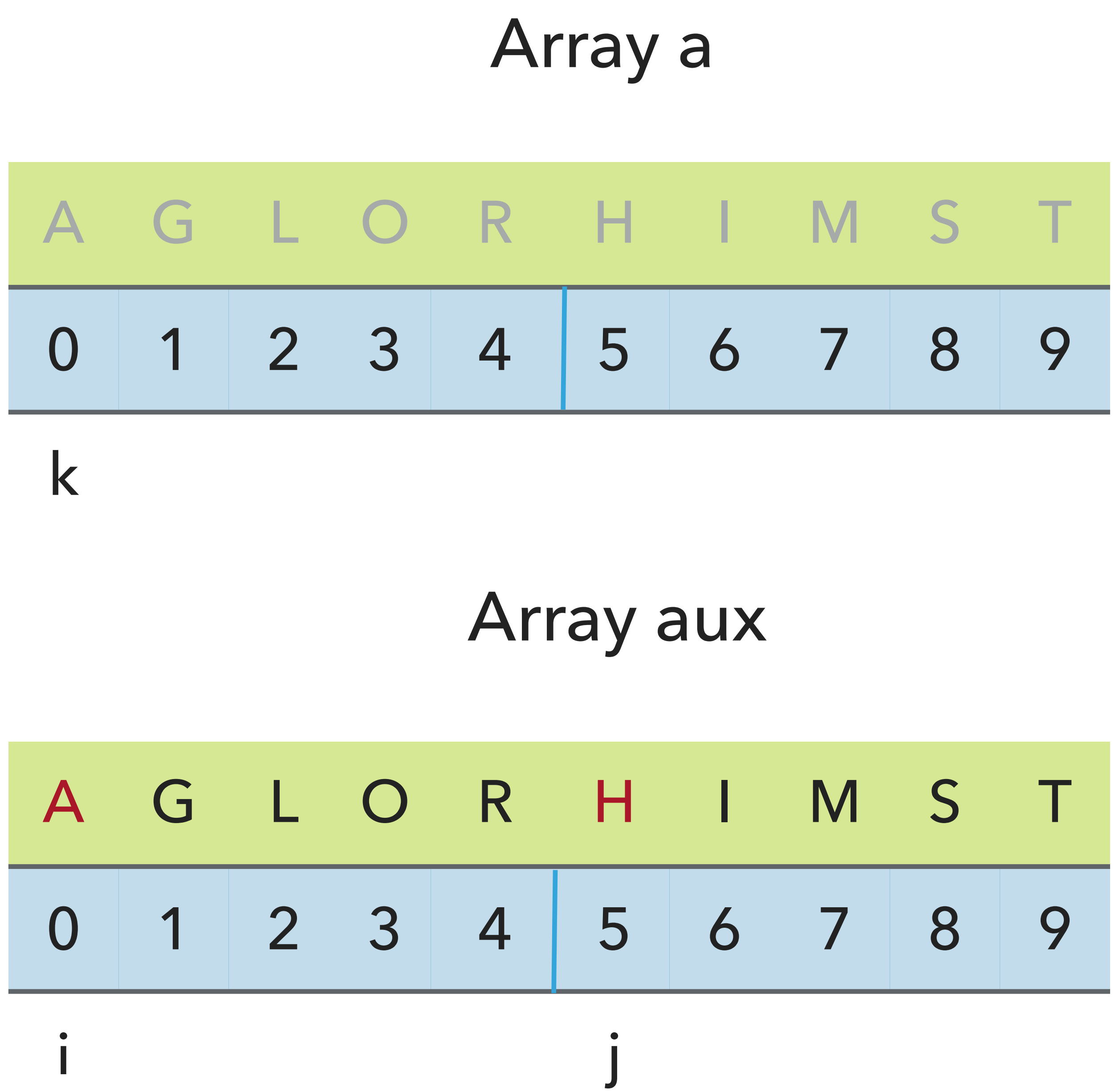
A	G	L	O	R	H	I	M	S	T
0	1	2	3	4	5	6	7	8	9

Array aux

A	G	L	O	R	H	I	M	S	T
0	1	2	3	4	5	6	7	8	9

Merging Example - copy elements back to original array in order

Maintain 3 indices: i, j, k



Merging Example - copy elements back to original array in order

Compare minimum in each subarray

Array a (sorted result)

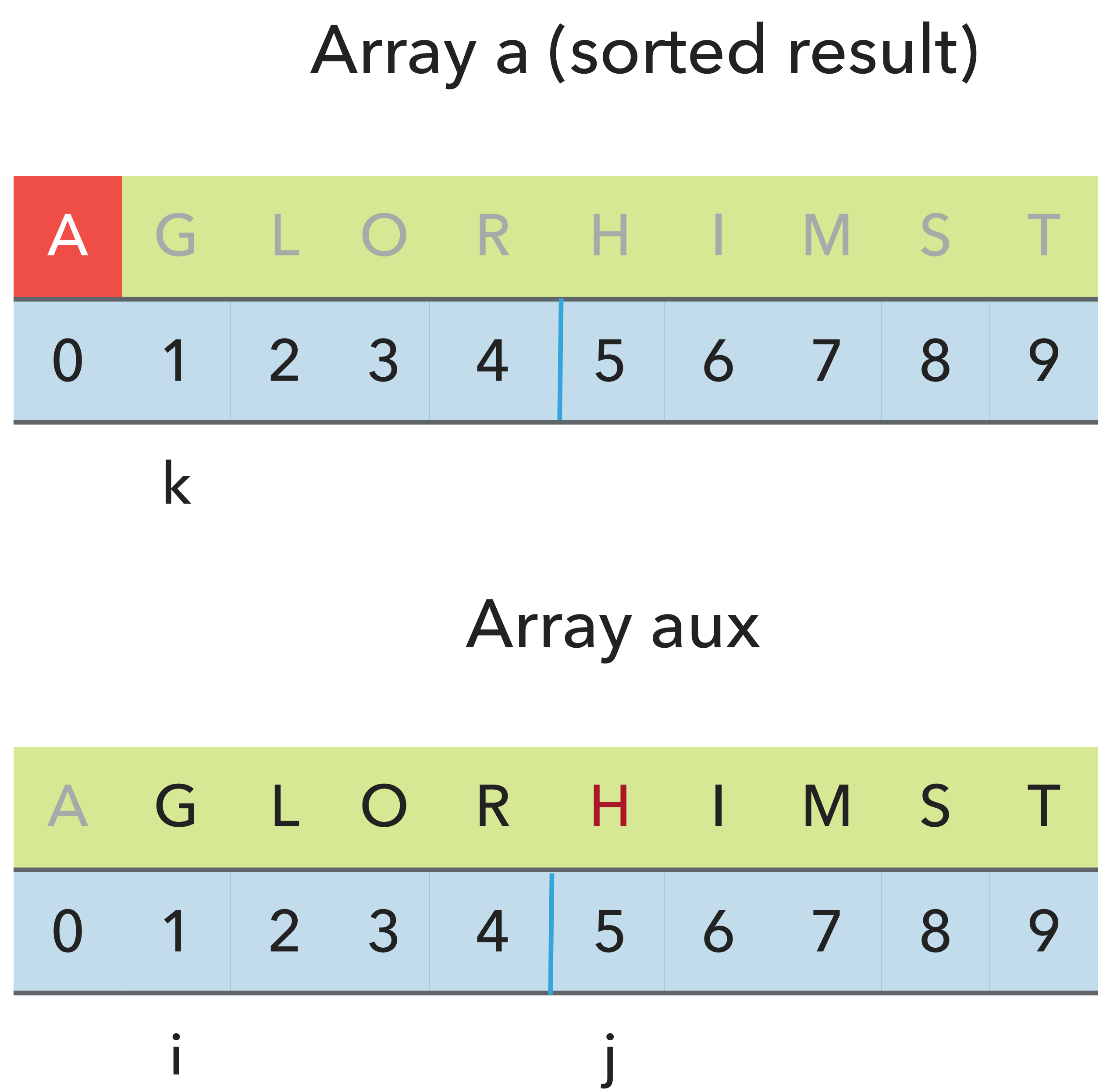
A	G	L	O	R	H	I	M	S	T
0	1	2	3	4	5	6	7	8	9
k									

Array aux

A	G	L	O	R	H	I	M	S	T
0	1	2	3	4	5	6	7	8	9
i					j				

Merging Example - copy elements back to original array in order

Copy smaller element back to a, increment indices i and j



Merging Example - copy elements back to original array in order

Compare

Array a (sorted result)

A	G	L	O	R	H	I	M	S	T
0	1	2	3	4	5	6	7	8	9
	k								

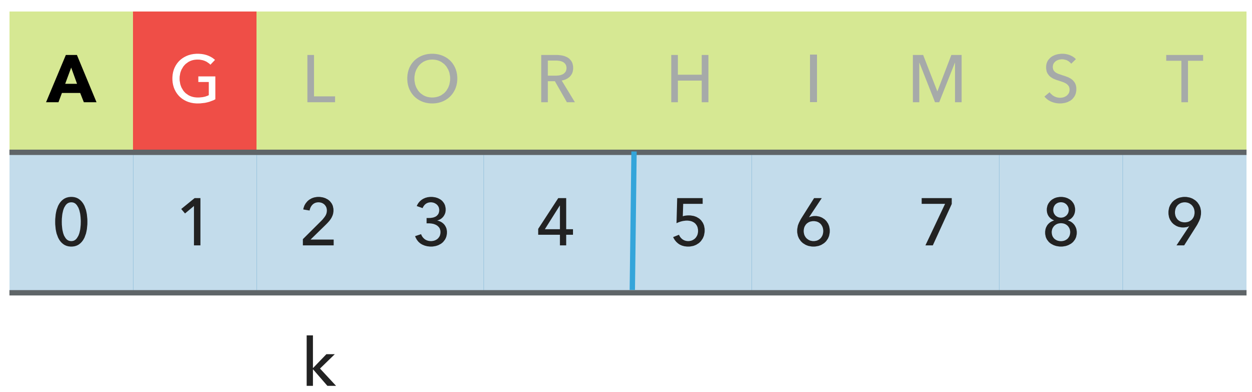
Array aux

A	G	L	O	R	H	I	M	S	T
0	1	2	3	4	5	6	7	8	9
	i				j				

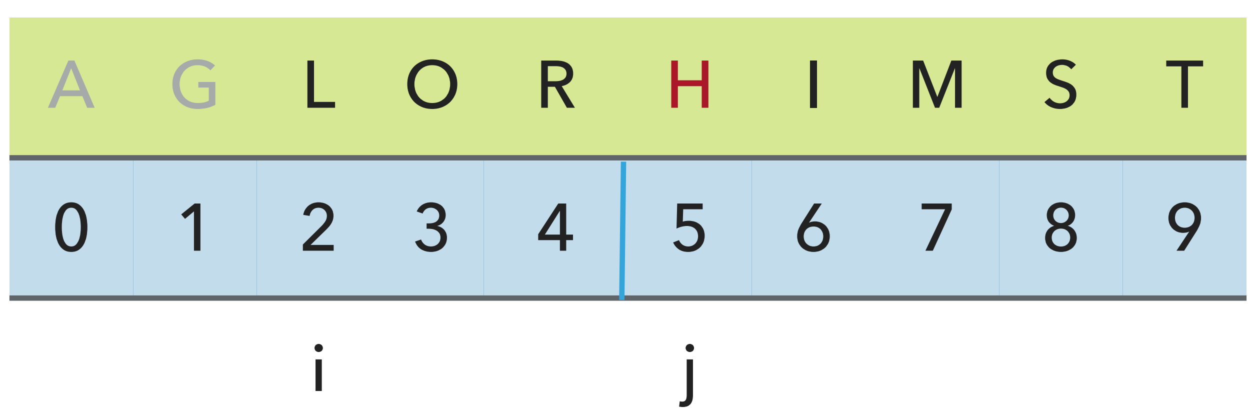
Merging Example - copy elements back to original array in order

Compare

Array a (sorted result)



Array aux



Merging Example - copy elements back to original array in order

Compare

Array a (sorted result)

A	**G**	L	O	R	H	I	M	S	T
0	1	2	3	4	5	6	7	8	9
		k							

Array aux

A	G	L	O	R	H	I	M	S	T
0	1	2	3	4	5	6	7	8	9
		i			j				

Merging Example - copy elements back to original array in order

Compare

Array a (sorted result)

A	G	H	O	R	H	I	M	S	T
0	1	2	3	4	5	6	7	8	9
			k						

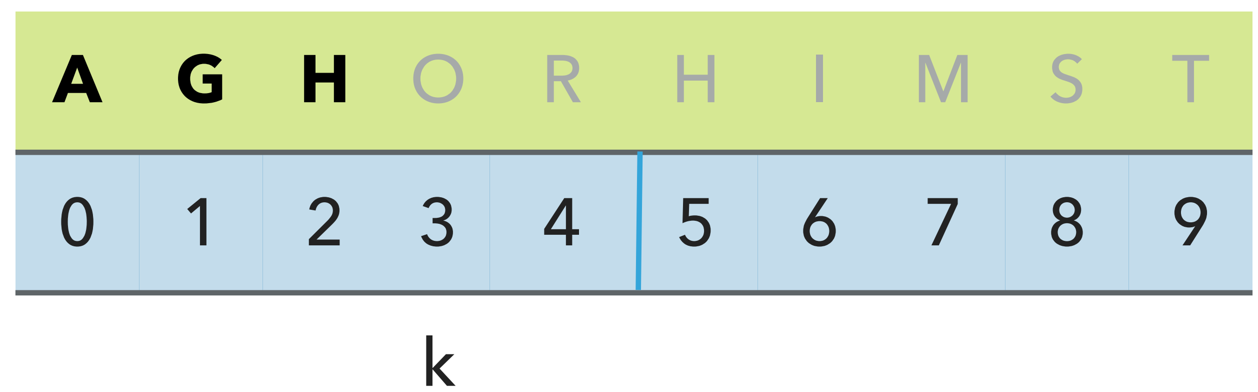
Array aux

A	G	L	O	R	H	I	M	S	T
0	1	2	3	4	5	6	7	8	9
		i				j			

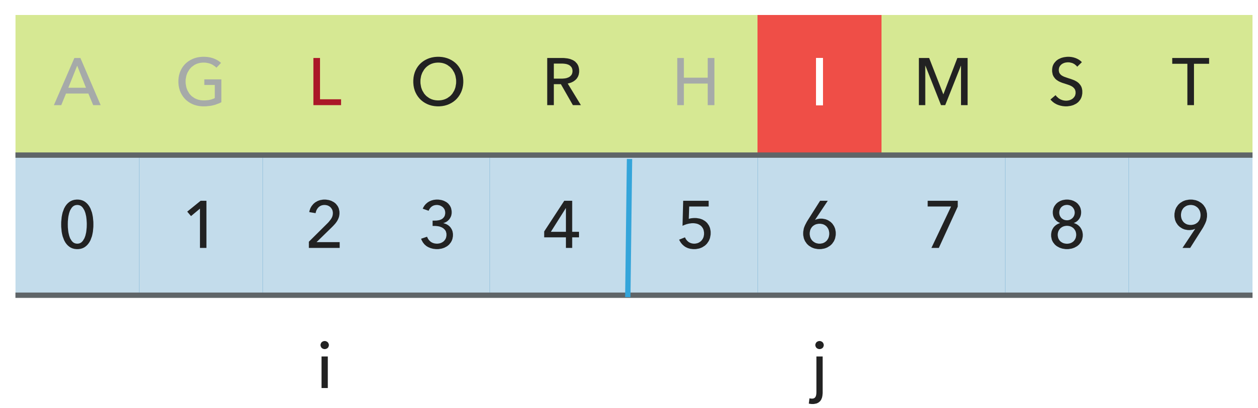
Merging Example - copy elements back to original array in order

Compare

Array a (sorted result)



Array aux



Merging Example - copy elements back to original array in order

Compare

Array a (sorted result)

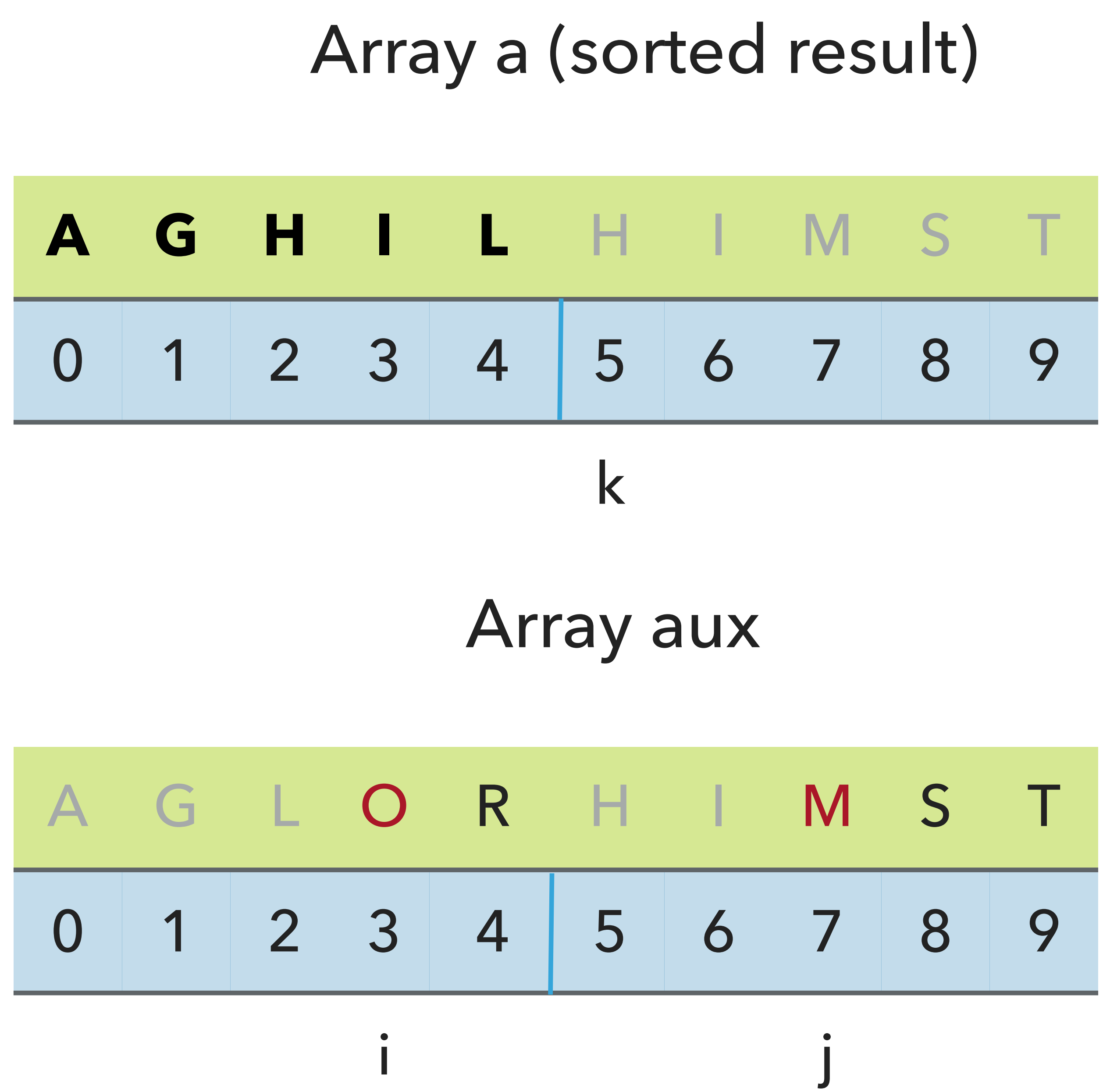
A	G	H	I	R	H	I	M	S	T
0	1	2	3	4	5	6	7	8	9
				k					

Array aux

A	G	L	O	R	H	I	M	S	T
0	1	2	3	4	5	6	7	8	9
		i					j		

Merging Example - copy elements back to original array in order

Compare



Merging Example - copy elements back to original array in order

Compare

Array a (sorted result)

A	G	H	I	L	M	I	M	S	T
0	1	2	3	4	5	6	7	8	9
						k			

Array aux

A	G	L	O	R	H	I	M	S	T
0	1	2	3	4	5	6	7	8	9
			i					j	

Merging Example - copy elements back to original array in order

Compare

Array a (sorted result)

A	G	H	I	L	M	O	M	S	T
0	1	2	3	4	5	6	7	8	9
							k		

Array aux

A	G	L	O	R	H	I	M	S	T
0	1	2	3	4	5	6	7	8	9
				i				j	

Merging Example - copy elements back to original array in order

Compare

Array a (sorted result)

A	**G**	**H**	**I**	**L**	**M**	**O**	**R**	S	T
0	1	2	3	4	5	6	7	8	9
								k	

Array aux

A	G	L	O	R	H	I	M	S	T
0	1	2	3	4	5	6	7	8	9
					i			j	

Merging Example - copy elements back to original array in order

Compare

Array a (sorted result)

A	**G**	**H**	**I**	**L**	**M**	**O**	**R**	**S**	T
0	1	2	3	4	5	6	7	8	9
									k

Array aux

A	G	L	O	R	H	I	M	S	T
0	1	2	3	4	5	6	7	8	9
					i				j

Merging Example - copy elements back to original array in order

Done

Array a (sorted result)

A	**G**	**H**	**I**	**L**	**M**	**O**	**R**	**S**	**T**
0	1	2	3	4	5	6	7	8	9
									k

Array aux

A	G	L	O	R	H	I	M	S	T
0	1	2	3	4	5	6	7	8	9
					i				j

Merging two already sorted halves into one sorted array

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi) {
    for (int k = lo; k <= hi; k++)   // copy to aux array
        aux[k] = a[k];

    int i = lo, j = mid+1;        // lo and mid+1 are the start of the 2 sorted halves

    for (int k = lo; k <= hi; k++) {
        if (i > mid) //ran out of elements in the left subarray
           a[k] = aux[j++];
        else if (j > hi) //ran out of elements in the right subarray
            a[k] = aux[i++];
        else if (less(aux[j], aux[i]))  // Compares left and right subarray
            a[k] = aux[j++];
        else
            a[k] = aux[i++];
    }
}
```

http://algs4.cs.princeton.edu

2.2 MERGING DEMO

Practice time

How many calls does `merge()` make to `less()` in order to merge two already sorted subarrays, each of length $n/2$ into a sorted array of length n?

A. $\sim 1/4n$ to $\sim 1/2n$

B. $\sim 1/2n$

C. $\sim 1/2n$ to n

D. $\sim n$

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, in
    for (int k = lo; k <= hi; k++)
        aux[k] = a[k];

    int i = lo, j = mid+1;
    for (int k = lo; k <= hi; k++) {
        if (i > mid) //ran out of elements in the left subarray
           a[k] = aux[j++];
        else if (j > hi) //ran out of elements in the right subarray
            a[k] = aux[i++];
        else if (less(aux[j], aux[i]))
            a[k] = aux[j++];
        else
            a[k] = aux[i++];
    }
}
```

Answer

How many calls does `merge()` make to `less()` in order to merge two already sorted subarrays, each of length $n/2$ into a sorted array of length n?

C. ~$1/2n$ to n, that is at most $n - 1$ or $O(n)$

Best case example
Merging [1,2,3] and [4,5,6] requires 3 calls to `less()` (1 with 4, 2 with 4, 3 with 4).

Worst case example
Merging [1,3,5] and [2, 4, 6] requires 5 calls to `less()` (1 with 2, 2 with 3, 3 with 4, 4 with 5, 5 with 6)

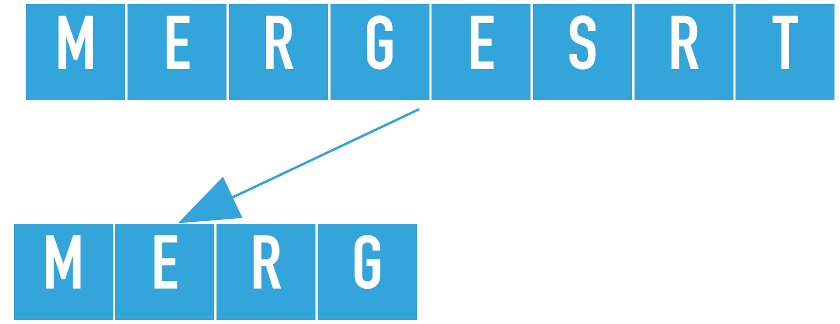
Mergesort - the quintessential example of divide-and-conquer

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;   // Computes midpoint
    sort(a, aux, lo, mid);          // Sort the first half
    sort(a, aux, mid+1, hi);        // Sort the second half
    merge(a, aux, lo, mid, hi);     // Merge the 2 halves
}

public static void sort(Comparable[] a) {
    Comparable[] aux = new Comparable[a.length];   // Create aux array
    sort(a, aux, 0, a.length - 1);                 // Recursively call sort
}
```

M	E	R	G	E	S	R	T

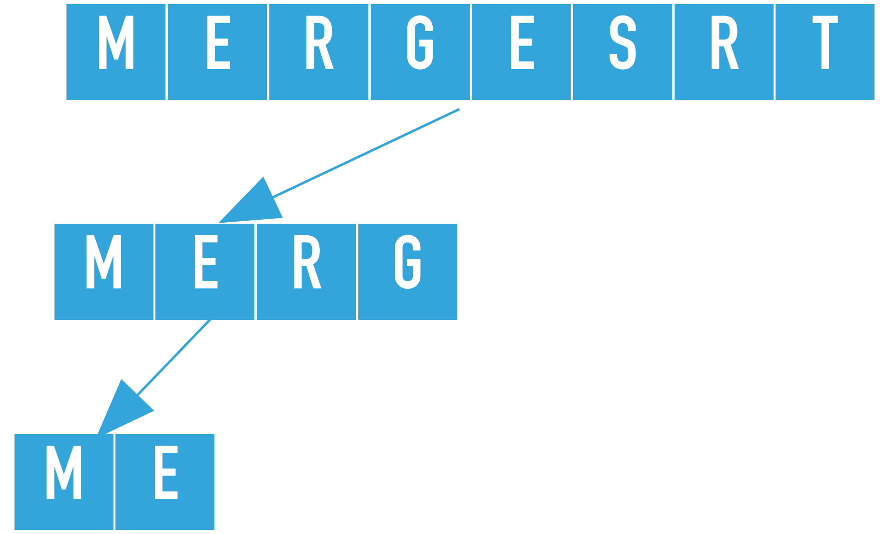

```java
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}

public static void sort(Comparable[] a) {
    Comparable[] aux = new Comparable[a.length];
    sort(a, aux, 0, a.length - 1);
}
```

`sort([M, E, R, G, E, S, R, T])` calls `sort([M, E, R, G, E, S, R, T], [null, null, null, null, null, null, null, null], 0, 7)` where the array of `nulls` is the auxiliary array, `lo = 0` and `hi = 7`.



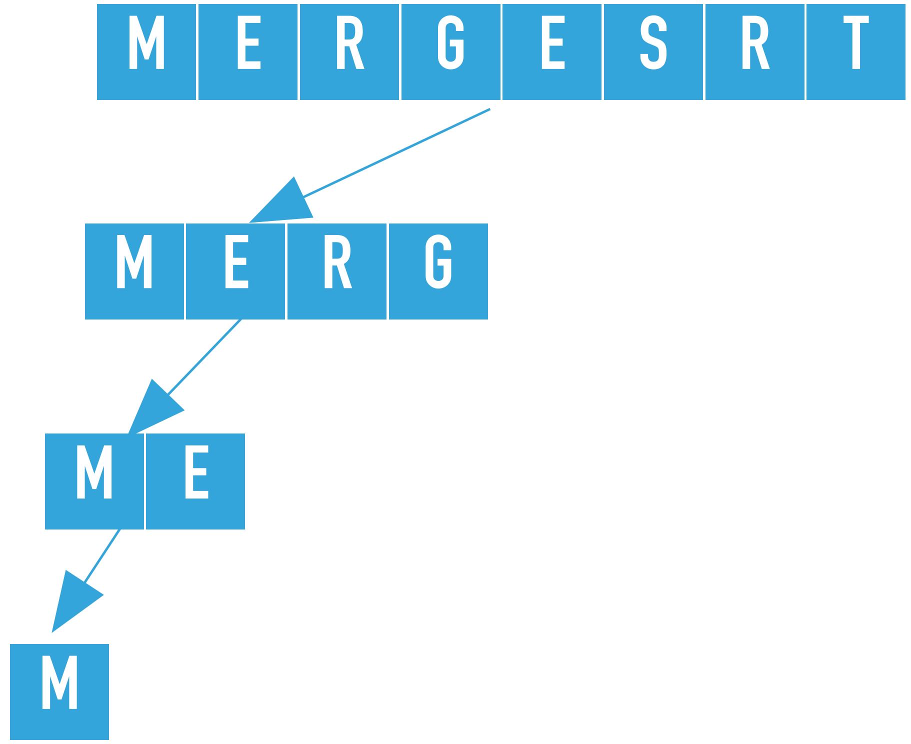

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([M, E, R, G, E, S, R, T], [null, null, null, null, null, null, null, null], 0, 7)` calculates the `mid = 3` and calls recursively `sort` on the left subarray, that is `sort([M, E, R, G, E, S, R, T], [null, null, null, null, null, null, null, null], 0, 3)`, where `lo = 0`, `hi = 3`



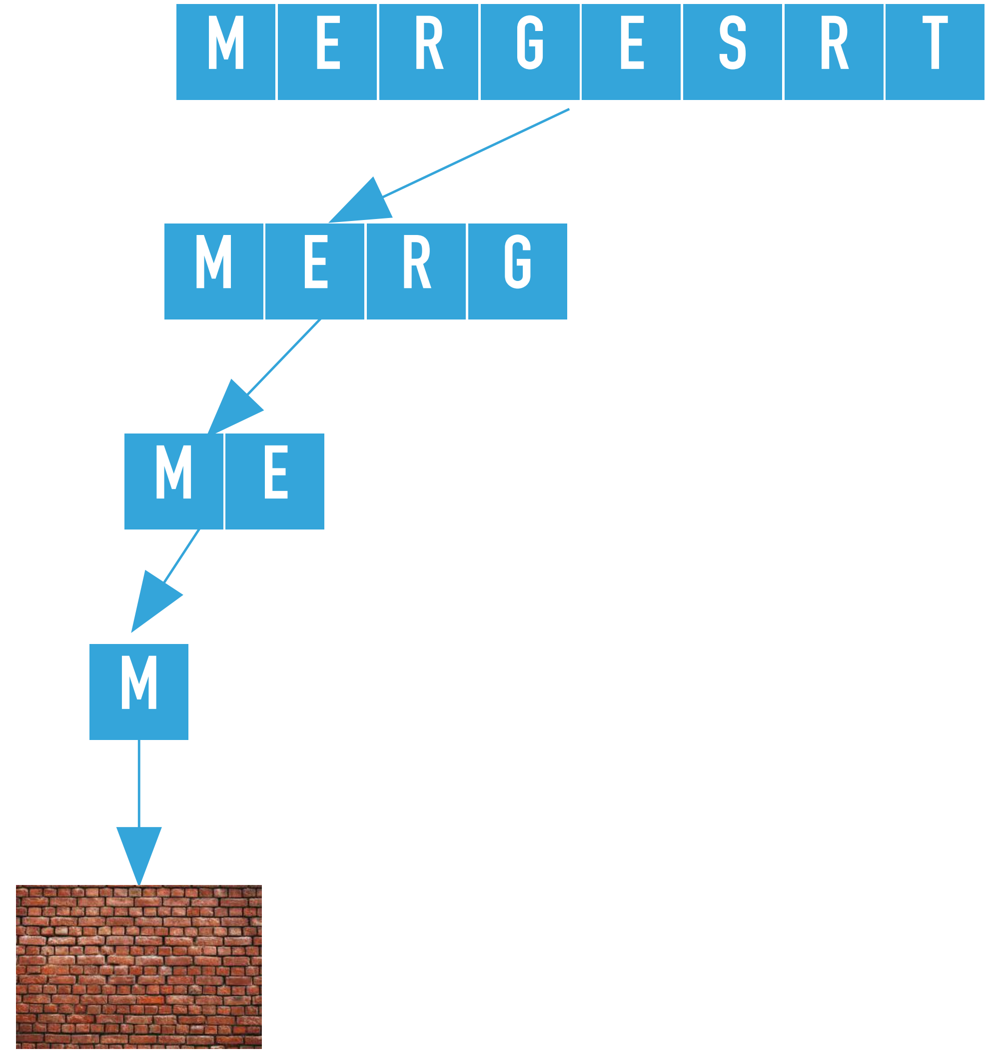

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([M, E, R, G, E, S, R, T], [null, null, null, null, null, null, null, null], 0, 3)` calculates the `mid = 1` and calls recursively `sort` on the left subarray, that is `sort([M, E, R, G, E, S, R, T], [null, null, null, null, null, null, null, null], 0, 1)`, where `lo = 0`, `hi = 1`



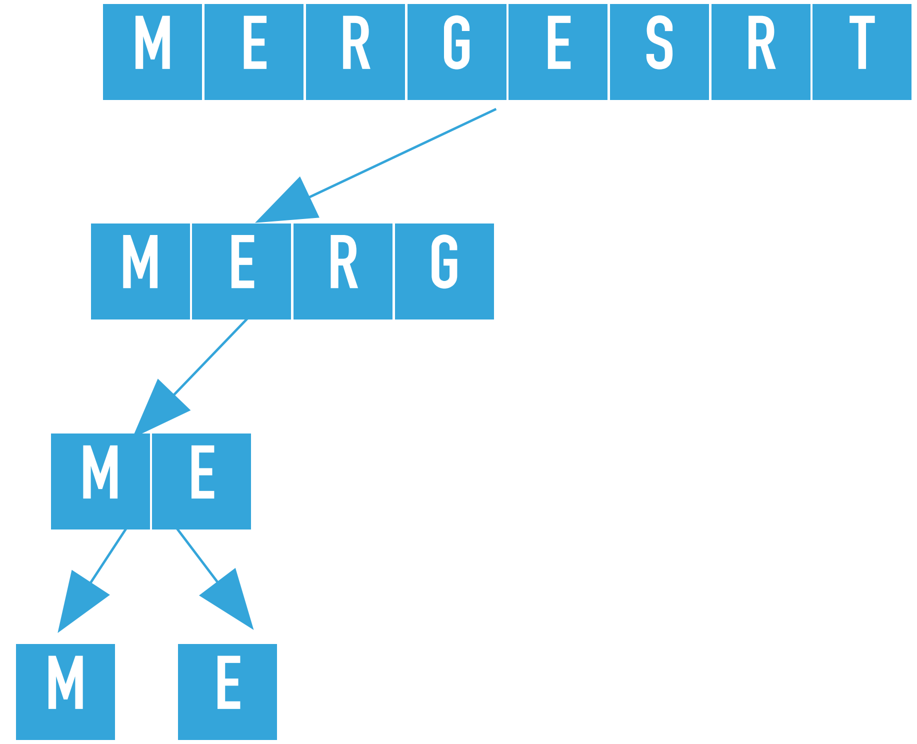

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([M, E, R, G, E, S, R, T], [null, null, null, null, null, null, null, null], 0, 1)` calculates the `mid = 0` and calls recursively `sort` on the left subarray, that is `sort([M, E, R, G, E, S, R, T], [null, null, null, null, null, null, null, null], 0, 0)`, where `lo = 0`, `hi = 0`



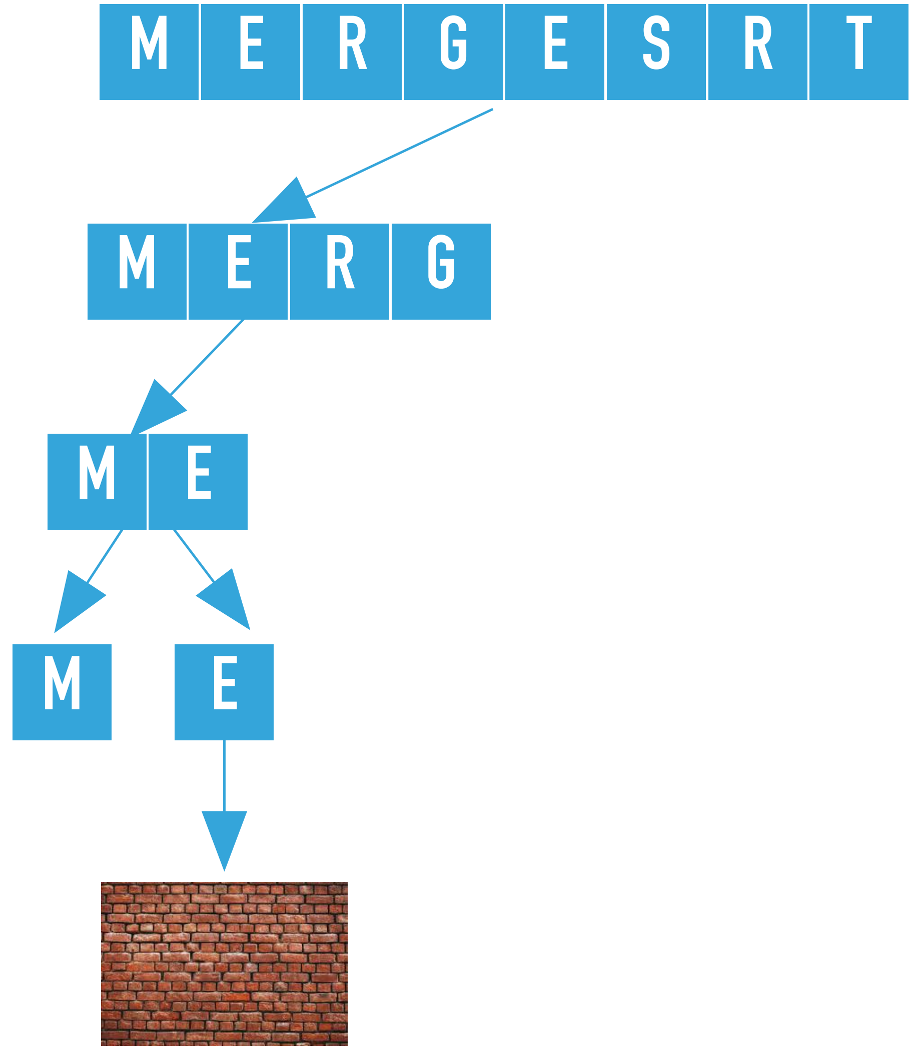

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([M, E, R, G, E, S, R, T], [null, null, null, null, null, null, null, null], 0, 0)` finds `hi <= lo` and returns.



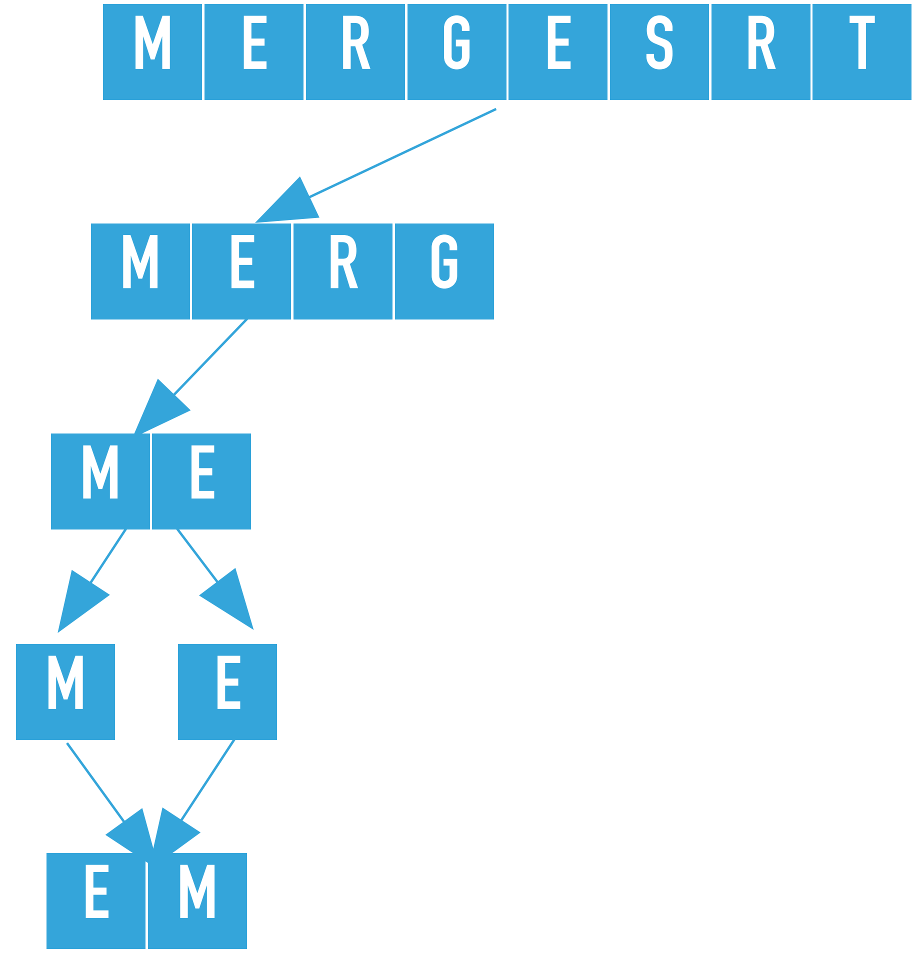
```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([M, E, R, G, E, S, R, T], [null, null, null, null, null, null, null, null], 0, 1)` calls recursively `sort` on the right subarray, that is `sort([M, E, R, G, E, S, R, T], [null, null, null, null, null, null, null, null], 1, 1)`, where `lo = 1`, `hi = 1`



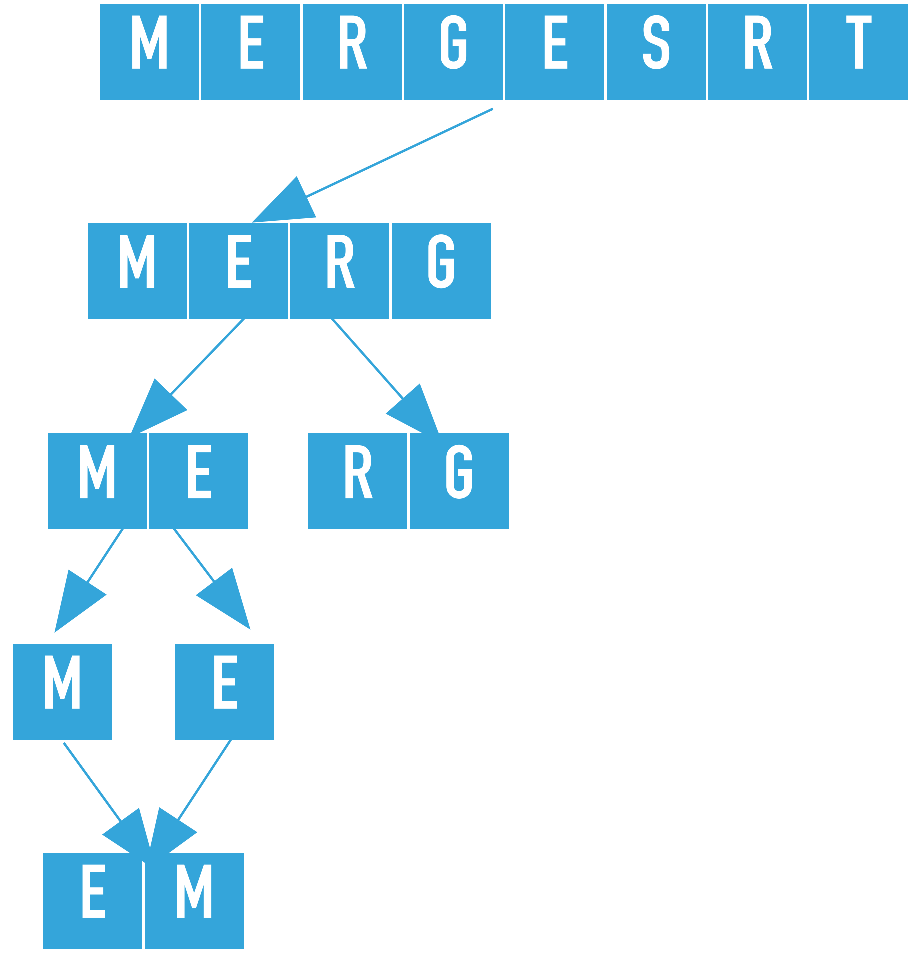

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([M, E, R, G, E, S, R, T], [null, null, null, null, null, null, null, null], 1, 1)` finds `hi <= lo` and returns.



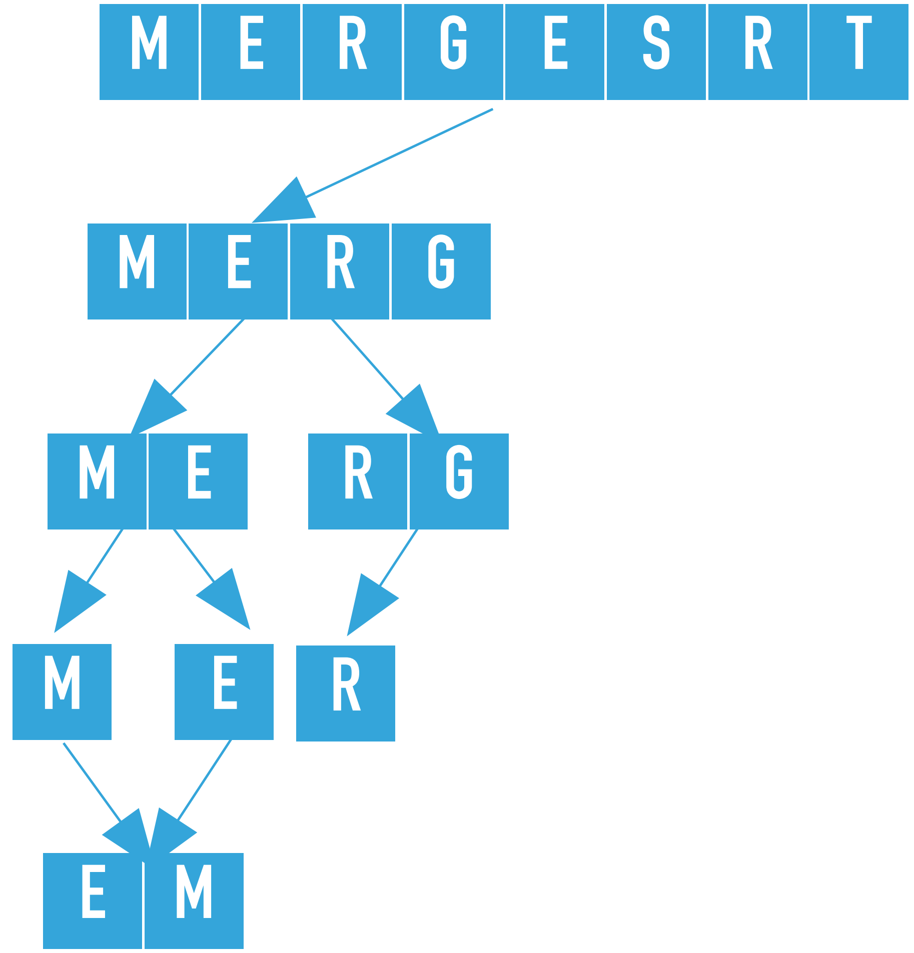

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([M, E, R, G, E, S, R, T], [null, null, null, null, null, null, null, null], 0, 1)` merges the two subarrays that is calls `merge([M, E, R, G, E, S, R, T], [null, null, null, null, null, null, null, null], 0, 0, 1)`, where `lo = 0`, `mid = 0`, and `hi = 1`. The resulting partially sorted array is `[E, M, R, G, E, S, R T]`.



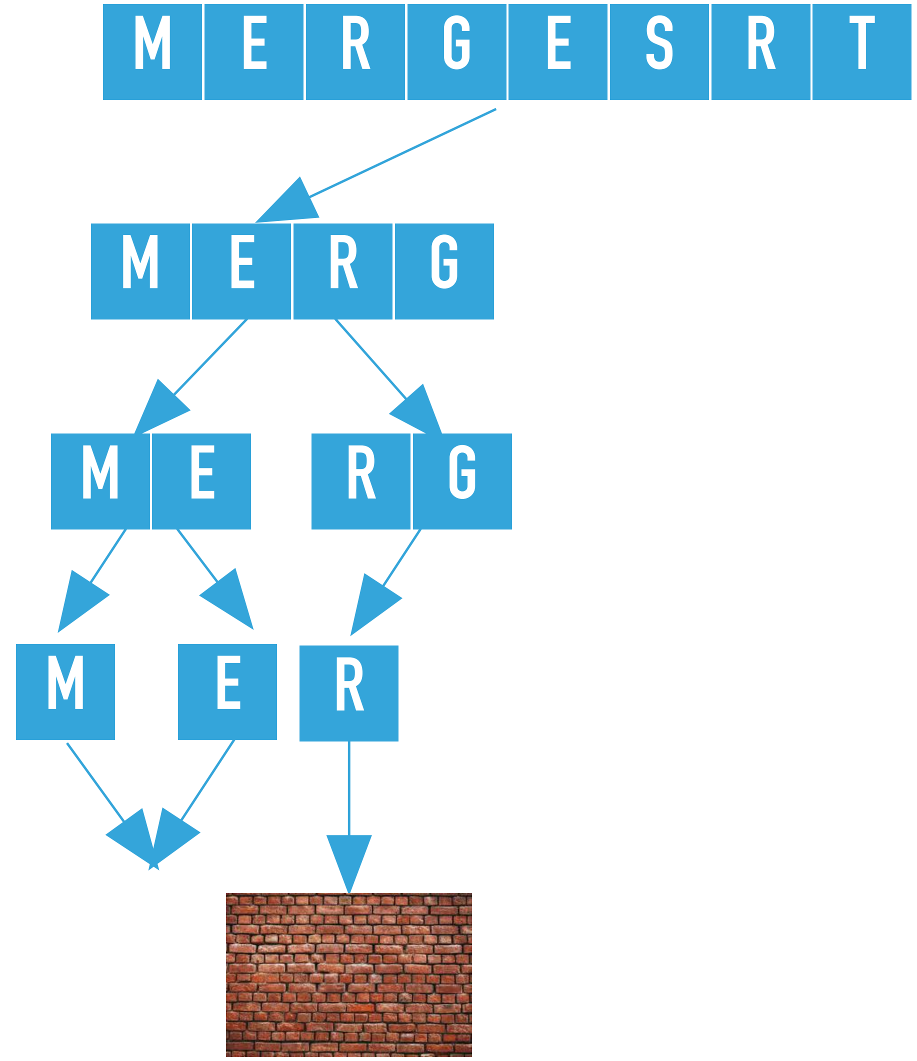

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, M, R, G, E, S, R, T], [M, E, null, null, null, null, null, null], 0, 3)` calls recursively `sort` on the right subarray, that is `sort([E, M, R, G, E, S, R, T], [M, E, null, null, null, null, null, null], 2, 3)`, where `lo = 2`, `hi = 3`



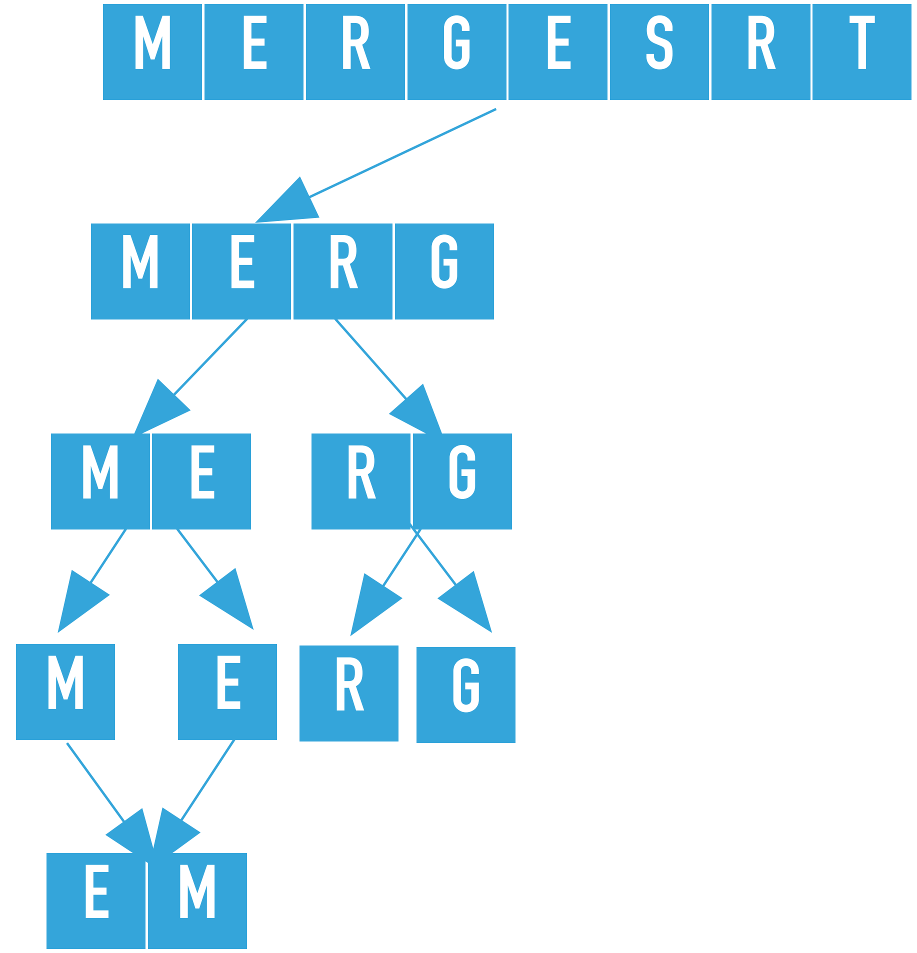

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, M, R, G, E, S, R, T], [M, E, null, null, null, null, null, null], 2, 3)` calculates the `mid = 2` and calls recursively `sort` on the left subarray, that is `sort([E, M, R, G, E, S, R, T], [M, E, null, null, null, null, null, null], 2, 2)`, where `lo = 2`, `hi = 2`



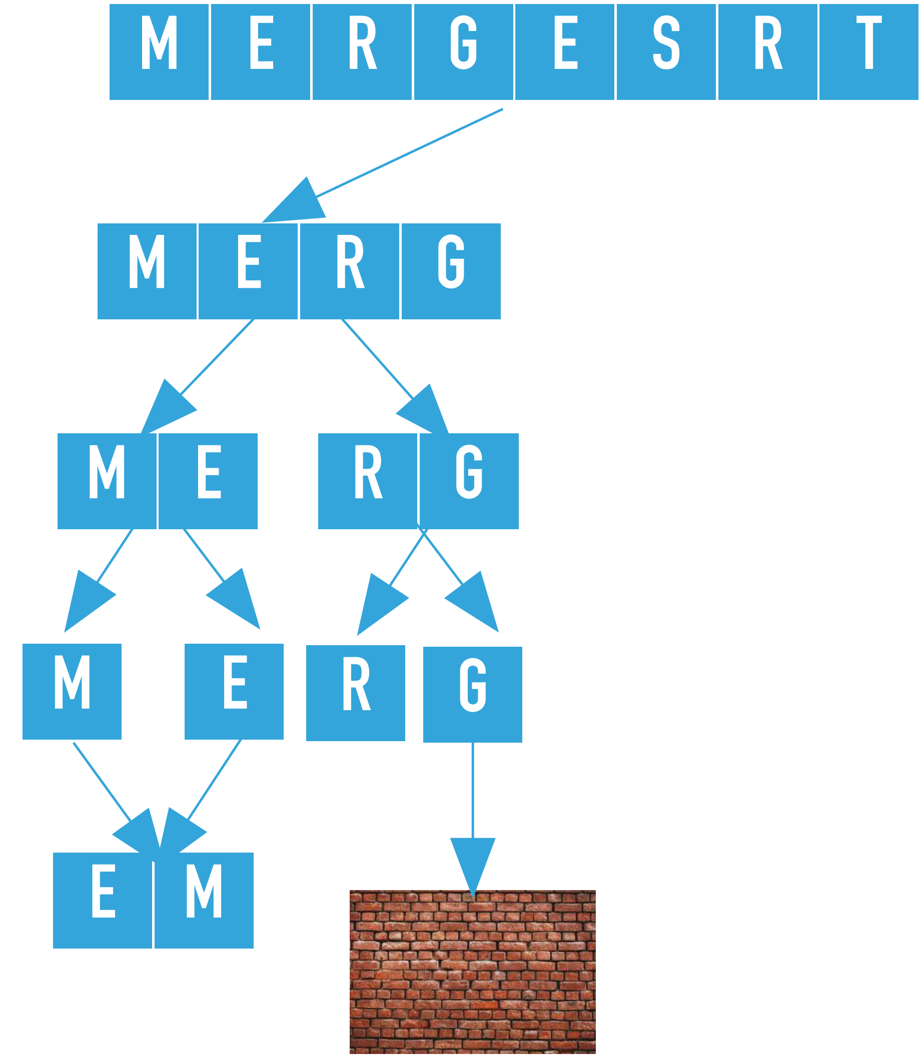

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, M, R, G, E, S, R, T], [M, E, null, null, null, null, null, null], 2, 2)` finds `hi <= lo` and returns.



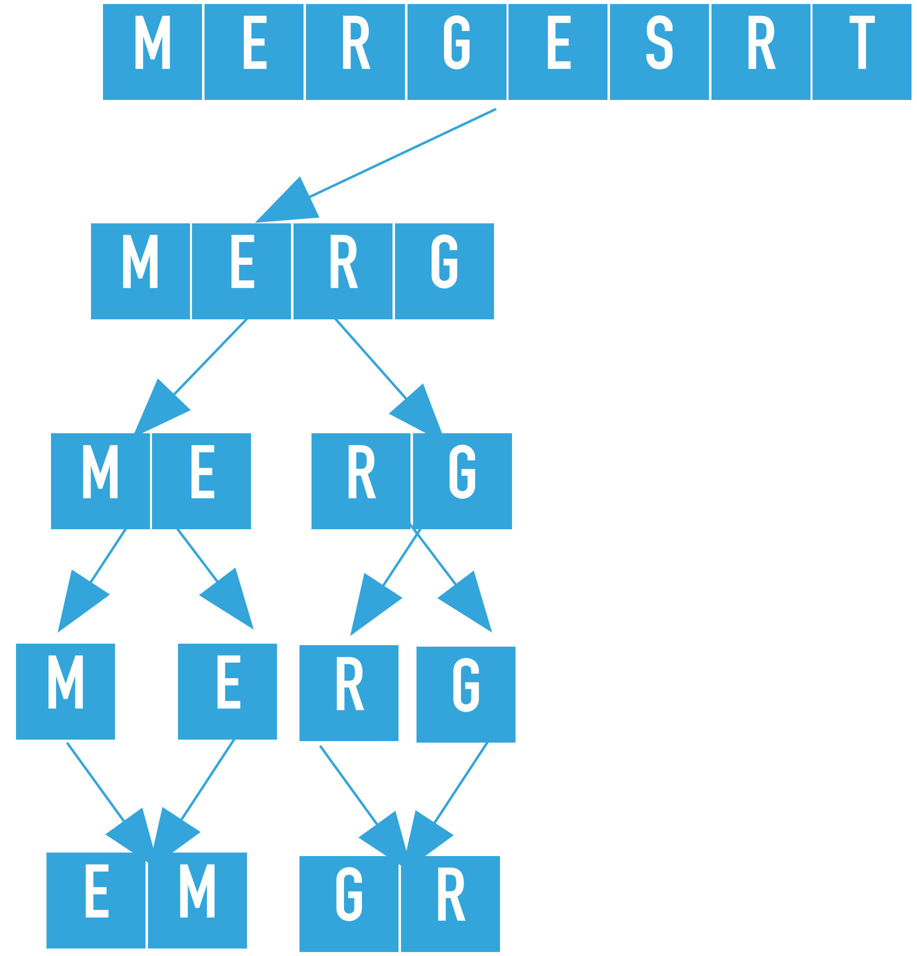

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, M, R, G, E, S, R, T], [M, E, null, null, null, null, null, null], 2, 3)` calls recursively `sort` on the right subarray, that is `sort([E, M, R, G, E, S, R, T], [M, E, null, null, null, null, null, null], 3, 3)`, where `lo = 3`, `hi = 3`



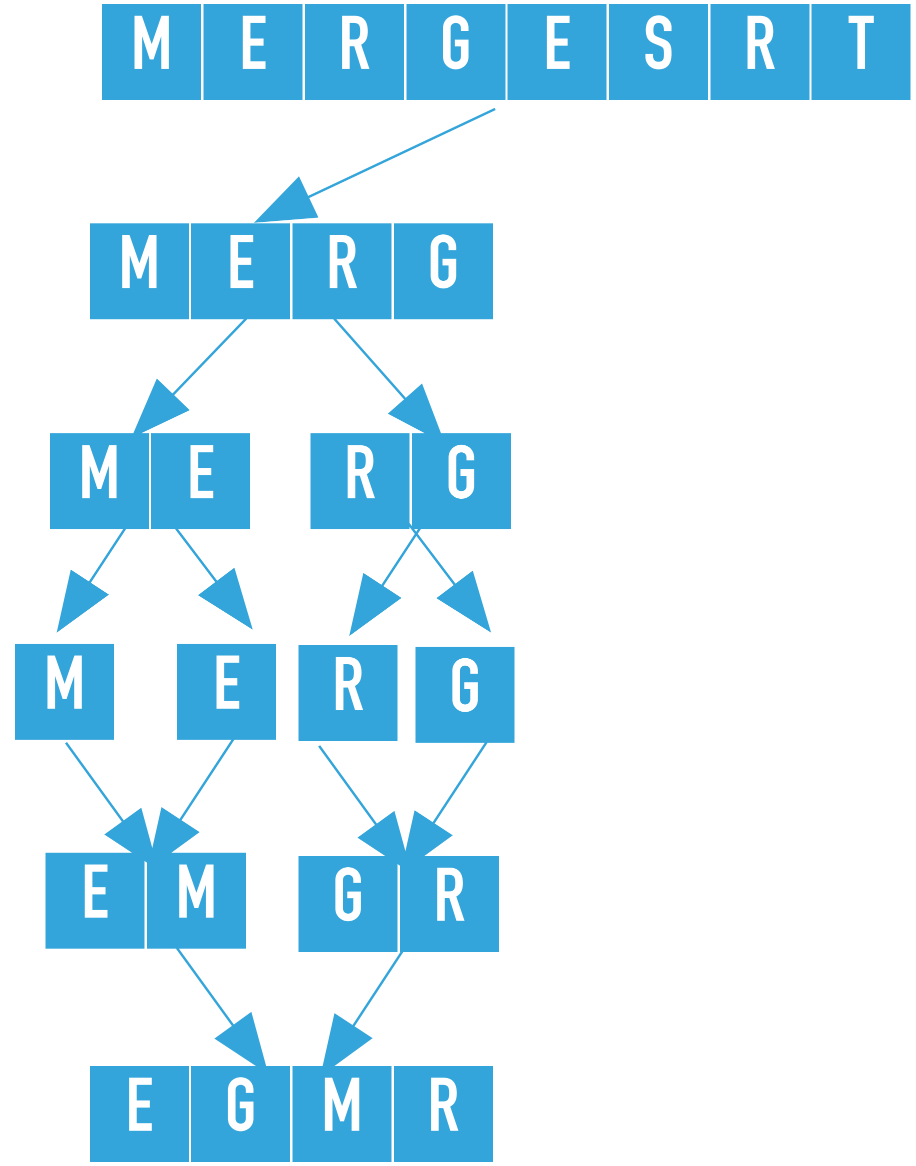

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, M, R, G, E, S, R, T], [M, E, null, null, null, null, null, null], 3, 3)` finds `hi <= lo` and returns.



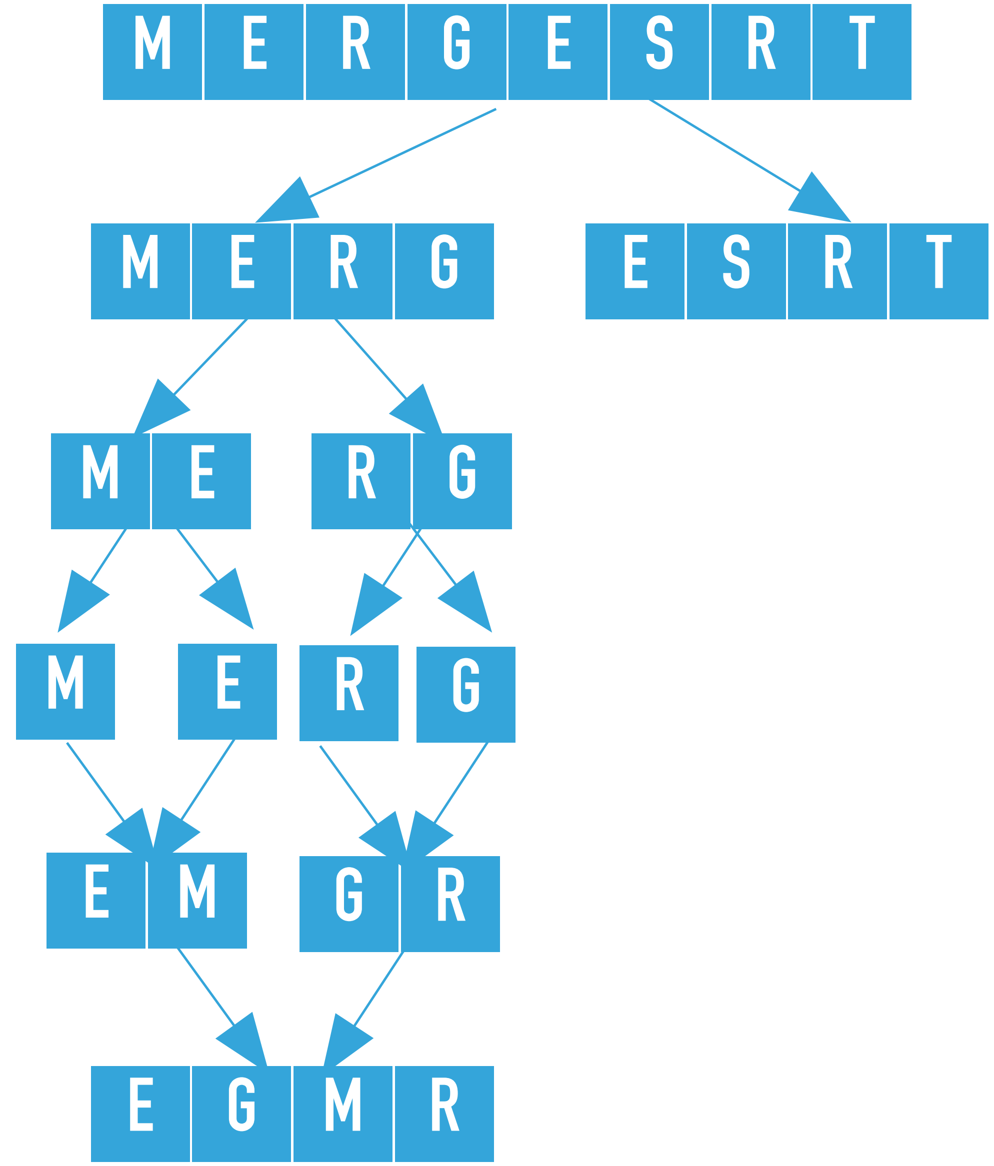

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, M, R, G, E, S, R, T], [M, E, null, null, null, null, null, null], 2, 3)` merges the two subarrays that is calls `merge([E, M, R, G, E, S, R, T], [M, E, null, null, null, null, null, null], 2, 2, 3)`, where `lo = 2`, `mid = 2`, and `hi = 3`. The resulting partially sorted array is `[E, M, G, R, E, S, R T]`.



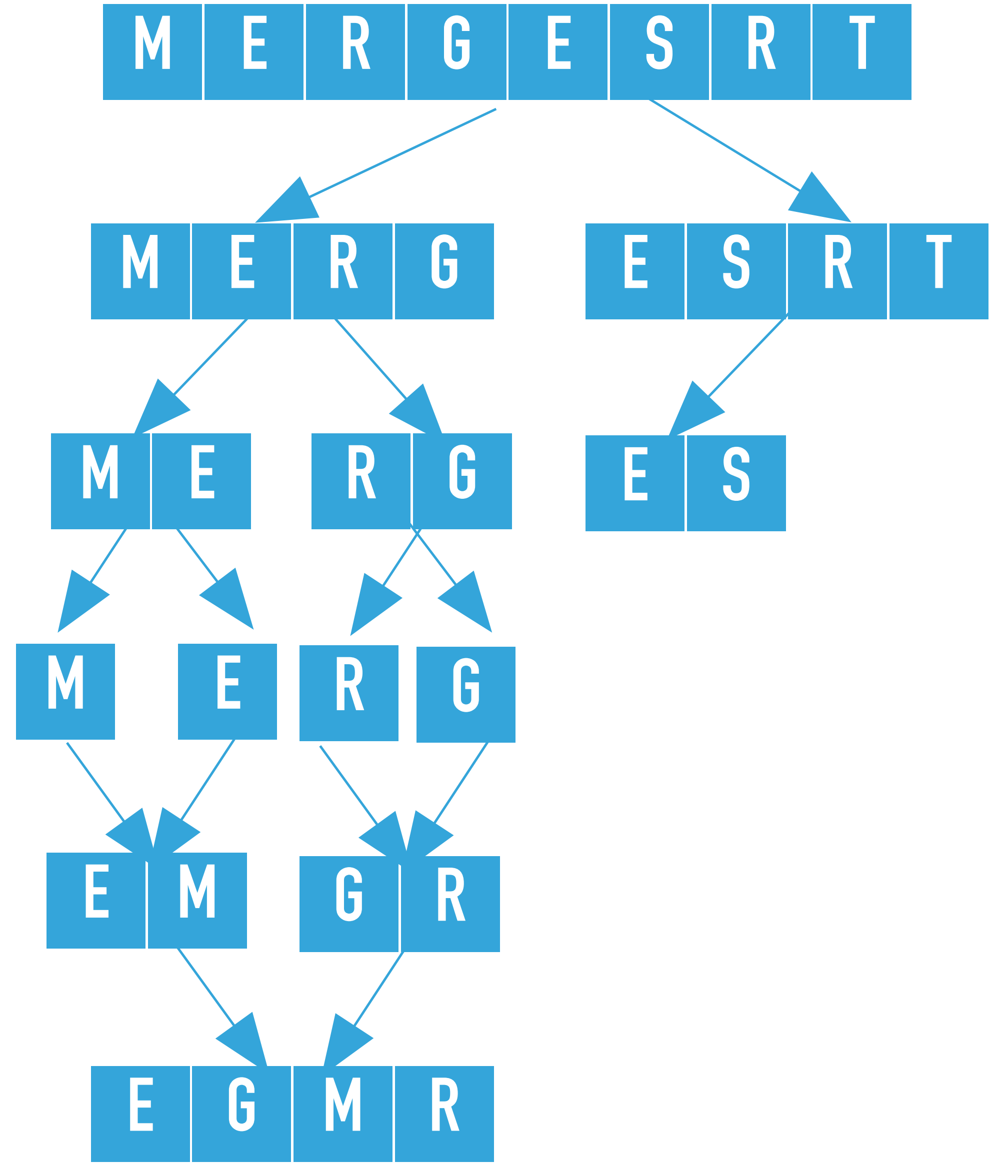

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, M, G, R, E, S, R, T], [M, E, R, G, null, null, null, null], 0, 3)` merges the two subarrays that is calls `merge([E, M, G, R, E, S, R, T], [M, E, R, G, null, null, null, null], 0, 1, 3)`, where `lo = 0`, `mid = 1`, and `hi = 3`. The resulting partially sorted array is `[E, G, M, R, E, S, R T]`.



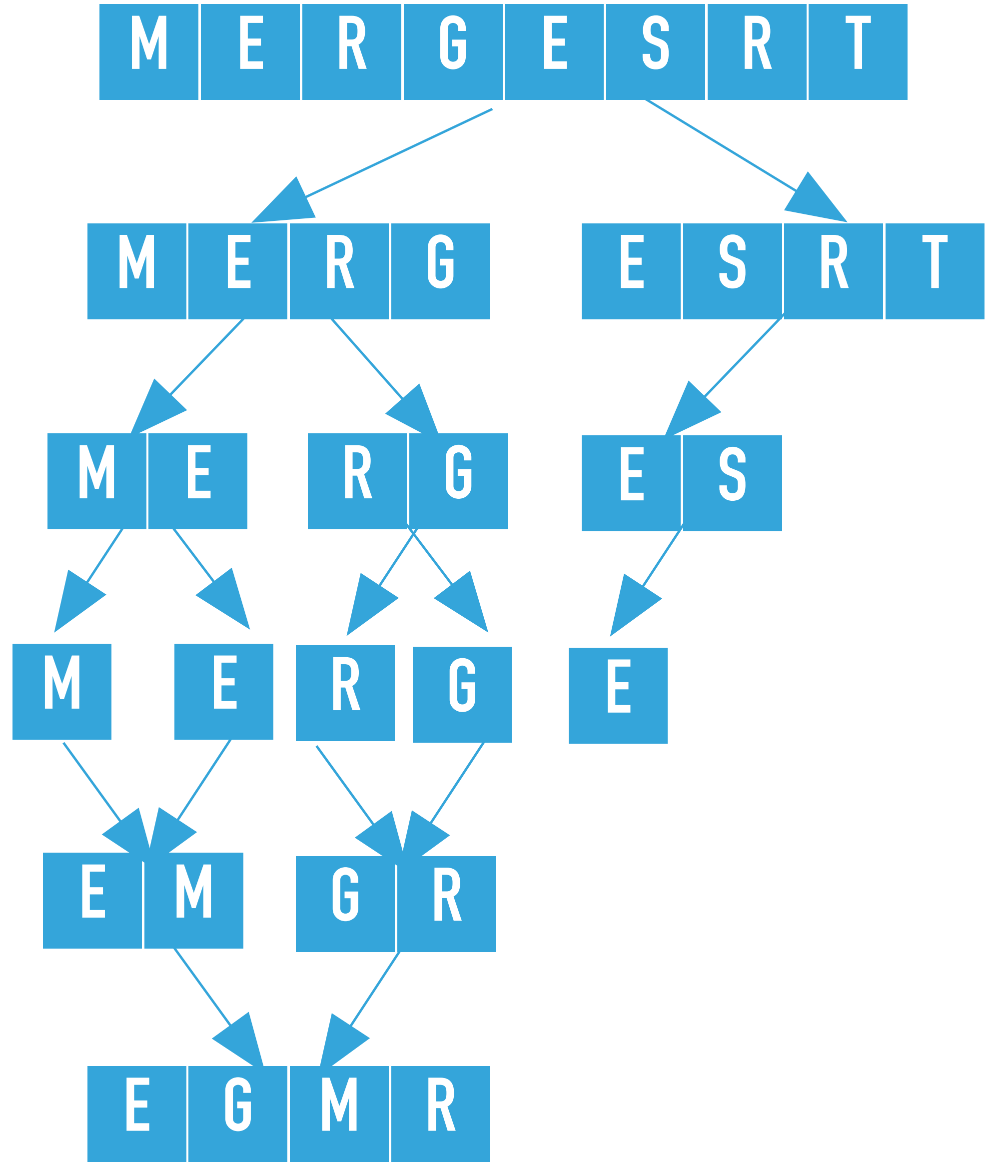

```java
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, G, M, R, E, S, R, T], [E, M, G, R, null, null, null, null], 0, 7)` calls recursively `sort` on the right subarray, that is `sort([E, G, M, R, E, S, R, T], [E, M, G, R, null, null, null, null], 4, 7)`, where `lo = 4`, `hi = 7`



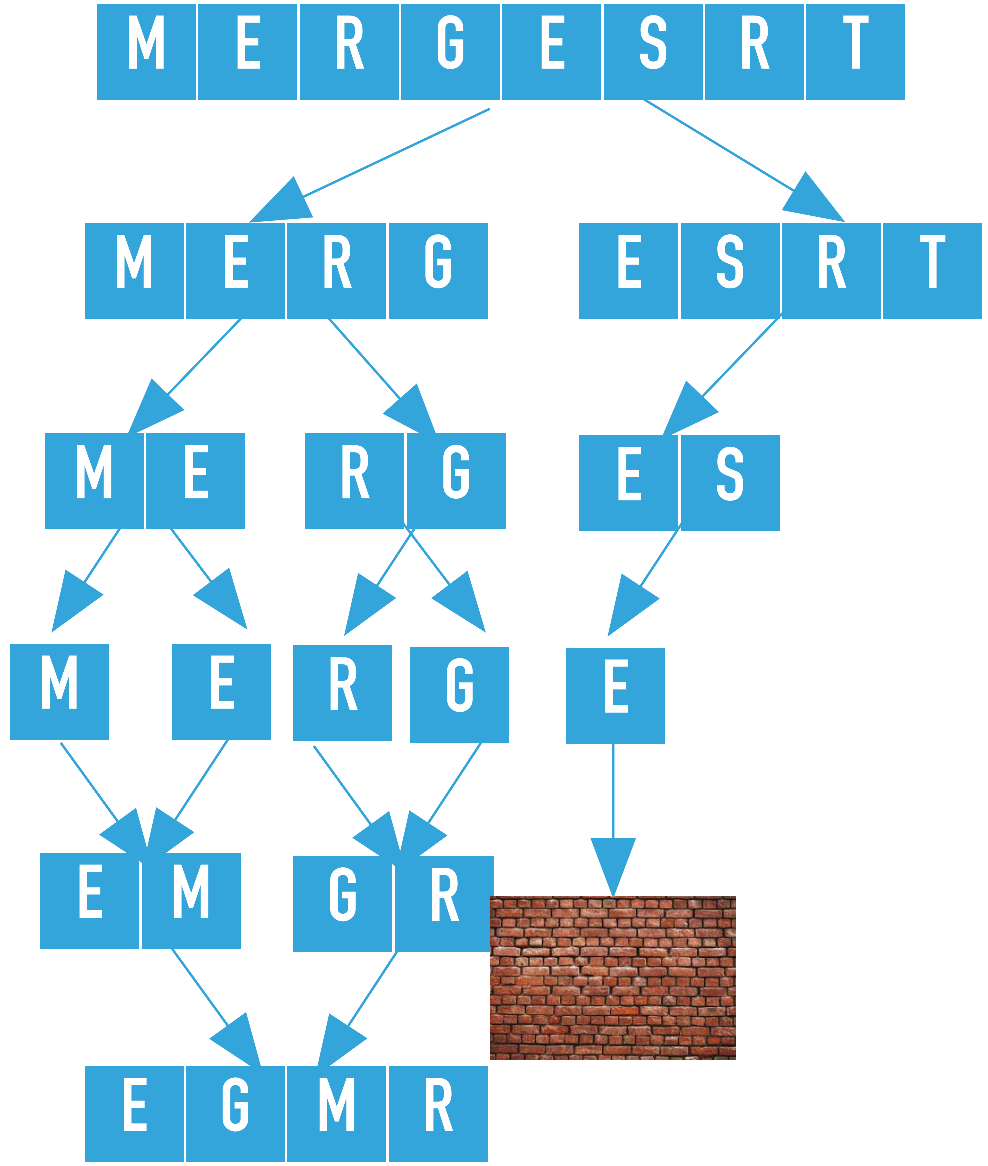

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, G, M, R, E, S, R, T], [E, M, G, R, null, null, null, null], 4, 7)` calculates the `mid = 5` and calls recursively `sort` on the left subarray, that is `sort([E, G, M, R, E, S, R, T], [E, M, G, R, null, null, null, null], 4, 5)`, where `lo = 4`, `hi = 5`.



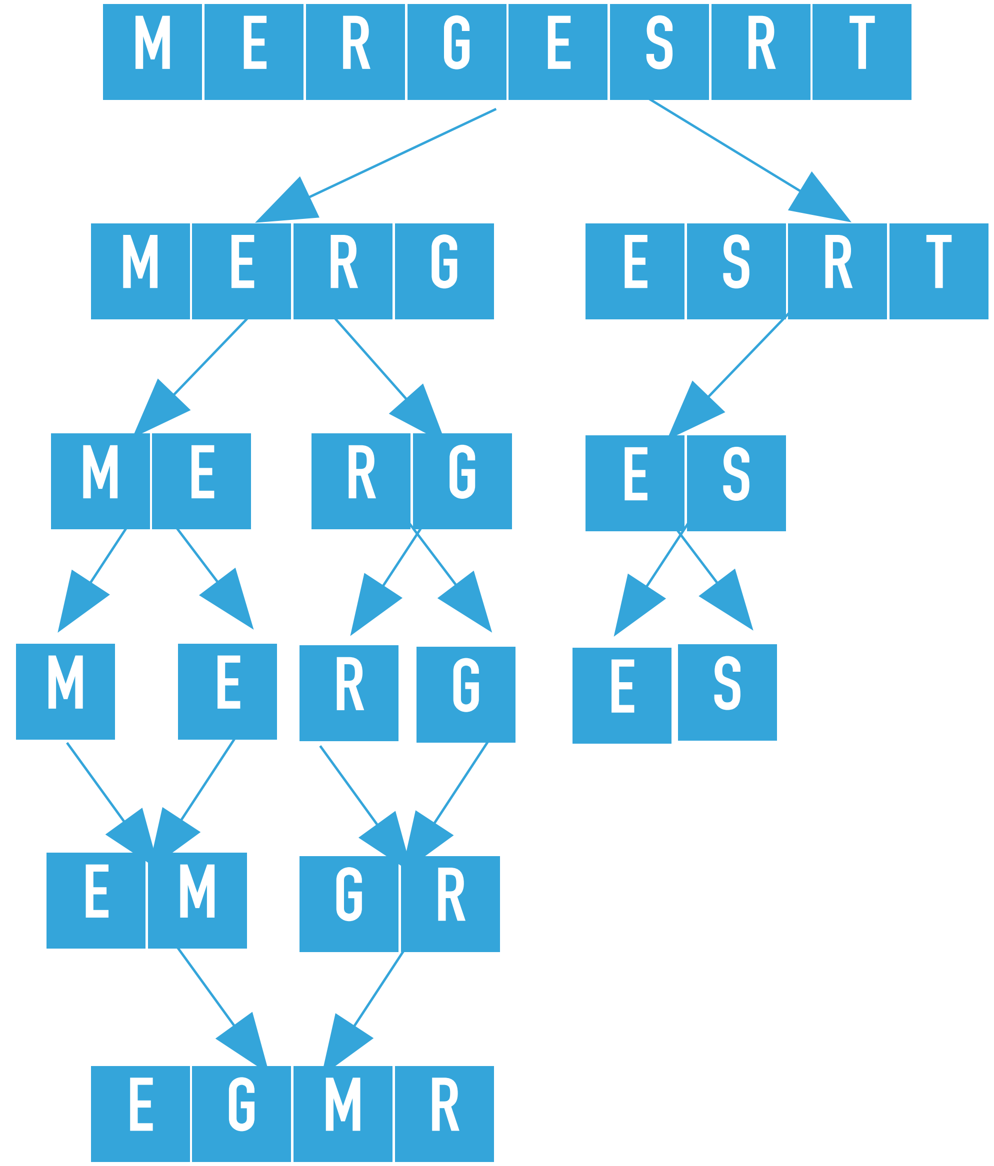

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, G, M, R, E, S, R, T], [E, M, G, R, null, null, null, null], 4, 5)` calculates the `mid = 4` and calls recursively `sort` on the left subarray, that is `sort([E, G, M, R, E, S, R, T], [E, M, G, R, null, null, null, null], 4, 4)`, where `lo = 4`, `hi = 4`.



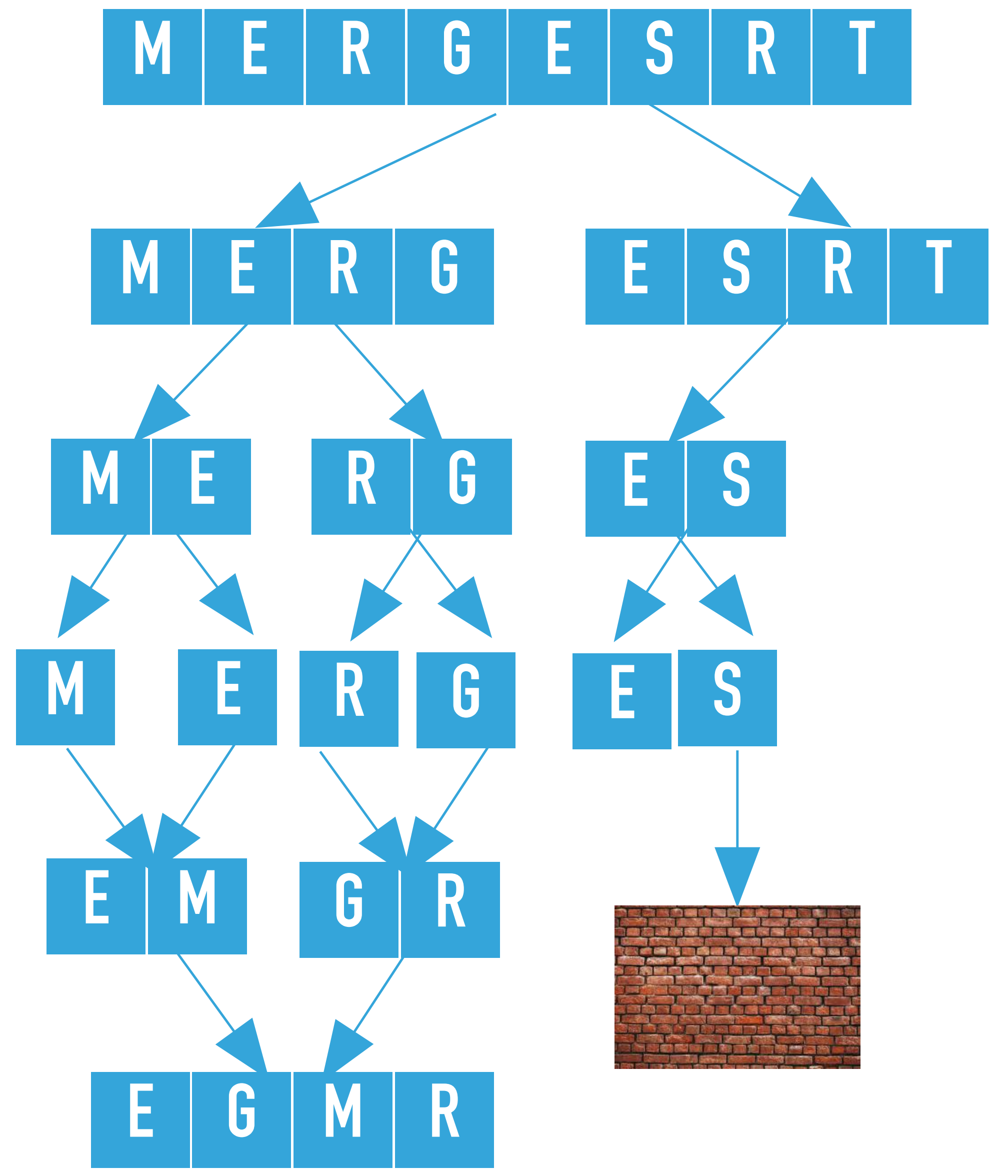

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, G, M, R, E, S, R, T], [E, M, G, R, null, null, null, null], 4, 4)` finds `hi <= lo` and returns.



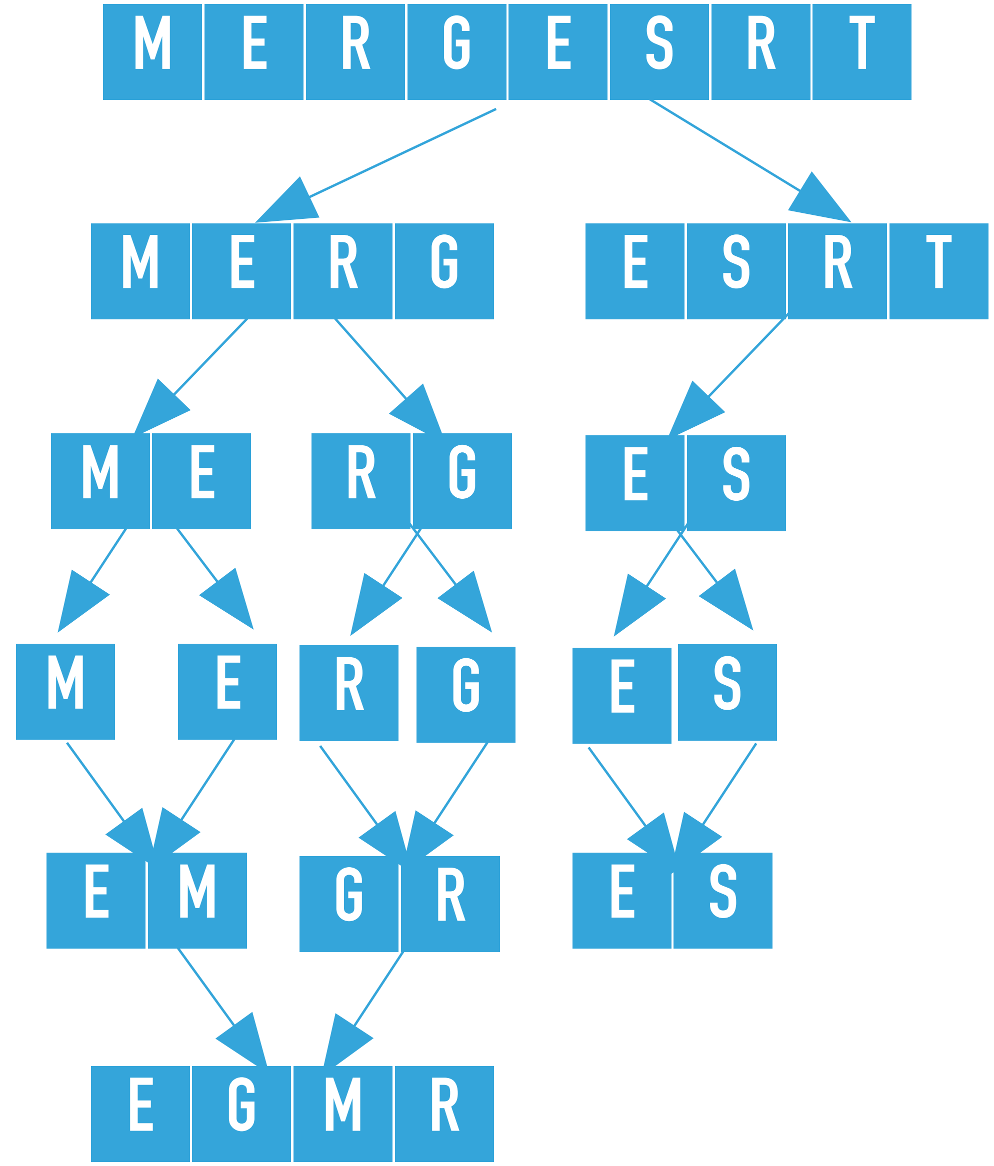

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, G, M, R, E, S, R, T], [E, M, G, R, null, null, null, null], 4, 5)` calls recursively `sort` on the right subarray, that is `sort([E, G, M, R, E, S, R, T], [E, M, G, R, null, null, null, null], 5, 5)`, where `lo = 5`, `hi = 5`



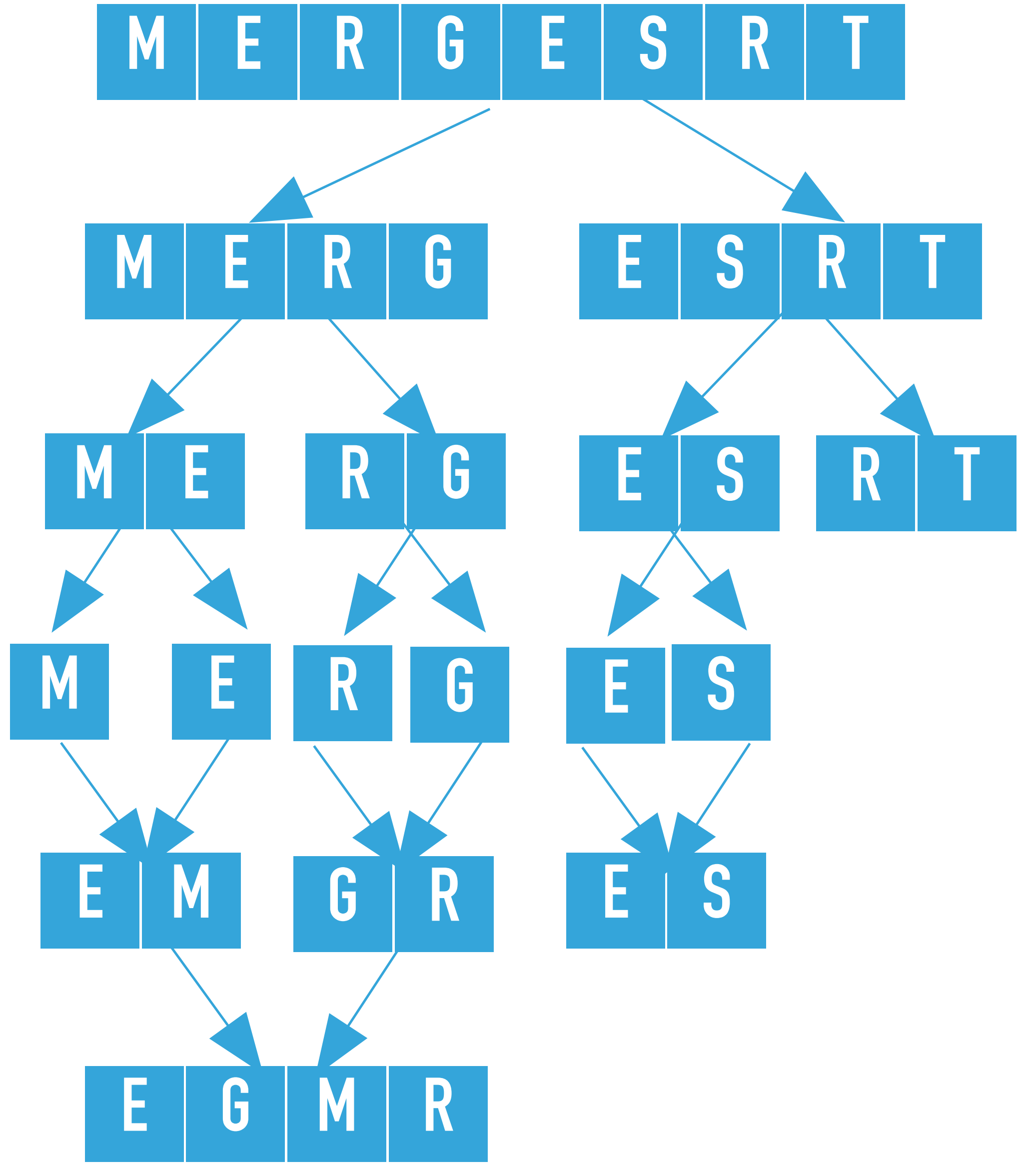

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, G, M, R, E, S, R, T], [E, M, G, R, null, null, null, null], 5, 5)` finds `hi <= lo` and returns.




```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, G, M, R, E, S, R, T], [E, M, G, R, null, null, null, null], 4, 5)` merges the two subarrays that is calls `merge([E, G, M, R, E, S, R, T], [E, M, G, R, null, null, null, null], 4, 4, 5)`, where `lo = 4`, `mid = 4`, and `hi = 5`. The resulting partially sorted array is `[E, G, M, R, E, S, R, T]`.

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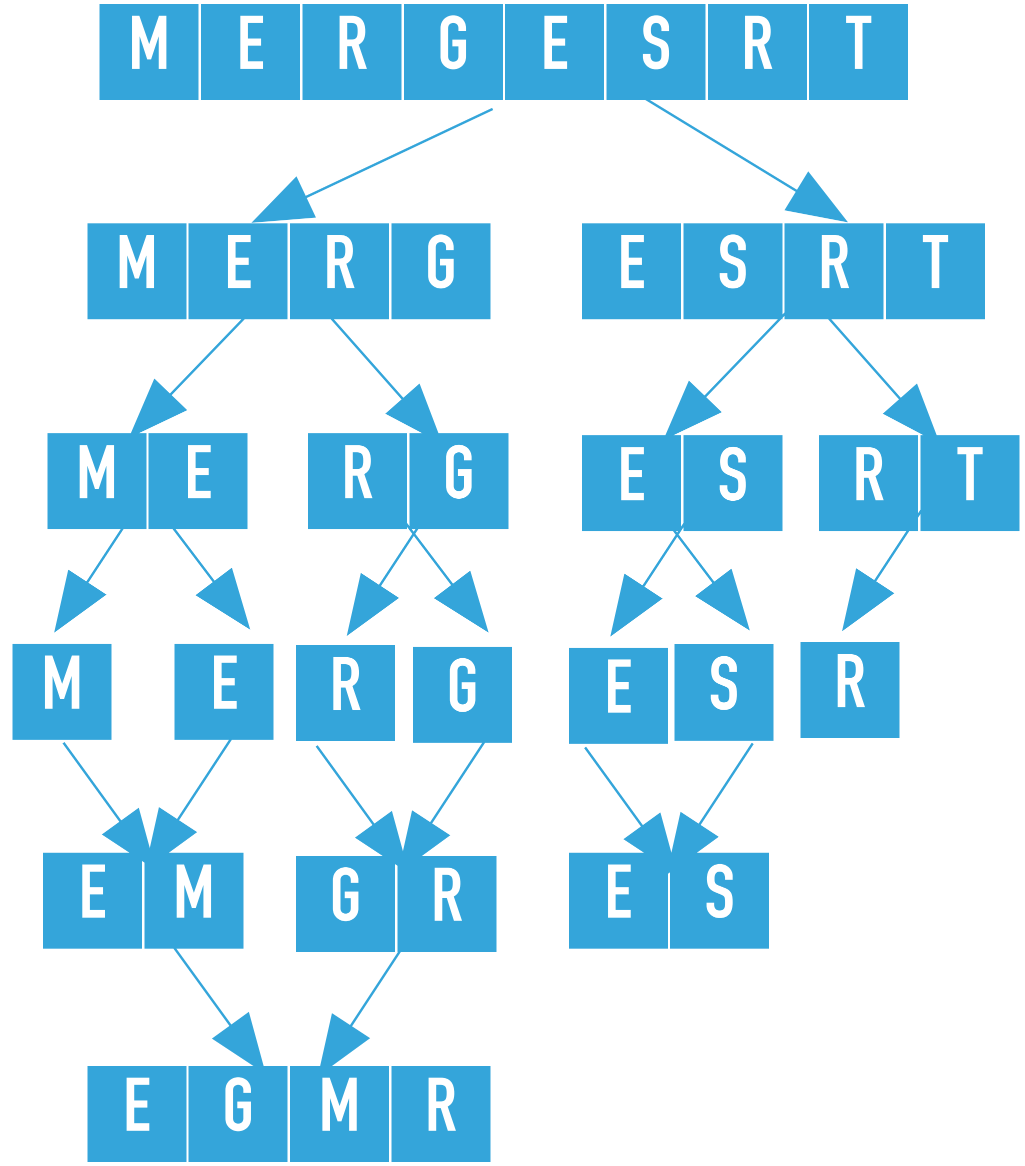
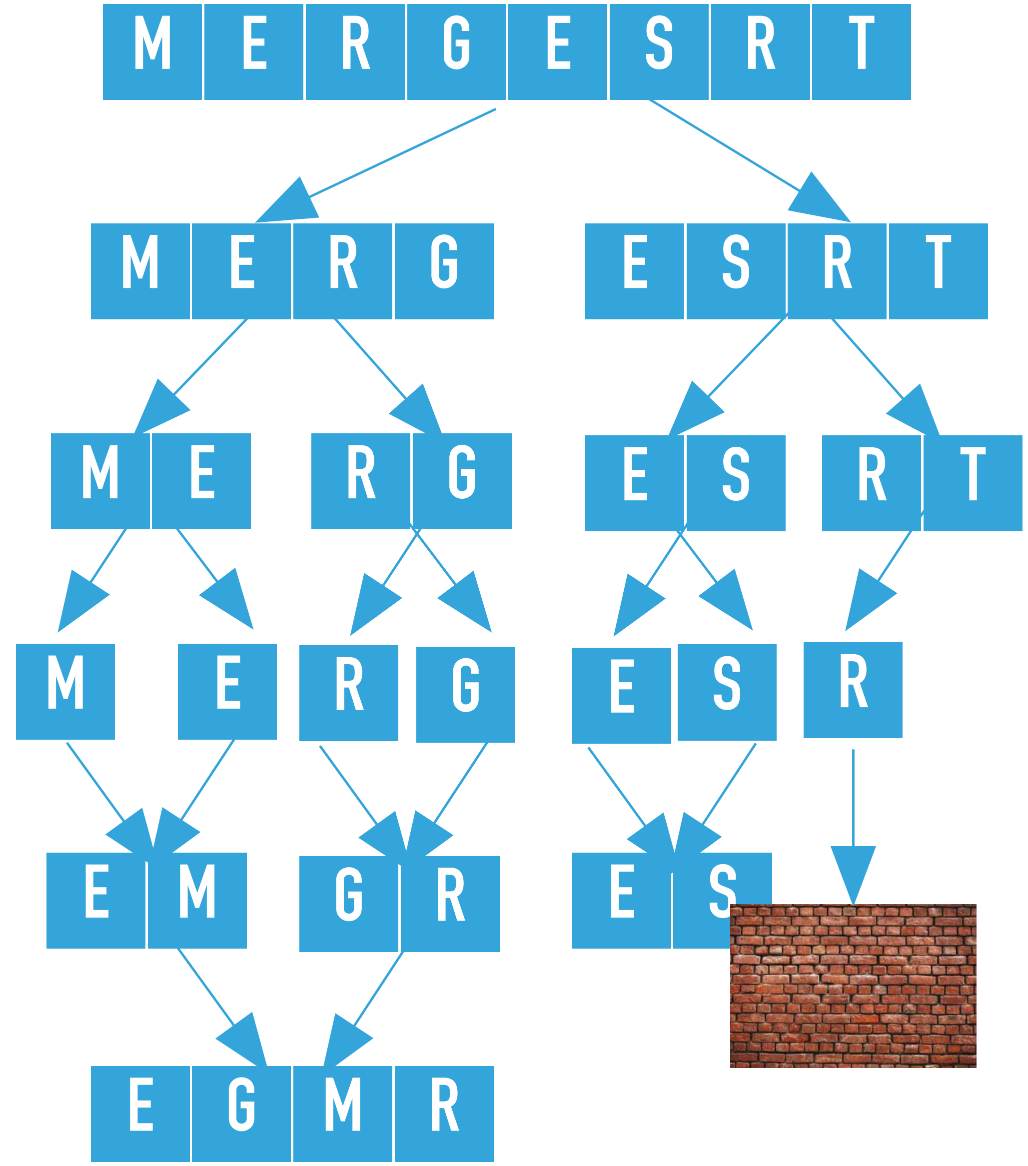
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, G, M, R, E, S, R, T], [E, M, G, R, E, S, null, null], 4, 7)` calls recursively `sort` on the right subarray, that is `sort([E, G, M, R, E, S, R, T], [E, M, G, R, E, S, null, null], 6, 7)`, where `lo = 6`, `hi = 7`



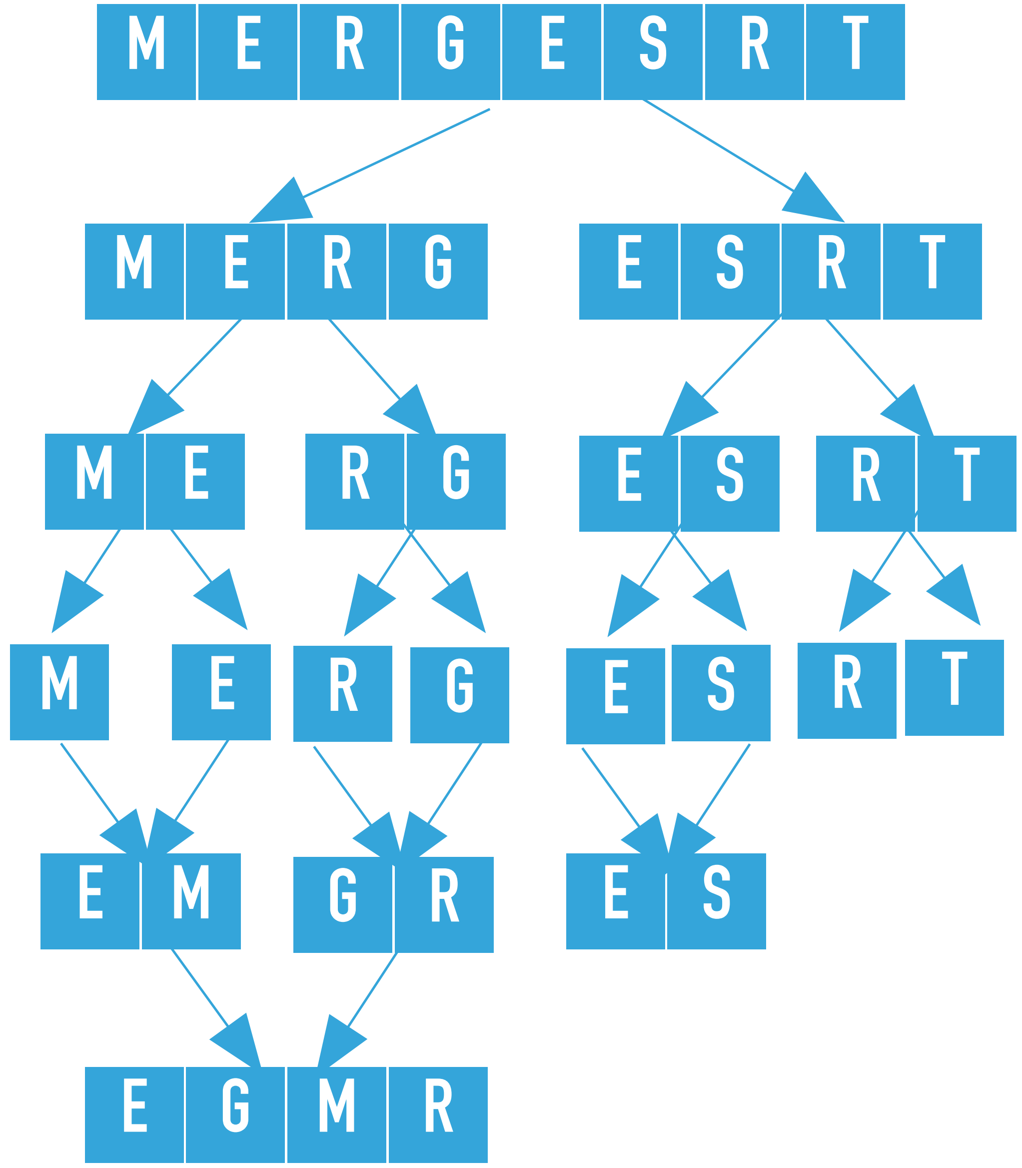

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, G, M, R, E, S, R, T], [E, M, G, R, E, S, null, null], 6, 7)` calculates the `mid = 6` and calls recursively `sort` on the left subarray, that is `sort([E, G, M, R, E, S, R, T], [E, M, G, R, E, S, null, null], 6, 6)`, where `lo = 6`, `hi = 6`.



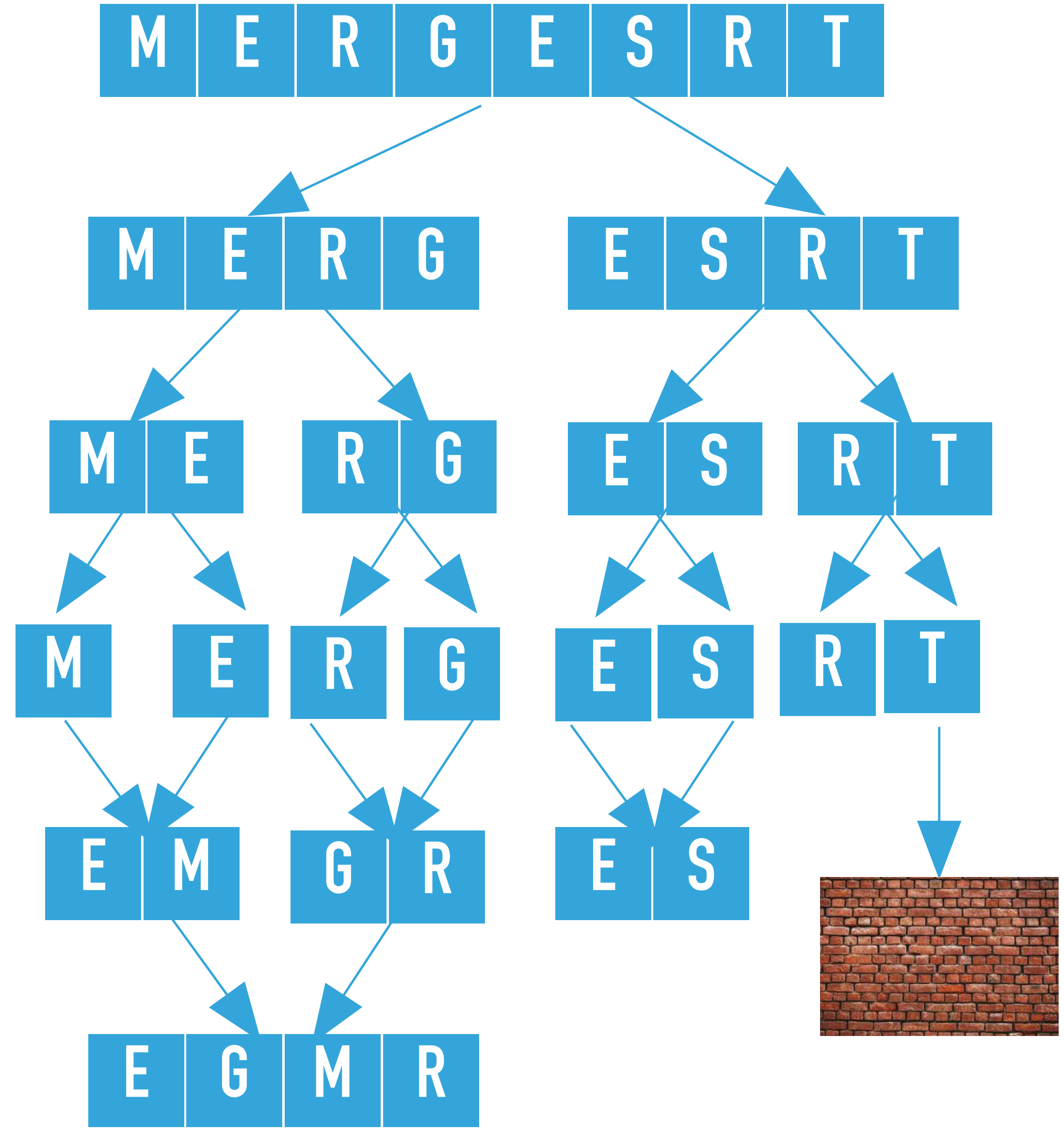

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, G, M, R, E, S, R, T], [E, M, G, R, E, S, null, null], 6, 6)` finds `hi <= lo` and returns.



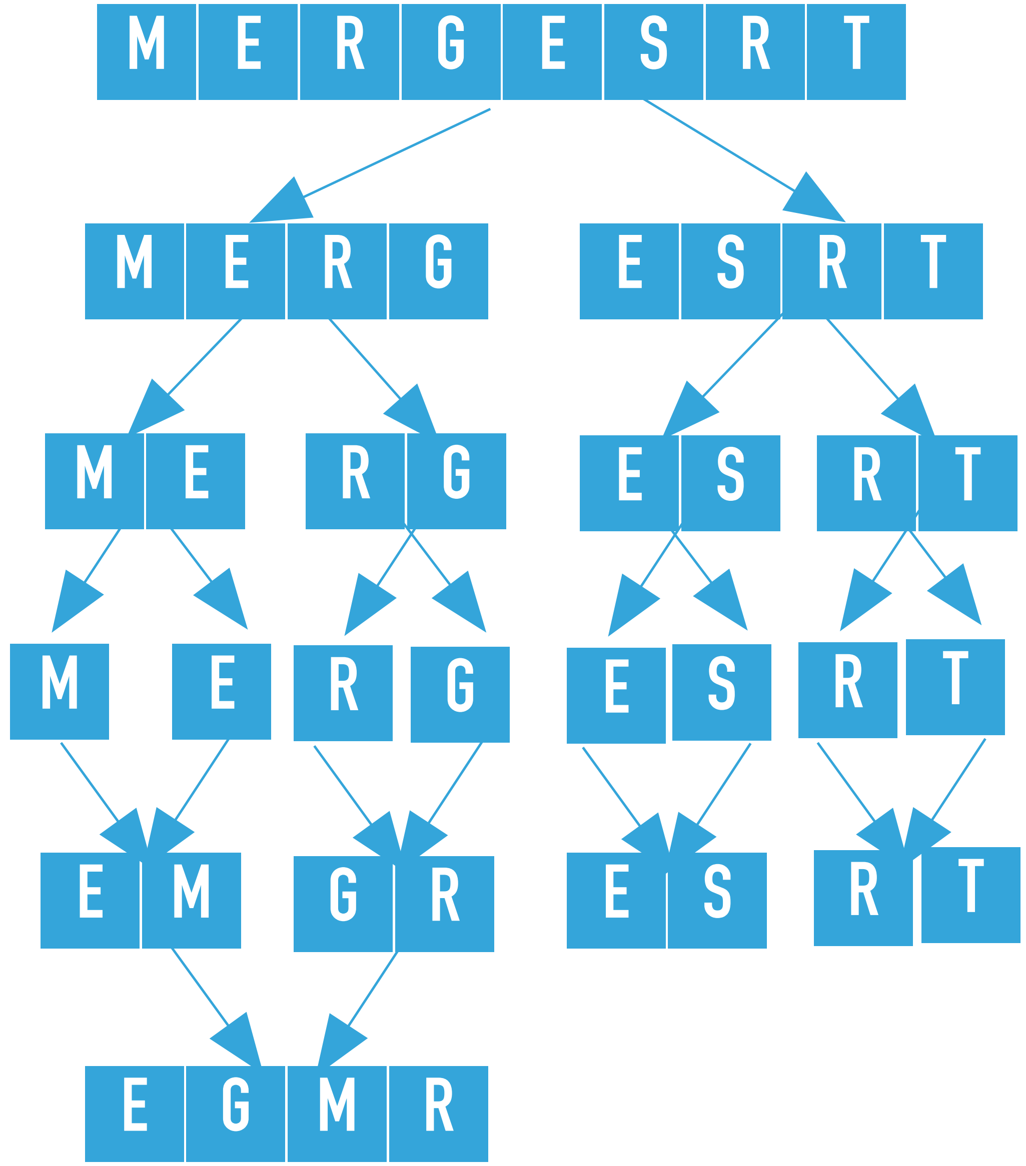

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, G, M, R, E, S, R, T], [E, M, G, R, E, S, null, null], 6, 7)` calls recursively `sort` on the right subarray, that is `sort([E, G, M, R, E, S, R, T], [E, M, G, R, E, S, null, null], 7, 7)`, where `lo = 7`, `hi = 7`




```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, G, M, R, E, S, R, T], [E, M, G, R, E, S, null, null], 7, 7)` finds `hi <= lo` and returns.




```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
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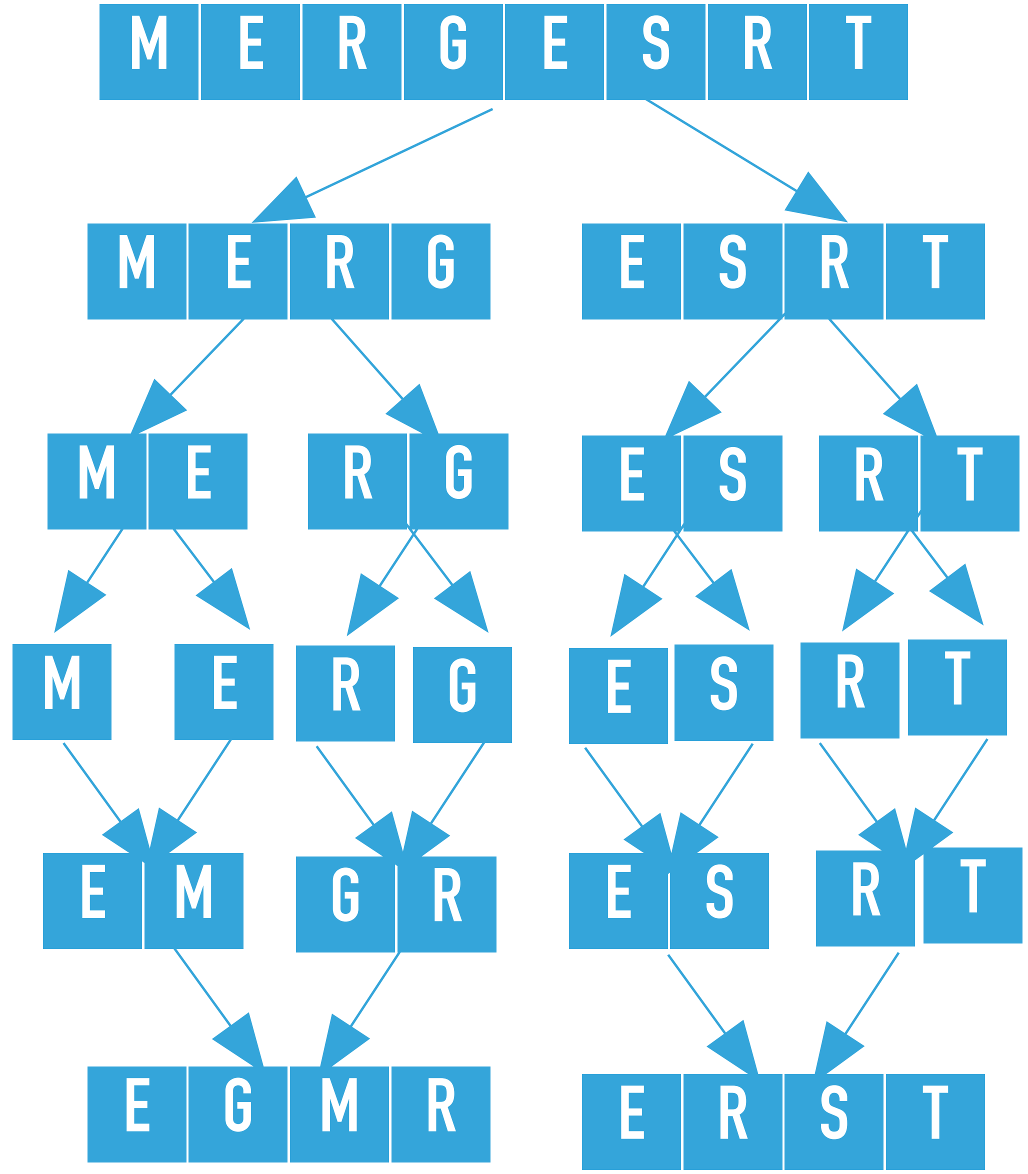
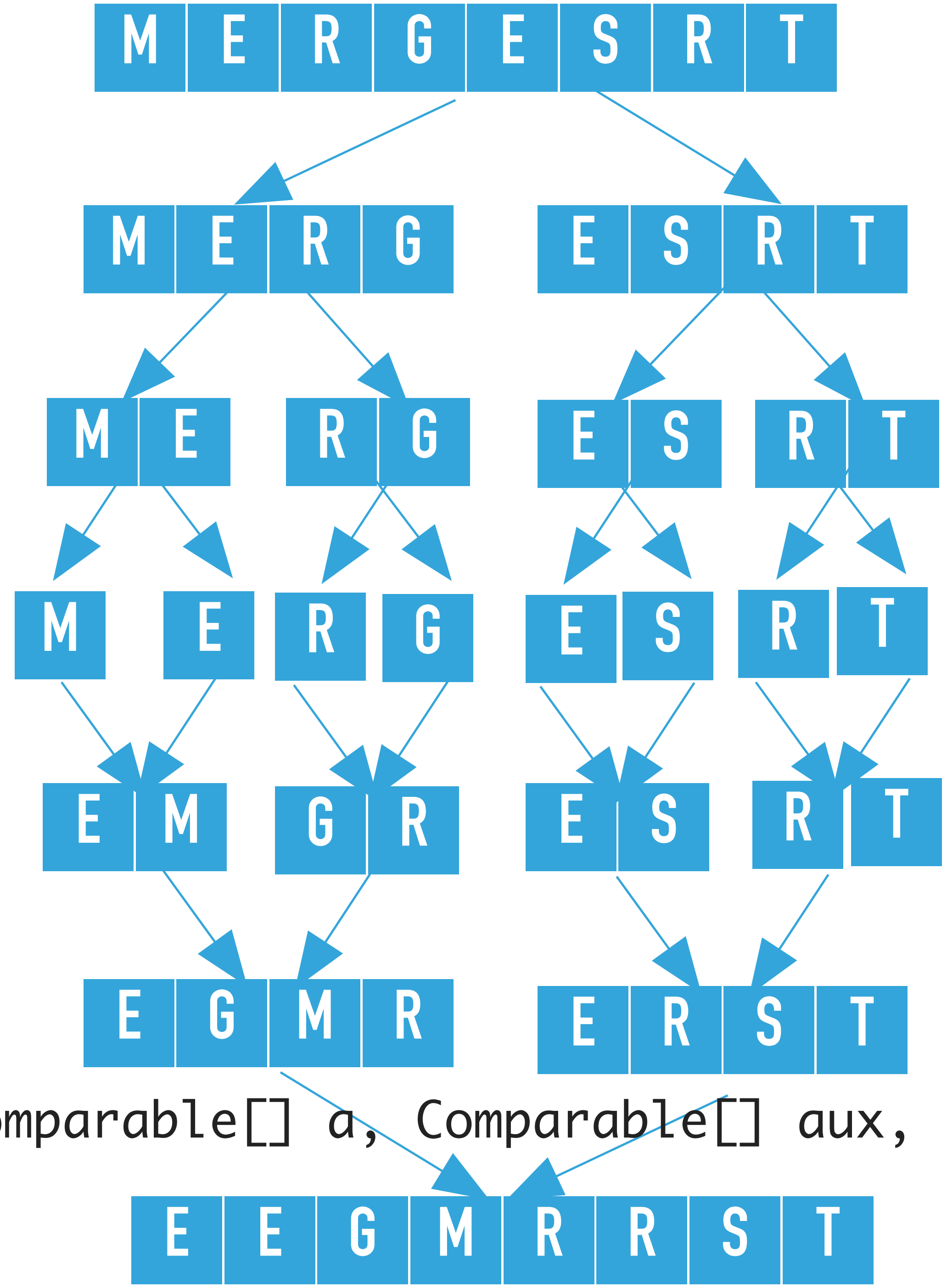
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, G, M, R, E, S, R, T], [E, M, G, R, E, S, null, null], 6, 7)` merges the two subarrays that is calls `merge([E, G, M, R, E, S, R, T], [E, M, G, R, E, S, null, null], 6, 6, 7)`, where `lo = 6`, `mid = 6`, and `hi = 7`. The resulting partially sorted array is `[E, G, M, R, E, S, R, T]`.




```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, G, M, R, E, S, R, T], [E, M, G, R, E, S, R, T], 4, 7)` merges the two subarrays that is calls `merge([E, G, M, R, E, S, R, T], [E, M, G, R, E, S, R, T], 4, 5, 7)`, where `lo = 4`, `mid = 5`, and `hi = 7`. The resulting partially sorted array is `[E, G, M, R, E, R, S, T]`.




```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi) {
    if (hi <= lo)
        return;
    int mid = lo + (hi - lo) / 2;
    sort(a, aux, lo, mid);
    sort(a, aux, mid+1, hi);
    merge(a, aux, lo, mid, hi);
}
```

`sort([E, G, M, R, E, R, S, T], [E, M, G, R, E, S, R, T], 0, 7)` merges the two subarrays that is calls `merge([E, G, M, R, E, R, S, T], [E, M, G, R, E, S, R, T], 0, 3, 7)`, where `lo = 0`, `mid = 3`, and `hi = 7`. The resulting sorted array is `[E, E, G, M, R, R, S, T]`.

Practice time

Which of the following subarray lengths will occur when running mergesort on an array of length 10?

A. { 1, 2, 3, 5, 10 }
B. { 2, 4, 6, 8, 10 }
C. { 1, 2, 5, 10 }
D. { 1,2,3,4,5,10}

Answer

Which of the following subarray lengths will occur when running mergesort on an array of length 10?

A. { 1, 2, 3, 5, 10 }

Good algorithms are better than supercomputers

- Your laptop executes 10^8 comparisons per second
- A supercomputer executes 10^{12} comparisons per second

	Insertion sort			Mergesort		
Computer	**Thousand inputs**	**Million inputs**	**Billion inputs**	**Thousand inputs**	**Million inputs**	**Billion inputs**
Home	Instant	2 hours	300 years	instant	1 sec	15 min
Supercomputer	Instant	1 second	1 week	instant	instant	instant

Analysis of comparisons

- We will assume that that n is a power of 2 ($n = 2^k$, where $k = log_2 n$) and the number of comparisons $T(n)$ to sort an array of length n with merge sort satisfies the recurrence:
  - $T(n) = T(n/2) + T(n/2) + (n - 1) = O(n \log n)$
- Number of array accesses (rather than exchanges, here) is also $O(n \log n)$.

Mergesort uses $\leq n \log n$ compares to sort an array of length n

If $n = 4$, 2 levels

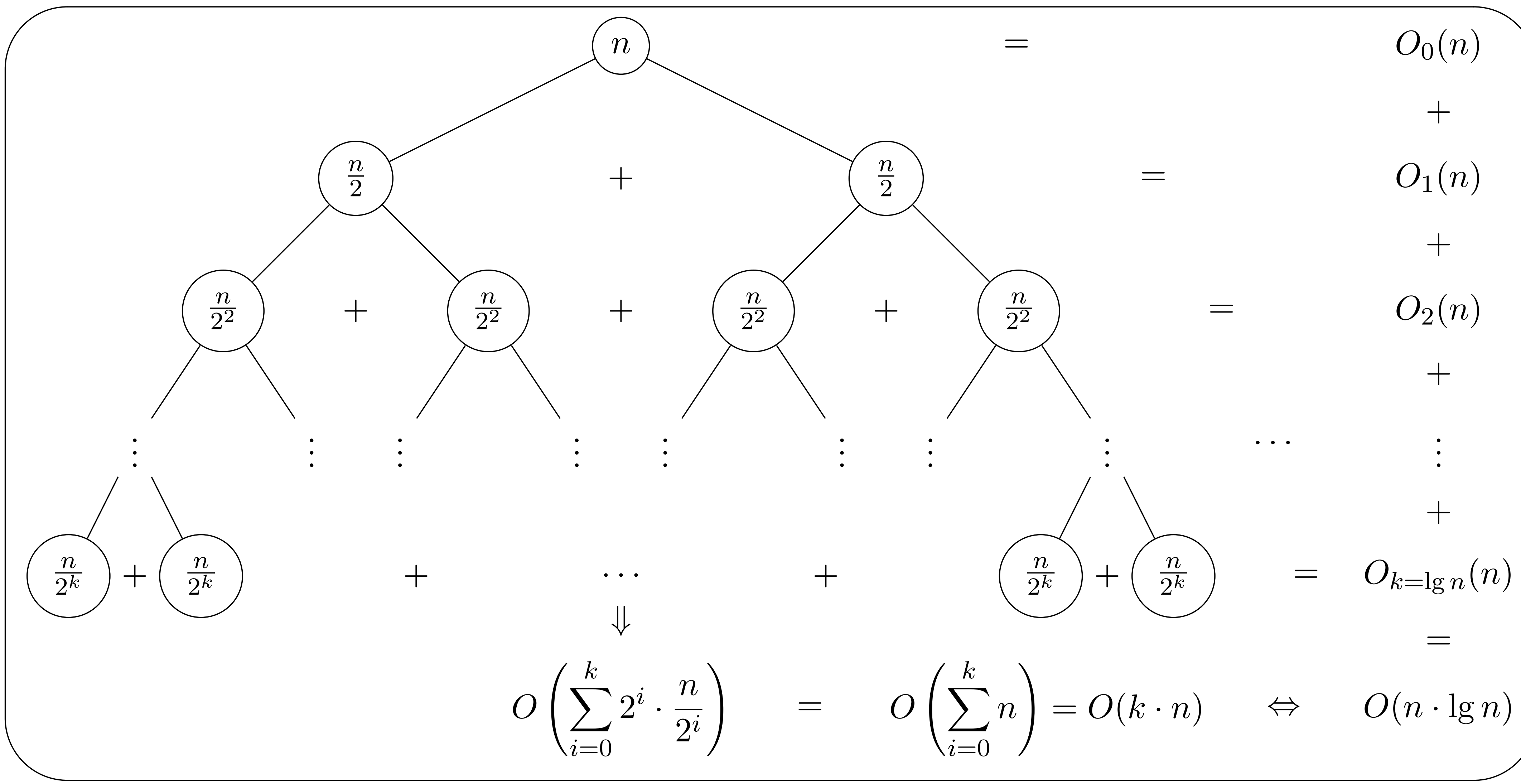
If $n = 8$, 3 levels

If $n = 16$, 4 levels

...

If $n = 2^k$, k levels,

or $k = log_2 n$



(log n levels) x (n comparisons) = O(nlogn)

http://www.texample.net/media/tikz/examples/PDF/merge-sort-recursion-tree.pdf

Any algorithm with the same structure takes $n \log n$ time

```
public static void f(int n) {
     if (n == 0)
         return;
     f(n/2);
     f(n/2);
     linear(n);
}
```

Mergesort basics

- Auxiliary memory is proportional to n, as `aux[]` needs to be of length n for the last merge.
- At its simplest form, merge sort is not an in-place algorithm.
- There are modifications for halting the size of the auxiliary array but in-place merge is very hard.
- Stable: Look into `merge()`, if equal keys, it takes them from the left subarray.
  - So is insertion sort, but not selection sort.

Practical improvements for Mergesort

- Use insertion sort for small subarrays.
- Stop if already sorted.
- Eliminate the copy to the auxiliary array by saving time (not space).

```
private static void sort(Comparable[] src, Comparable[] dst, int lo, int hi) {
   if (hi <= lo + 7) {
        insertionSort(dst, lo, hi);
        return;
    }

    int mid = lo + (hi - lo) / 2;
    sort(dst, src, lo, mid);
    sort(dst, src, mid+1, hi);

    if (!less(src[mid+1], src[mid])) {
      for (int i = lo; i <= hi; i++) dst[i] = src[i];
       return;
    }

    merge(src, dst, lo, mid, hi);
}
```

- For years, Java used this version to sort Collections of objects.

Sorting: the story so far

	In place	Stable	Best	Average	Worst	Remarks
Selection	X		$O(n^2)$	$O(n^2)$	$O(n^2)$	n exchanges
Insertion	X	X	$O(n)$	$O(n^2)$	$O(n^2)$	Use for small arrays or partially
Merge sort		X	$O(n \log n)$	$O(n \log n)$	$O(n \log n)$	Guaranteed performance; stable

Lecture 13: Mergesort

- Mergesort

Readings:

- Textbook:
  - Chapter 2.2 (pages 270–277)
- Website:
  - Mergesort: https://algs4.cs.princeton.edu/22mergesort/
  - Code: https://algs4.cs.princeton.edu/22mergesort/Merge.java.html

Practice Problems:

- 2.2.1-2.2.2, 2.2.11

Readings:

- Textbook:
  - Chapter 2.1 (pages 244–262), Chapter 2.1 (Page 247), Chapter 2.5 (Pages 338-339)
- Website:
  - Elementary sorts: https://algs4.cs.princeton.edu/21elementary/
  - Code: https://algs4.cs.princeton.edu/21elementary/Selection.java.html and https://algs4.cs.princeton.edu/21elementary/Insertion.java.html
- Oracle documentation:
  - Collections: https://docs.oracle.com/javase/tutorial/collections/intro/index.html
  - Comparable: https://docs.oracle.com/javase/8/docs/api/java/lang/Comparable.html
  - Comparator: https://docs.oracle.com/javase/8/docs/api/java/util/Comparator.html

Practice Problems:

- 2.1.1-2.1.8