CS062

DATA STRUCTURES AND ADVANCED PROGRAMMING

34: Undirected Graphs

Alexandra Papoutsaki
LECTURES

Mark Kampe
LABS

Lecture 34: Undirected Graphs

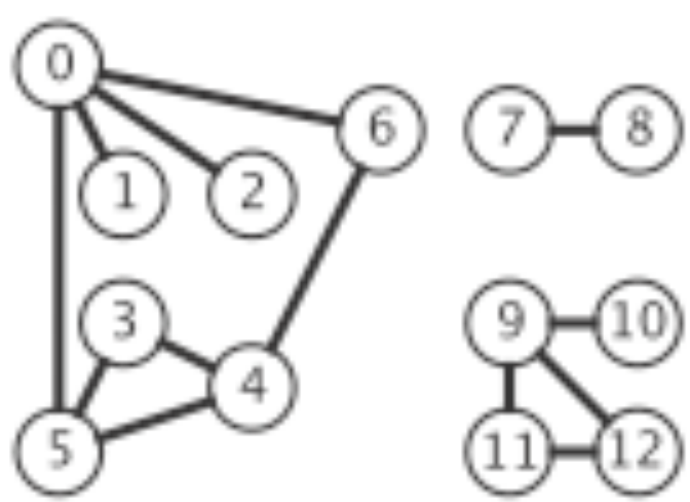
- Graph API
- Depth-First Search
- Breadth-First Search
- Connected Components

Some slides adopted from Algorithms 4th Edition or COS226

Graph representation

- Vertex representation: Here, integers between 0 and V-1.
    - We will use a symbol table to map between names and integers.

```
0 5
4 3
0 1
9 12
6 4
5 4
0 2
11 12
9 10
0 6
7 8
9 11
5 3
```



Basic Graph API

- `public class Graph`
- `Graph(int V)`: create an empty graph with V vertices.
- `void addEdge(int v, int w)`: add an edge v-w.
- `Iterable<Integer> adj(int v)`: return vertices adjacent to v.
- `int V()`: number of vertices.
- `int E()`: number of edges.

Example of how to use the Graph API to process the graph

- ```
  public static int degree(Graph g, int v){
      int count = 0;
      for(int w : g.adj(v))
          count++;
      return count;
  }
  ```
  ```

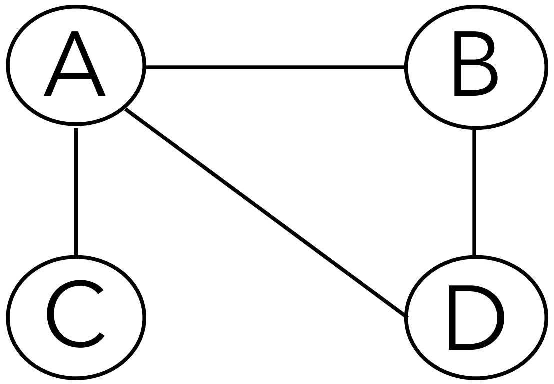
Graph density

- In a simple graph (no parallel edges or loops), if $|V| = n$, then:
  - minimum number of edges is 0 and
  - maximum number of edges is $n(n-1)/2$.
- Dense graph -> edges closer to maximum.
- Sparse graph -> edges closer to minimum.

Graph representation: adjacency matrix

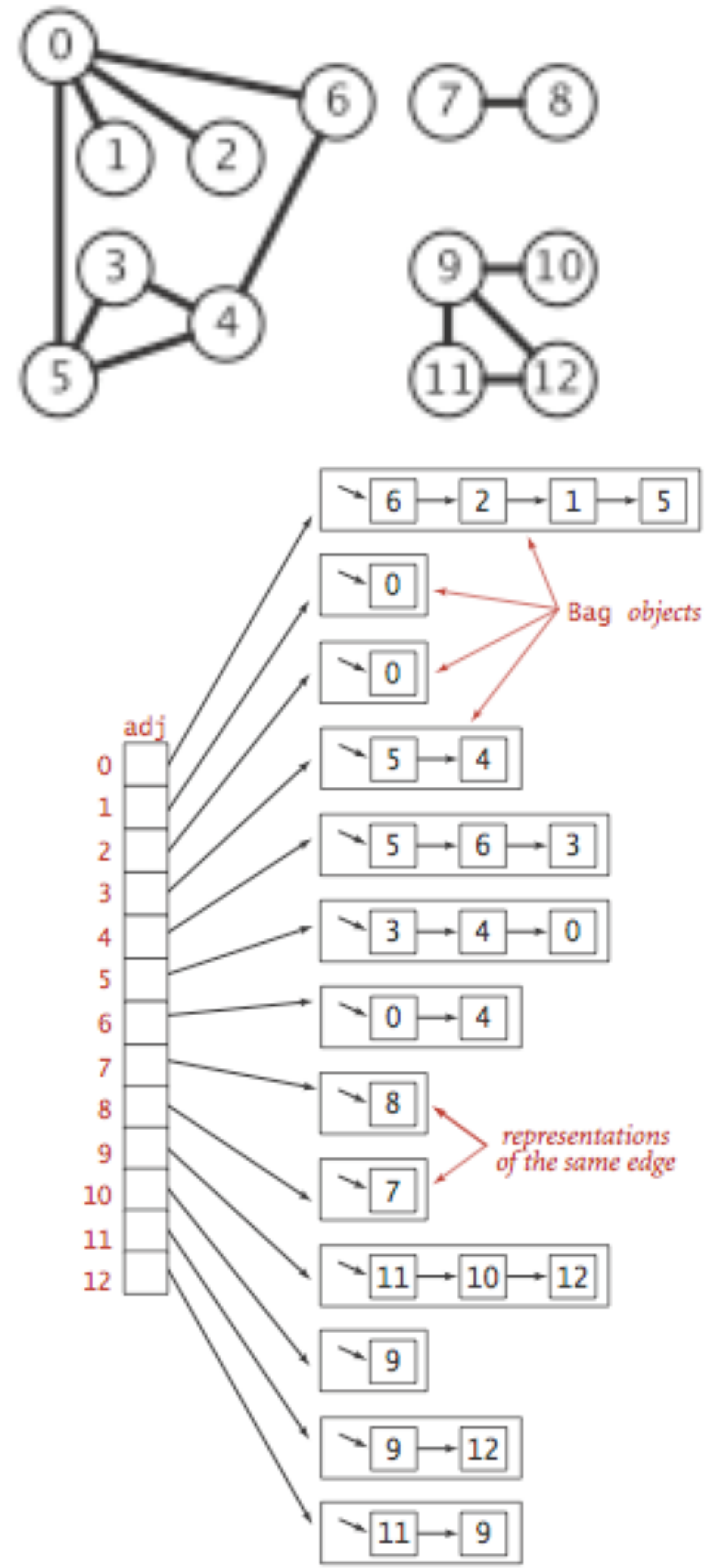
- Maintain a $|V|$-by-$|V|$ boolean array;
  for each edge `v-w`:
  - `adj[v][w] = adj[w][v] = true;` (1).
- Good for dense graphs (edges close to $|V|^2$).
- Constant time for lookup of an edge.
- Constant time for adding an edge.
- $|V|$ time for iterating over vertices adjacent to v.
- Symmetric, therefore wastes space in undirected graphs ($|V|^2$).
- Not widely used in practice.

	A	B	C	D
A	0	1	1	1
B	1	0	0	1
C	1	0	0	0
D	1	1	0	0



Graph representation: adjacency list

- Maintain vertex-indexed array of lists.
- Good for sparse graphs (edges proportional to $|V|$) which are much more common in the real world.
- Algorithms based on iterating over vertices adjacent to v.
- Space efficient ($|E| + |V|$).
- Constant time for adding an edge.
- Lookup of an edge or iterating over vertices adjacent to v is $degree(v)$.



Adjacency-list graph representation in Java

```
public class Graph {

    private final int V;
    private int E;
    private Bag<Integer>[] adj;

    //Initializes an empty graph with V vertices and 0 edges.
    public Graph(int V) {
        this.V = V;
        this.E = 0;
        adj = (Bag<Integer>[]) new Bag[V];
        for (int v = 0; v < V; v++) {
            adj[v] = new Bag<Integer>();
        }
    }

    //Adds the undirected edge v-w to this graph. Parallel edges and self-loops allowed
    public void addEdge(int v, int w) {
        E++;
        adj[v].add(w);
        adj[w].add(v);
    }


    //Returns the vertices adjacent to vertex v.
    public Iterable<Integer> adj(int v) {
       return adj[v];
    }
```

A bag is a collection where removing items is not supported—its purpose is to provide clients with the ability to collect items and then to iterate through the collected items

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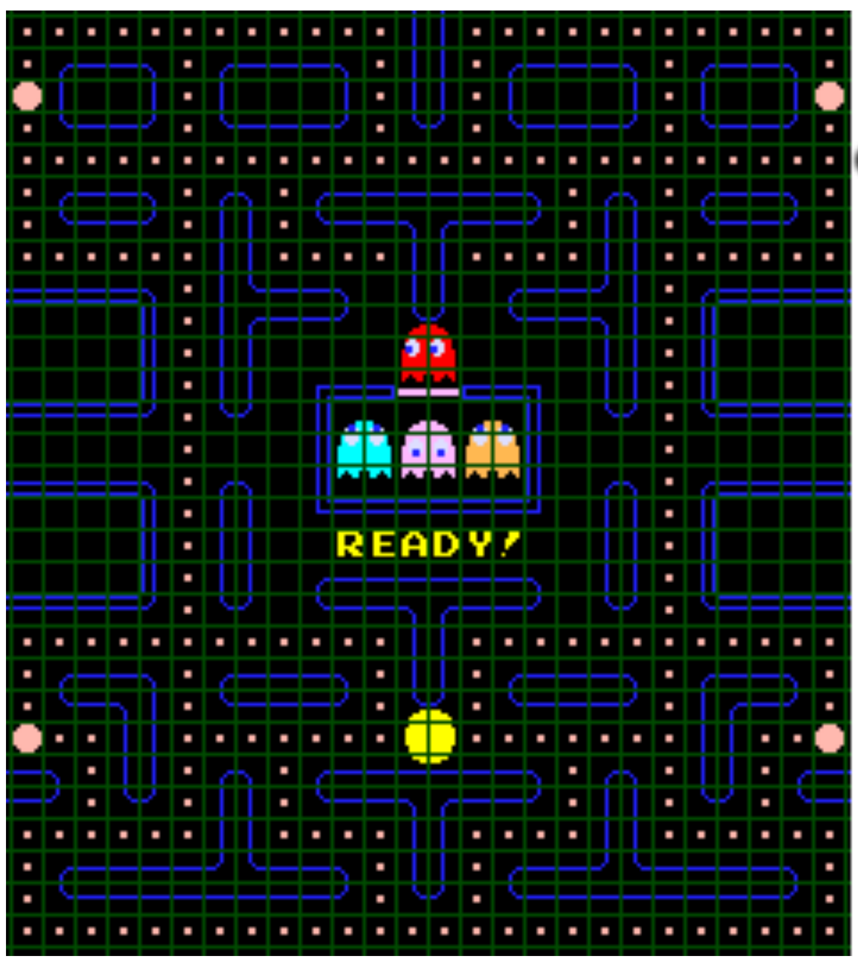
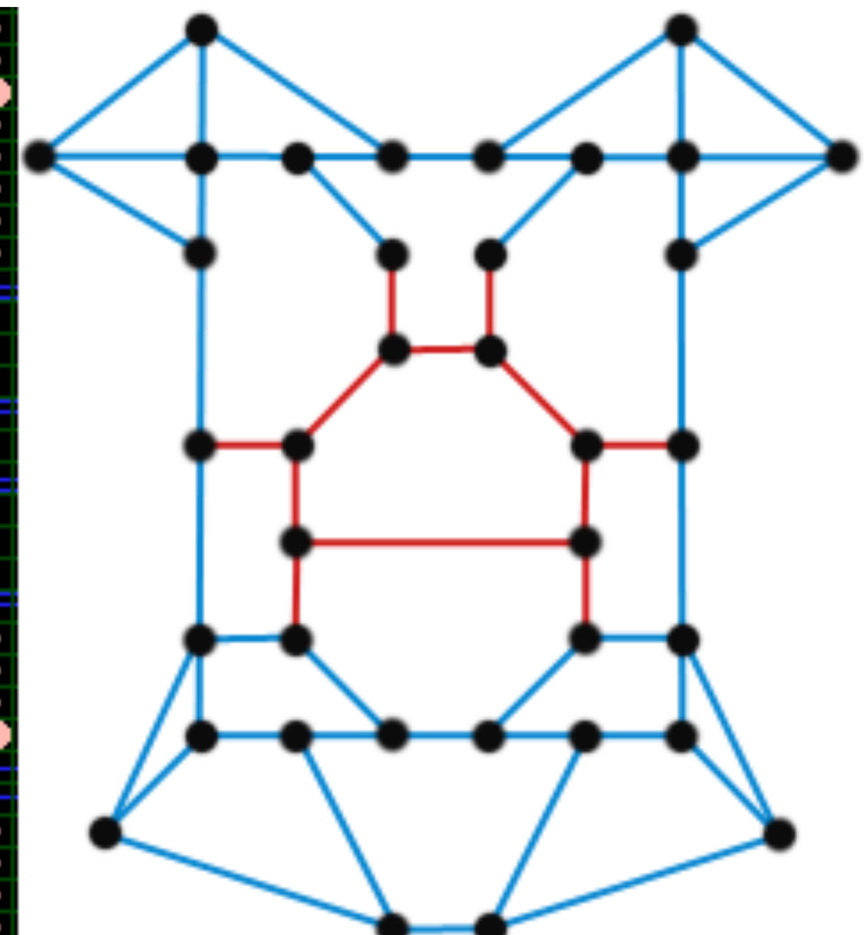
Mazes as graphs

- Vertex = intersection; edge = passage

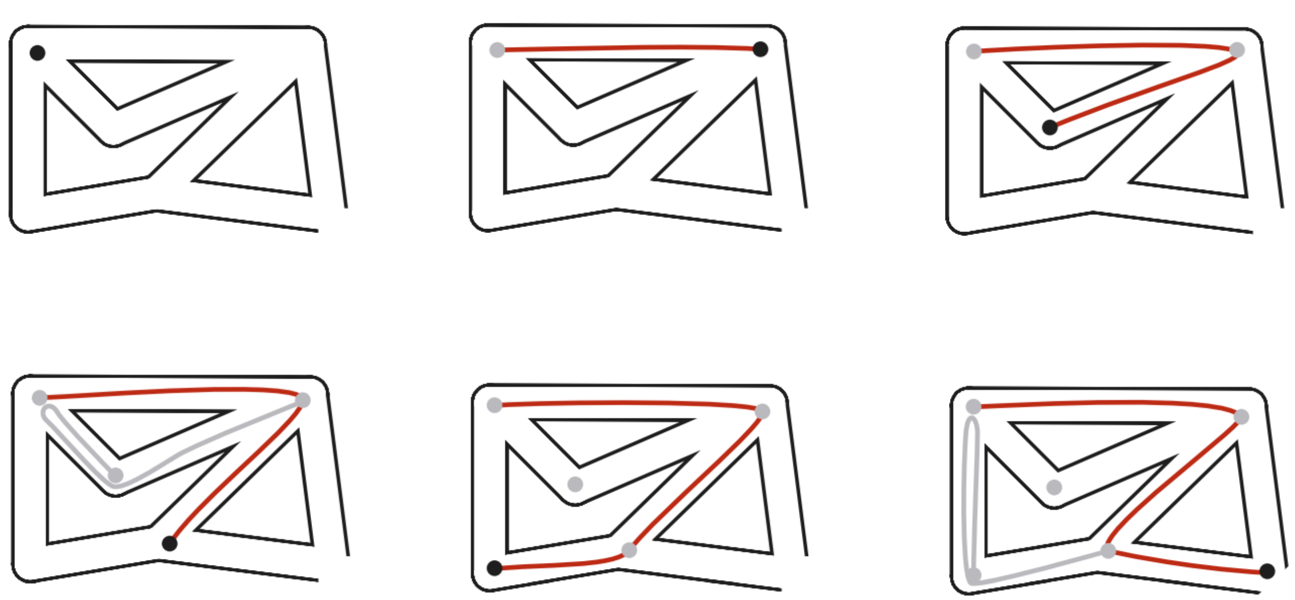




http://oatzy.blogspot.com/2011/09/playing-with-pac-man.html

How to survive a maze: a lesson from a Greek myth

- Theseus escaped from the labyrinth after killing the Minotaur with the following strategy instructed by Ariadne:
  - Unroll a ball of string behind you.
  - Mark each newly discovered intersection.
  - Retrace steps when no unmarked options.
- Also known as the Trémaux algorithm.



Depth-first search

- Goal: Systematically traverse a graph.
- DFS (to visit a vertex v)
  - Mark vertex v.
  - Recursively visit all unmarked vertices w adjacent to v.
- Typical applications:
  - Find all vertices connected to a given vertex.
  - Find a path between two vertices.

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4.1 DEPTH-FIRST SEARCH DEMO

Depth-first search

- Goal: Find all vertices connected to s (and a corresponding path).
- Idea: Mimic maze exploration.
- Algorithm:
  - Use recursion (ball of string).
  - Mark each visited vertex (and keep track of edge taken to visit it).
  - Return (retrace steps) when no unvisited options.

- When started at vertex s, DFS marks all vertices connected to s (and no other).

Depth-first search in Java

```
public class DepthFirstSearch {
     private boolean[] marked;    // marked[v] = is there an s-v path?
     private int[] edgeTo;        // edgeTo[v] = previous vertex on path from s to v

    public DepthFirstSearch(Graph G, int s) {
        marked = new boolean[G.V()];
        edgeTo = new int[G.V()];
        dfs(G, s);
    }

    // depth first search from v
    private void dfs(Graph G, int v) {
        marked[v] = true;
        for (int w : G.adj(v)) {
            if (!marked[w]) {
                edgeTo[w] = v;
                dfs(G, w);
            }
        }
    }
```

Depth-first search Analysis

- DFS marks all vertices connected to `s` in time proportional to $|V| + |E|$ in the worst case.
    - Initializing arrays `marked` and `edgeTo` takes time proportional to $|V|$.
    - Each adjacency-list entry is examined exactly once and there are $2E$ such edges (two for each edge).
- Once we run DFS, we can check if vertex `v` is connected to `s` in constant time. We can also find the `v`-`s` path (if it exists) in time proportional to its length.

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Breadth-first search

- BFS (from source vertex s)
  - Put s on a queue and mark it as visited.
  - Repeat until the queue is empty:
    - Dequeue vertex v.
    - Enqueue each of v's unmarked neighbors and mark them.

- Basic idea: BFS traverses vertices in order of distance from s.

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4.1 BREADTH-FIRST SEARCH DEMO

Breadth-first search in Java

```
public class BreadthFirstPaths {
   private boolean[] marked;  // marked[v] = is there an s-v path
    private int[] edgeTo;      // edgeTo[v] = previous edge on shortest s-v path
    private int[] distTo;      // distTo[v] = number of edges shortest s-v path

    public BreadthFirstPaths(Graph G, int s) {
        marked = new boolean[G.V()];
        distTo = new int[G.V()];
        edgeTo = new int[G.V()];
        bfs(G, s);
   }

   private void bfs(Graph G, int s) {
        Queue<Integer> q = new Queue<Integer>();
        distTo[s] = 0;
        marked[s] = true;
        q.enqueue(s);

        while (!q.isEmpty()) {
            int v = q.dequeue();
            for (int w : G.adj(v)) {
                if (!marked[w]) {
                    edgeTo[w] = v;
                    distTo[w] = distTo[v] + 1;
                    marked[w] = true;
                    q.enqueue(w);
                }
            }
        }
    }
```

Breadth-first search

- DFS: Put unvisited vertices on a stack.
- BFS: Put unvisited vertices on a queue.
- Shortest path problem: Find path from s to t that uses the fewest number of edges.
    - E.g., calculate the fewest numbers of hops in a communication network.
    - E.g., calculate the Kevin Bacon number or Erdös number.
- BFS computes shortest paths from s to all vertices in a graph in time proportional to $|E| + |V|$
    - The queue always consists of zero or more vertices of distance k from s, followed by zero or more vertices of k+1.

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Connectivity queries

- Goal: Preprocess graph to answer questions of the form "is `v` connected to `w`" in constant time.
- `public class CC`
- `CC(Graph G)`: find connected components in `G`.
- `boolean connected(int v, int w)`: are `v` and `w` connected?
- `int count()`: number of connected components.
- `int id(int v)`: component identifier for vertex `v`.

Connected components

- Goal: Partition vertices into connected components.
- Connected Components
  - Initialize all vertices as unmarked.
  - For each unmarked vertex, run DFS to identify all vertices discovered as part of the same component.

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4.1 Connected Components Demo

Connected Components in Java

```
public class CC {
    private boolean[] marked;   // marked[v] = has vertex v been marked?
    private int[] id;           // id[v] = id of connected component containing v
    private int[] size;         // size[id] = number of vertices in given component
    private int count;          // number of connected components

    public CC(Graph G) {
        marked = new boolean[G.V()];
        id = new int[G.V()];
        size = new int[G.V()];
        for (int v = 0; v < G.V(); v++) {
            if (!marked[v]) {
                dfs(G, v);
                count++;
            }
        }
    }

    private void dfs(Graph G, int v) {
        marked[v] = true;
        id[v] = count;
        size[count]++;
        for (int w : G.adj(v)) {
            if (!marked[w]) {
                dfs(G, w);
            }
        }
    }
```

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Readings:

- Textbook: Chapter 4.1 (Pages 522-556)
- Website:
  - https://algs4.cs.princeton.edu/41graph/

Practice Problems:

- 4.1.1-4.1.6, 4.1.9, 4.1.11