CS062

DATA STRUCTURES AND ADVANCED PROGRAMMING

31–32: Hash tables

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Lecture 31-32: Hash tables

- Hash functions
- Separate chaining
- Linear Probing

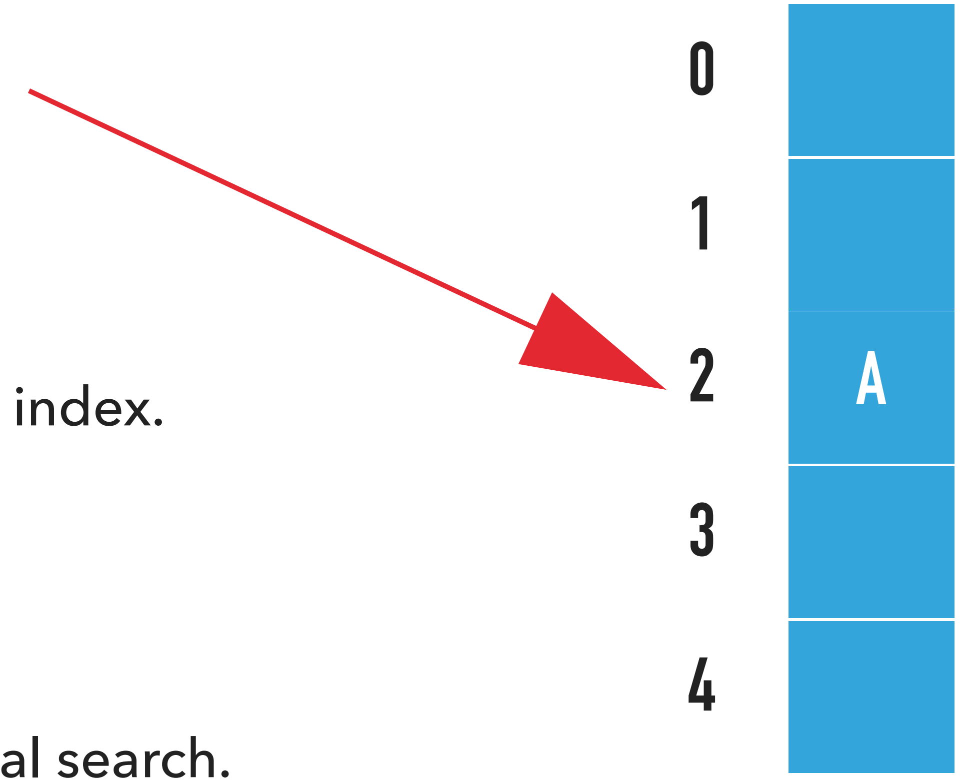
Some slides adopted from Algorithms 4th Edition or COS226

Summary for symbol table operations

	Worst case			Average case		
	Search	**Insert**	**Delete**	**Search**	**Insert**	**Delete**
Sequential search (unordered	n	n	n	$n/2$	n	$n/2$
Binary search (ordered array)	$\log n$	n	n	$\log n$	$n/2$	$n/2$
BST	n	n	n	$1.39 \log n$	$1.39 \log n$	?
2-3 search tree	$c \log n$	$c \log n$	$c \log n$	$c \log n$	$c \log n$	$c \log n$
Red-black BSTs	$2 \log n$	$2 \log n$	$2 \log n$	$1 \log n$	$1 \log n$	$1 \log n$

Basic plan for hashing

- Save items in a key-indexed table (index is a function of the key).
- Hash function: Method for computing array index from key.
    - `hash(“A”) = 2`
    - `hash(“B”) = 2 ???`
- Issues:
    - Computing the hash function.
    - Method for checking whether two keys are equal.
    - How to handle collisions when two keys hash to same index.
- Space-time tradeoff:
    - If no space limitation: hash function with key as index.
    - If no time limitation: collision resolution with sequential search.
    - If space and time limitation (real world): hashing



Computing hash function

- Ideal scenario: Take any key and uniformly "scramble" it to produce a symbol table index.
- Requirements:
  - Computing the hash function efficiently.
  - Every symbol table index is equally likely for each key.
- Although thoroughly researched, still problematic in practical applications.
- Examples: Hashing phone numbers or social security numbers.
  - Bad: if we choose the first three digits (area code/geographic region and time).
  - Better: if we choose the last three digits.
- Practical challenge: Need different approach for each key type.

Hashing in Java

- All Java classes inherit a method `hashCode()`, which returns an integer.
- Requirement: If `x.equals(y)` then it should be `x.hashCode()==y.hashCode()`.
- Ideally: If !`x.equals(y)` then it should be `x.hashCode()!=y.hashCode()`.
- Default implementation: Memory address of x.
  - Need to override it for custom types.
  - Already done for us for `Integer`, `Double`, etc.

Equality test in Java

- **Requirement:** For any objects x, y, and z.
  - **Reflexive:** `x.equals(x)` is `true`.
  - **Symmetric:** `x.equals(y)` iff `y.equals(x)`.
  - **Transitive:** if `x.equals(y)` and `y.equals(z)` then `x.equals(z)`.
  - **Non-null:** if `x.equals(null)` is `false`.
- If you don't override it the default implementation checks whether x and y refer to the same object in memory.

Java implementations of `equals()` for user-defined types

```
public final class Date {
      private final int month;
      private final int day;
      private final int year;
      …
      public boolean equals(Object y) {
          if (y == this) return true;
          if (y == null) return false;
          if (y.getClass() != this.getClass()) return false;
          Date that = (Date) y;
          return (this.day == that.day &&
                  this.month == that.month &&
                  this.year == that.year);
      }
}
```

General equality test recipe in Java

- Optimization for reference equality.
  - `if (y == this) return true;`
- Check against null.
  - `if (y == null) return false;`
- Check that two objects are of the same type.
  - `if (y.getClass() != this.getClass()) return false;`
- Cast them.
  - `Date that = (Date) y;`
- Compare each significant field.
  - `return (this.day == that.day && this.month == that.month && this.year == that.year);`
  - If a field is a primitive type, use `==`.
  - If a field is an object, use `equals()`.
  - If field is an array of primitives, use `Arrays.equals()`.
  - If field is an area of objects, use `Arrays.deepEquals()`.

Java implementations of hashCode()

-
  ```
  public final class Integer {
       private final int value;
       …
       public int hashCode() {
            return (value);
       }
  }
  ```

-
  ```
  public final class Boolean {
       private final boolean value;
       …
       public int hashCode() {
            if(value)    return 1231;
            else return 1237;
       }
  }
  ```

Implementing hash code for arrays

- 31x+y rule.
  - Initialize hash to 1.
  - Repeatedly multiply hash by 31 and add next integer in array.

  -
    ```
    public class Arrays {
        …
        public static int hashCode(int[] a) {
            int hash = 1;
            for (int i=0; i<a.length; i++) {
                hash = 31*hash + a[i];
            return hash;
        }
    }
    ```

Implementing hash code for strings

- Treat a string as an array of characters.
  - Initialize hash to 0.

  -
    ```
    public final class String {
         private final char[] s;
         private int hash = 0;
         …
         public int hashCode() {
              int h = hash;
              if (h != 0) return h;
              for (int i=0; i< length; i++) {
                  h = s[i] + (31 * h);
              hash = h;
              return h;
         }
    }
    ```
- Not foolproof, e.g., both Aa and BB hash to 2112. Actually, 2^n strings of length $2n$ hash to the same value!

Java implementations of hashCode() for user-defined types

-
  ```
public final class Date {
     private final int month;
     private final int day;
     private final int year;
     …
     public int hashCode() {
          int hash = 1;
          hash = 31*hash + ((Integer) month).hashCode();
          hash = 31*hash + ((Integer) day).hashCode();
          hash = 31*hash + ((Integer) year).hashCode();
          return hash;
          //could be also written as
          //return Objects.hash(month, day, year);
     }
}
```

General hash code recipe in Java

- Combine each significant field using the 31x+y rule.
- Shortcut 1: use `Objects.hash()` for all fields (except arrays).
- Shortcut 2: use `Arrays.hashCode()` for primitive arrays.
- Shortcut 3: use `Arrays.deepHashCode()` for object arrays.

Modular hashing

- Hash code: an `int` between -2^{31} and $2^{31}-1$
- Hash function: an `int` between 0 and $m-1$, where m is the hash table size (typically a prime number or power of 2).
-
```
private int hash (Key key){
    return key.hashCode() % m;
}
```
    - Bug! Might map to negative number.
-
```
private int hash (Key key){
    return Math.abs(key.hashCode()) % m;
}
```
    - Very unlikely bug. For a hash code of -2^{31}, `Math.abs` will return a negative number.
-
```
private int hash (Key key){
    return (key.hashCode() & 0x7fffffff) % m;
}
```
    - Correct.

Uniform hashing assumption

- Uniform hashing assumption: Each key is equally likely to hash to an integer between 0 and $m - 1$.
- Mathematical model: balls & bins. Toss n balls uniformly at random into m bins.
- Bad news: Expect two balls in the same bin after ~$\sqrt(\pi m/2)$ tosses.
  - Birthday problem: In a random group of 23 or more people, more likely than not that two people will share the same birthday.
- Good news: load balancing
  - When $n = m$, expect most loaded bin has ~$\ln m / \ln \ln n$ balls.
  - When $n > > m$, the number of balls in each bin is "likely close" to n/m.

Lecture 31-32: Hash tables

- Hash functions
- Separate chaining
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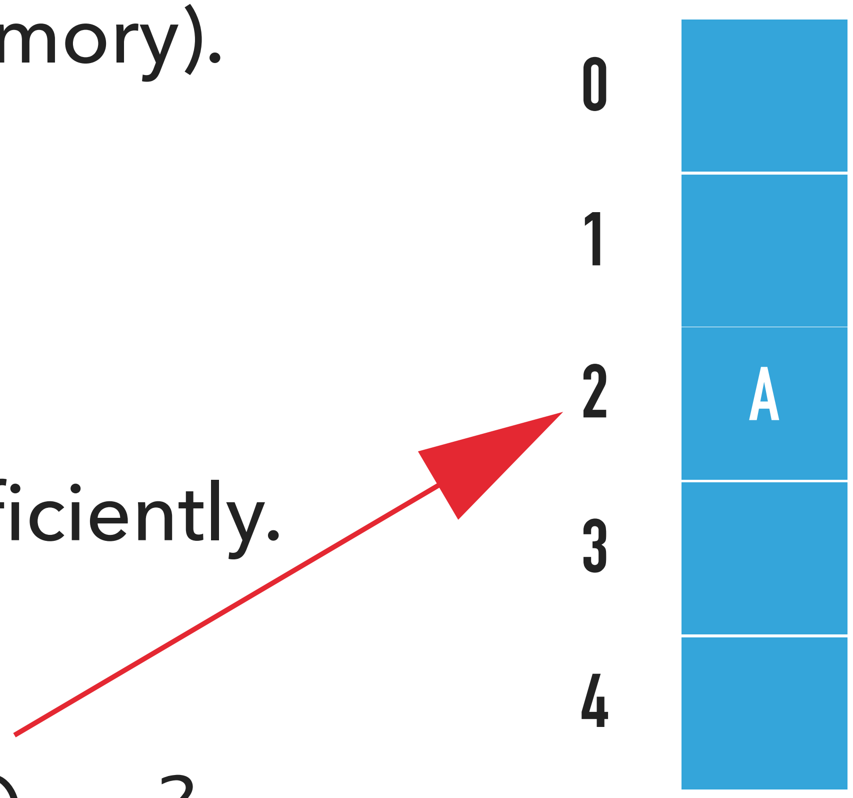
Collisions are unavoidable

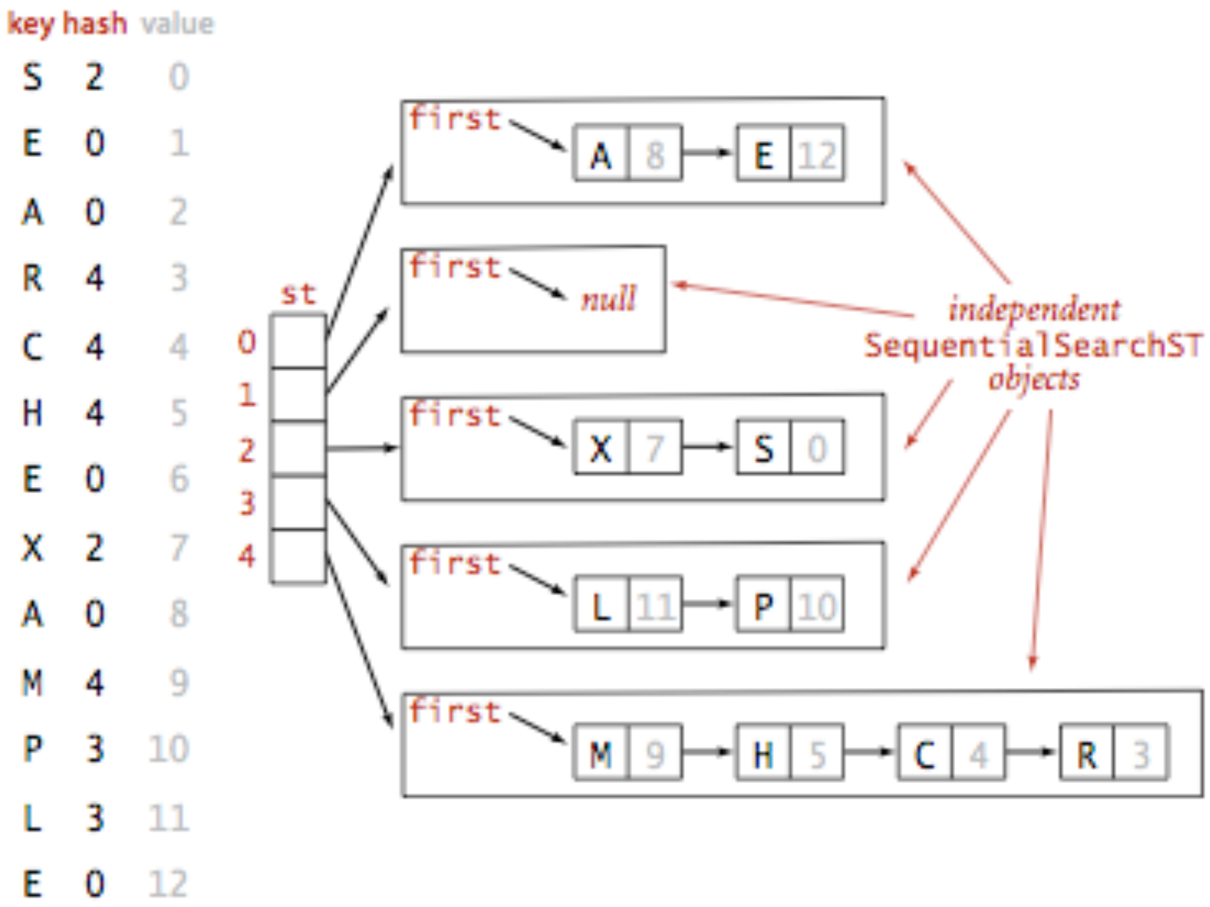
- Collision: Two distinct keys hash to the same index.
  - Birthday problem: Can't avoid collisions (unless you have at least quadratic memory).
  - Coupon collector + load balancing: collisions will be evenly distributed.
- Challenge: how to deal with collisions efficiently.



- `hash("A") = 2`
- `hash("B") = 2 ???`

Separate Chaining

- Use an array of $m < n$ distinct lists [H.P. Luhn, IBM 1953].
  - **Hash**: Map key to integer i between 0 and $m - 1$.
  - **Insert**: Put at front of i-th chain (if not already there).
  - **Search**: Need to only search the i-th chain.



key	hash	value
S	2	0
E	0	1
A	0	2
R	4	3
C	4	4
H	4	5
E	0	6
X	2	7
A	0	8
M	4	9
P	3	10
L	3	11
E	0	12

Hashing with separate chaining for standard indexing client

Symbol table with separate chaining implementation

```
public class SeparateChainingLiteHashST<Key, Value> {

    private int m = 128;  // hash table size
    private Node[] st = new Node[m];
    // array of linked-list symbol tables. Node is inner class that holds keys and values of type Object

    public Value get(Key key) {
        int i = hash(key);
        for (Node x = st[i]; x != null; x = x.next;)
            if (key.equals(x.key)) return (Value) x.val;
        return null;
    }

    public void put(Key key, Value val) {
        int i = hash(key);
        for (Node x = st[i]; x != null; x = x.next;)
            if (key.equals(x.key)) {
                x.val = val;
                return;
        }
        st[i] = new Node(key, val, st[i];
    }
```

Analysis

- Under uniform hashing assumption, length of each chain is $\sim n/m$.
- Consequence: Number of probes (calls to either `equals()` or `hashCode()`) for search/insert is proportional to n/m (m times faster than sequential search).
  - m too large -> too many empty chains.
  - m too small -> chains too long.
  - Typical choice: $m \sim 1/4n$ -> constant time per operation.

Resizing in a separate-chaining hash table

- Goal: Average length of chain n/m = constant lookup.
  - Double hash table size when $n/m \geq 8$.
  - Halve hash table size when $n/m \leq 2$.
  - Need to rehash all keys when resizing (hash code does not change, but hash changes).

Deletion in a separate-chaining hash table

- Find key in chain and remove it along with its associated value.

Summary for symbol table operations

	Worst case			Average case		
	Search	**Insert**	**Delete**	**Search**	**Insert**	**Delete**
Sequential search (unordered list)	n	n	n	$n/2$	n	$n/2$
Binary search (ordered array)	$\log n$	n	n	$\log n$	$n/2$	$n/2$
BST	n	n	n	$1.39 \log n$	$1.39 \log n$	?
2-3 search tree	$c \log n$	$c \log n$	$c \log n$	$c \log n$	$c \log n$	$c \log n$
Red-black BSTs	$2 \log n$	$2 \log n$	$2 \log n$	$1 \log n$	$1 \log n$	$1 \log n$
Separate chaining	n	n	n	$3 - 5$	$3 - 5$	$3 - 5$

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Open addressing

- Alternate approach to handle collisions.
- Maintain keys and values in two parallel arrays.
- When a new key collides, find next empty slot and put it there.
- If the array is full, the search would not terminate.

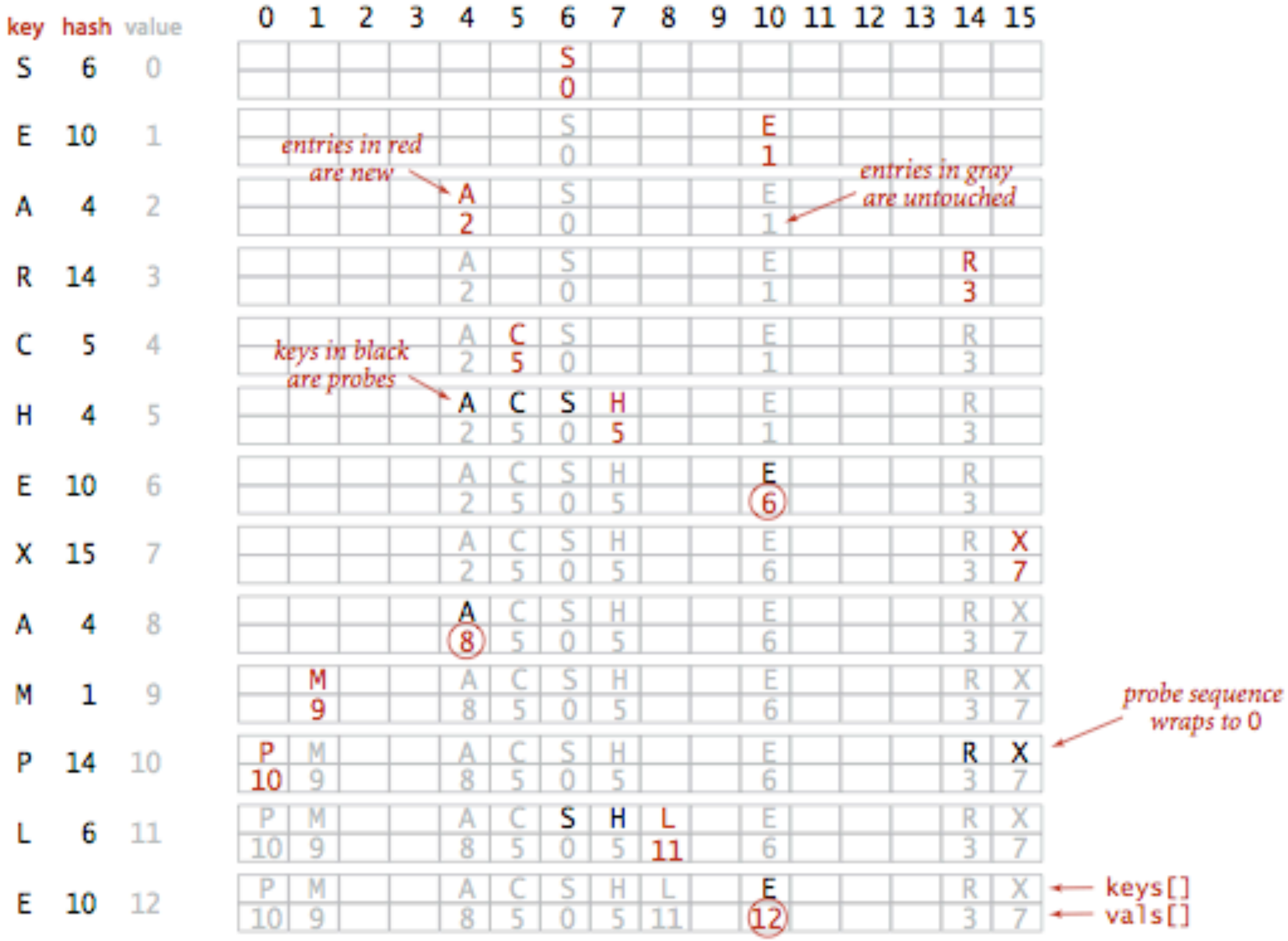
Linear probing

- Hash: Map key to integer i between 0 and $m - 1$.
- Insert: Put at index i if free. If not, try $i + 1$, $i + 2$, etc.
- Search: Search table index i. If occupied but no match, try $i + 1$, $i + 2$, etc
  - If you find a gap then you know that it does not exist.
- Table size m **must** be greater than the number of key-value pairs n.

http://algs4.cs.princeton.edu

3.4 LINEAR PROBING DEMO

Linear probing



key	hash	value	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
S	6	0							S 0									
E	10	1							S 0				E 1					
A	4	2					A 2		S 0				E 1					
R	14	3					A 2		S 0				E 1				R 3	
C	5	4					A 2	C 5	S 0				E 1				R 3	
H	4	5					A 2	C 5	S 0	H 5			E 1				R 3	
E	10	6					A 2	C 5	S 0	H 5			E 6				R 3	
X	15	7					A 2	C 5	S 0	H 5			E 6				R 3	X 7
A	4	8					A 8	C 5	S 0	H 5			E 6				R 3	X 7
M	1	9		M 9			A 8	C 5	S 0	H 5			E 6				R 3	X 7
P	14	10	P 10	M 9			A 8	C 5	S 0	H 5			E 6				R 3	X 7
L	6	11	P 10	M 9			A 8	C 5	S 0	H 5	L 11		E 6				R 3	X 7
E	10	12	P 10	M 9			A 8	C 5	S 0	H 5	L 11		E 12				R 3	X 7

Trace of linear-probing ST implementation for standard indexing client

Symbol table with linear probing implementation

```
public class LinearProbingHashST<Key, Value> {

    private int m = 32768;  // hash table size
    private Value[] Vals = (Value[]) new Object[m];
    private Key[] Vals = (Key[]) new Object[m];

    public Value get(Key key) {
        for (int i = hash(key); keys[i] != null; i = (i+1) % m;)
            if (key.equals(keys[i])) return vals[i];
        return null;
    }

    public void put(Key key, Value val) {
        int i;
        for (int i = hash(key); keys[i] != null; i = (i+1) % m;)
            if (key.equals(keys[i])){
                break;
        }
        keys[i] = key;
        vals[i] = val;
    }
```

Clustering

- Cluster: a contiguous block of keys.
- Observation: new keys likely to hash in middle of big clusters.

Analysis

- **Proposition**: Under uniform hashing assumption, the average number of probes in a linear-probing hash table of size m that contains $n = \alpha m$ keys is at most
  - $1/2(1 + \frac{1}{1-a})$ for search hits and
  - $1/2(1 + \frac{1}{(1-a)^2})$ for search misses and insertions.
  - [Knuth 1963]
- **Parameters**:
  - m too large -> too many empty array entries.
  - m too small -> search time becomes too long.
  - Typical choice: $\alpha = n/m \sim 1/2$ -> constant time per operation.

Resizing in a linear probing hash table

- Goal: Fullness of array (load factor) $n/m \leq 1/2$.
  - Double hash table size when $n/m \geq 1/2$.
  - Halve hash table size when $n/m \leq 1/8$.
  - Need to rehash all keys when resizing (hash code does not change, but hash changes).
- Deletion not straightforward.

Summary for symbol table operations

	Worst case			Average case		
	Search	**Insert**	**Delete**	**Search**	**Insert**	**Delete**
Sequential search (unordered list)	n	n	n	$n/2$	n	$n/2$
Binary search (ordered array)	$\log n$	n	n	$\log n$	$n/2$	$n/2$
BST	n	n	n	$1.39 \log n$	$1.39 \log n$	?
2-3 search tree	$c \log n$	$c \log n$	$c \log n$	$c \log n$	$c \log n$	$c \log n$
Red-black BSTs	$2 \log n$	$2 \log n$	$2 \log n$	$1 \log n$	$1 \log n$	$1 \log n$
Separate chaining	n	n	n	$3-5$	$3-5$	$3-5$
Linear probing	n	n	n	$3-5$	$3-5$	$3-5$

Separate chaining vs linear probing

- Separate chaining:
  - Performance degrades gracefully as number of keys increases.
  - Clustering less sensitive to poorly-designed hash function.
    - Potentially fewer probes.
- Linear probing:
  - Less wasted space.
  - Better cache performance (locality).

Hashing: variations on the theme

- Two-probe hashing (separate chaining variant):
    - Hash to two positions, insert key in shorter of the two chains.
    - Reduces expected length of longest chain to $\log \log n$.
- Double hashing (linear probing variant):
    - Use linear probing, but skip a variable amount, not just 1 each time you have collision.
    - Effectively eliminates clustering.
    - Can allow table to become nearly full.
    - More difficult to implement delete.
- Cuckoo hashing (linear probing variant):
    - Hash to two positions, insert key into either position. If occupied, reinsert displayed key into its alternative position and recur.
    - Constant worst case time for search.

Hash tables vs balanced search trees

- Hash tables:
    - Simpler to code.
    - No effective alternative of unordered keys.
    - Faster for simple keys (a few arithmetic operations versus $\log n$ compares).
- Balanced search trees:
    - Stronger performance guarantee.
    - Support for ordered symbol table operations.
    - Easier to implement `compareTo()` than `hashCode()`.
- Java includes both:
    - Balanced search trees: `java.util.TreeMap`, `java.util.TreeSet`.
    - Hash tables: `java.util.HashMap`, `java.util.IdentityHashMap`.

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Readings:

- Textbook: Chapter 3.4 (Pages 458-477)
- Website:
    - https://algs4.cs.princeton.edu/34hash/

Practice Problems:

- 3.4.1-3.4.13