CS062

DATA STRUCTURES AND ADVANCED PROGRAMMING

25–26: Binary Search Trees

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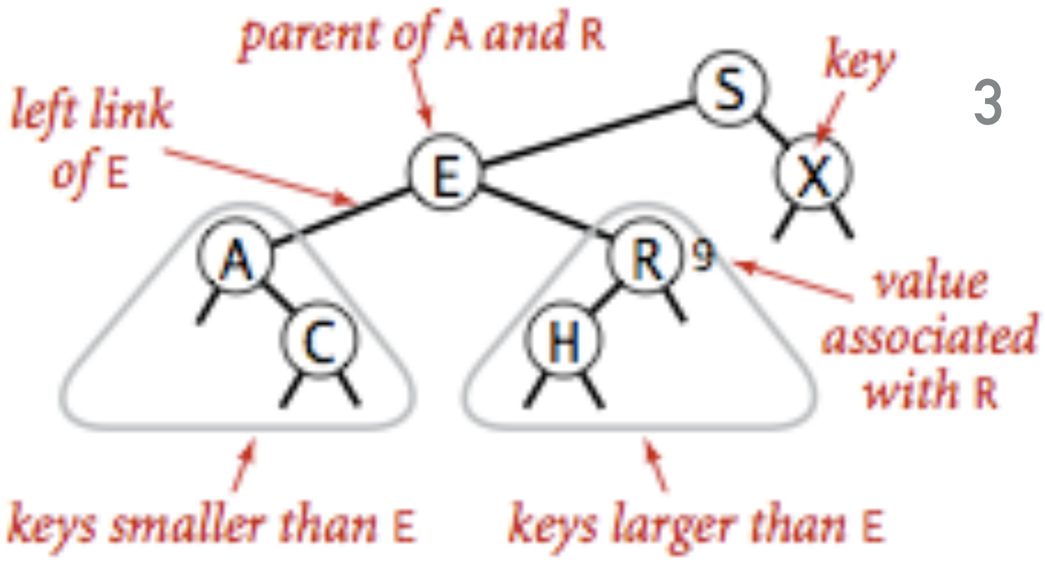
Lecture 25-26: Binary Search Trees

- Binary Search Trees
- Ordered Operations
- Deletion in BSTs

Some slides adopted from Algorithms 4th Edition or COS226

Definitions



- Binary Search Tree: A binary tree in symmetric order.
- Symmetric order: Each node has a key, and every node's key is:
  - Larger than all keys in its left subtree.
  - Smaller than all keys in its right subtree.
- Our textbook uses BSTs to implement symbol tables, therefore each node holds a key-value pair. Other implementations (like today's lab) hold only a key.

Differences between heaps and BSTs

	Heap	**BST**
Supported operations	Insert, delete max	insert, search, delete, ordered operations
What is inserted	Keys	Key-value pairs
Underlying data structure	(Resizing) array	Linked nodes
Tree shape	Complete binary tree	Depends on data
Ordering of keys	Heap-ordered	Symmetrically-ordered
Duplicate keys allowed?	Yes	No*

*: depends on implementation.

BST representation

- We will use an inner class Node that is composed by:
  - A Key that is comparable and a `Value`
  - A reference to the root nodes of the left (smaller keys) and right (larger keys) subtrees.
  - Potentially, the total number of nodes in the subtree that has root this node.
- A BST has a reference to a Node `root`.

Node representation

```
private class Node {
    private Key key;            // sorted by key
    private Value val;          // associated data
    private Node left, right;   // left and right subtrees
    private int size;           // number of nodes in subtree

    public Node(Key key, Value val, int size) {
        this.key = key;
        this.val = val;
        this.size = size;
    }
}
```

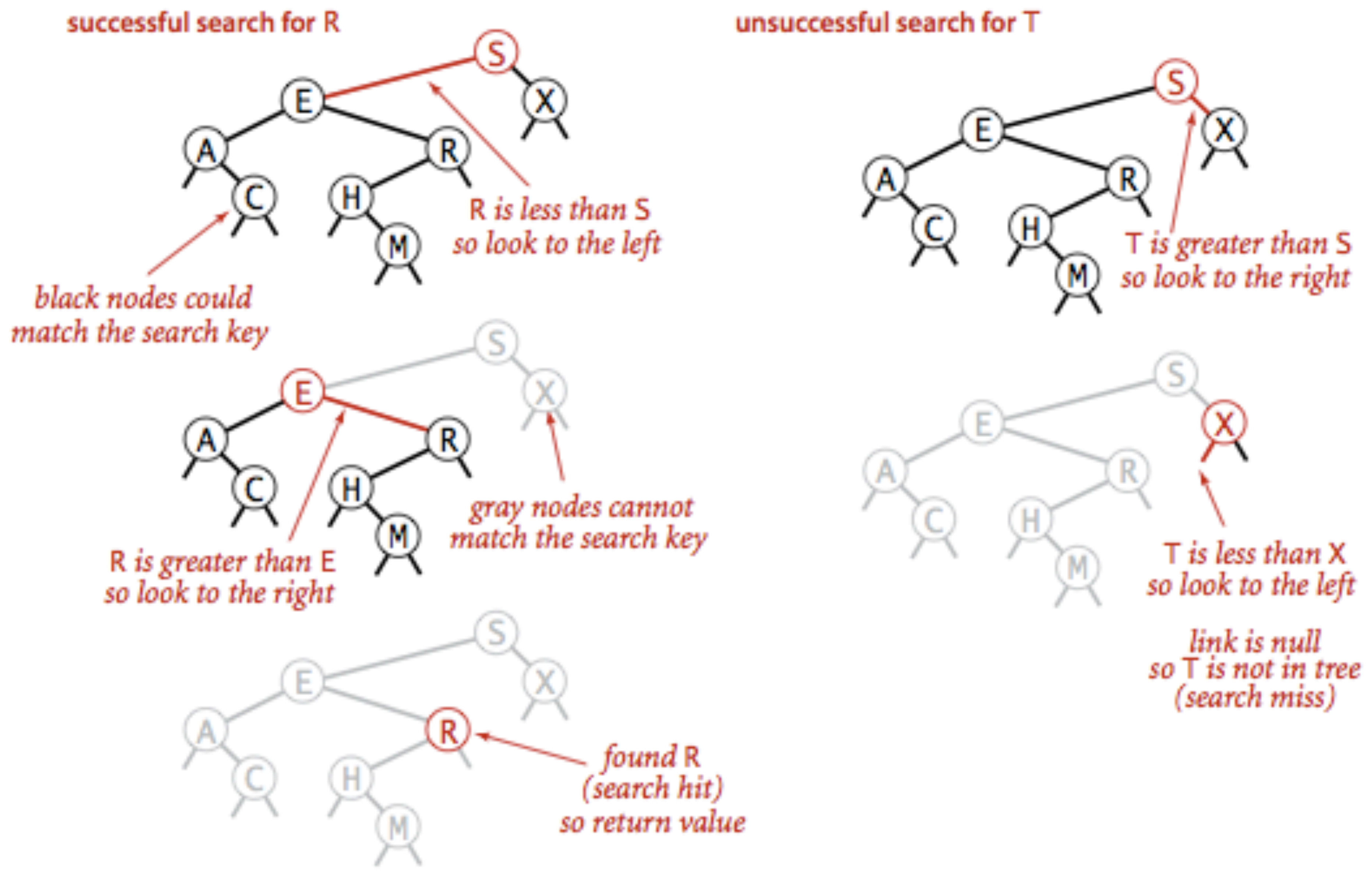
http://algs4.cs.princeton.edu

3.2 BINARY SEARCH TREE DEMO

Search

- If less go left.
- If greater go right.
- If equal, search hit.
- Return value corresponding to given key, or `null` if no such key.
  - In other implementations, you return the last node you reached.
- Number of compares is equal to the depth of the node + 1.

Search example



Successful (left) and unsuccessful (right) search in a BST

Search - iterative implementation

```
public Value get(Key key) {
      Node x = root;
      while (x != null) {
            int cmp = key.compareTo(x.key);
            if (cmp < 0)
                    x = x.left;
            else if (cmp > 0)
                    x = x.right;
            else if (cmp == 0)
                    return x.val;
       }
       return null;
}
```

Search - recursive implementation

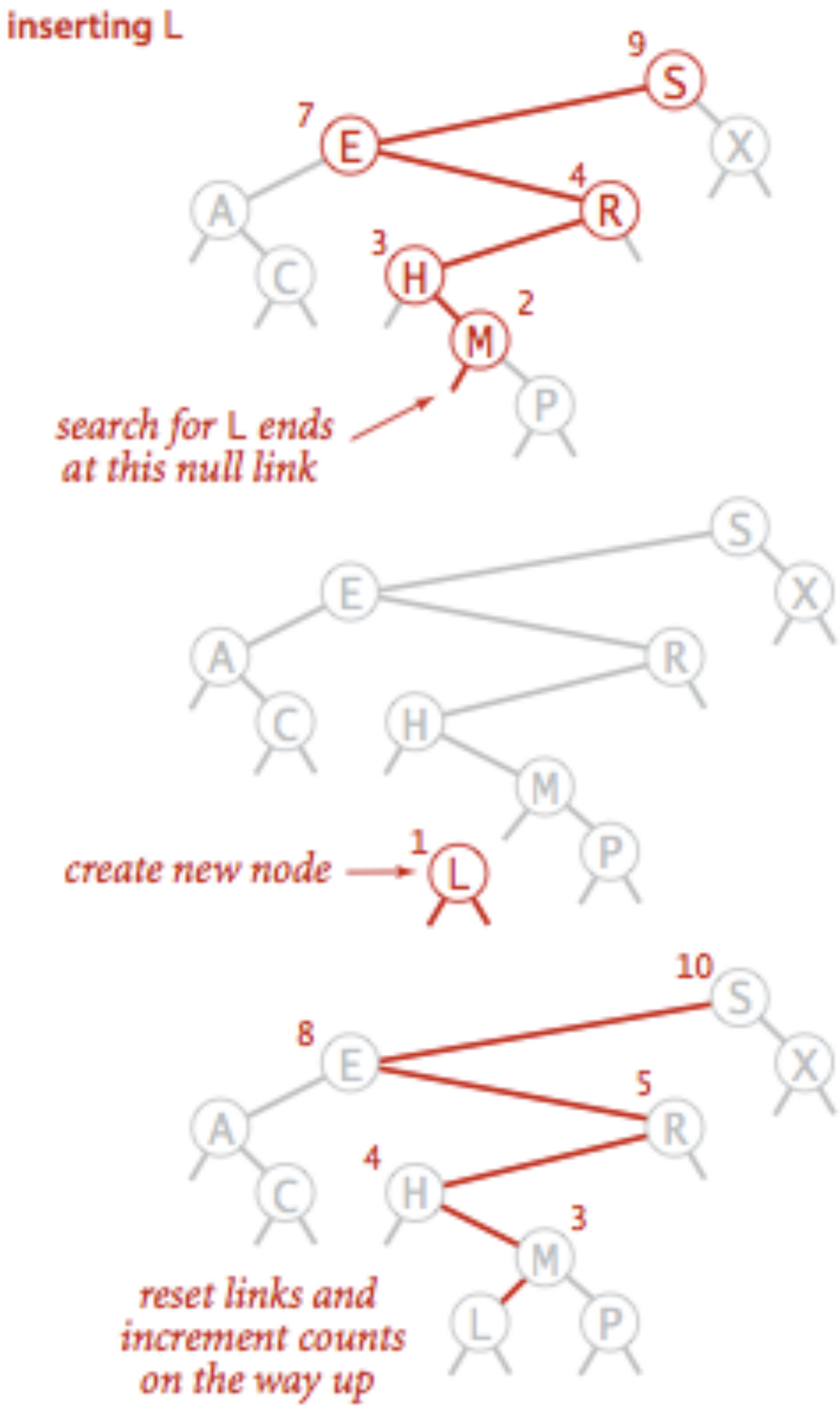
-
  ```
  public Value get(Key key) {
       return get(root, key);
  }
  ```

-
  ```
  private Value get(Node x, Key key) {
       if (x == null)
             return null;
       int cmp = key.compareTo(x.key);
       if (cmp < 0)
           return get(x.left, key);
       else if (cmp > 0)
           return get(x.right, key);
       else
           return x.val;
  }
  ```

Insert

- If less go left.
- If greater go right.
- If null, insert.
- If already exists, update value.
- Number of compares is equal to the depth of the node + 1.

Insert example



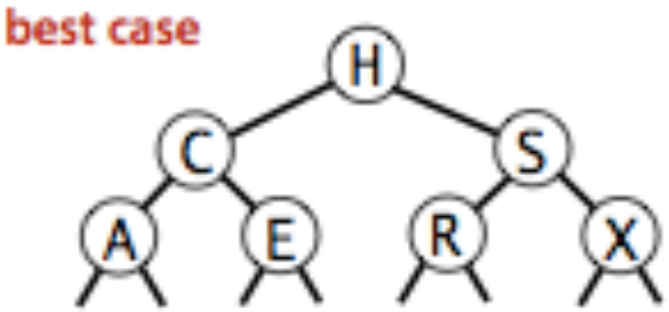
Insertion into a BST

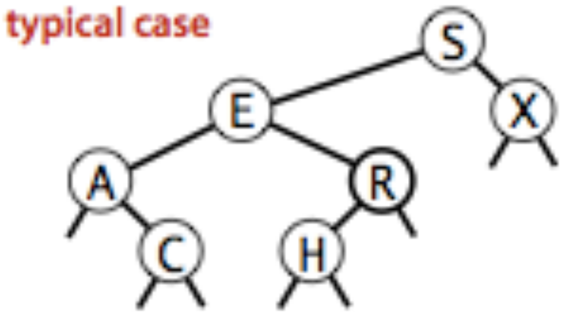
Insert

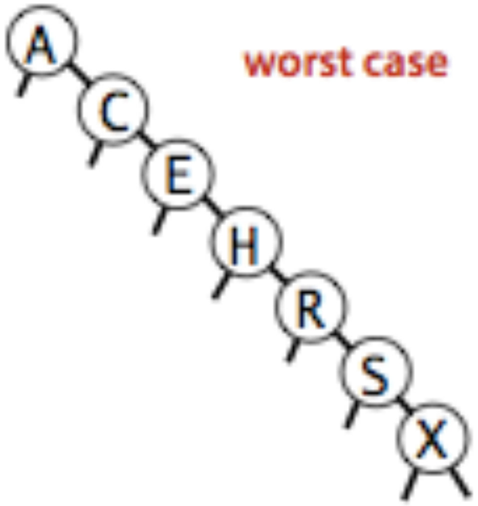
```
public void put(Key key, Value val) {
     root = put(root, key, val);
}
private Node put(Node x, Key key, Value val) {
     if (x == null)
           return new Node(key, val, 1);
     int cmp = key.compareTo(x.key);
     if (cmp < 0)
         x.left = put(x.left, key, val);
     else if (cmp > 0)
         x.right = put(x.right, key, val);
     else
         x.val = val;
     x.size = 1 + size(x.left) + size(x.right);
     return x;
}
```

Tree shape

- The same set of keys can result to different BSTs based on their order of insertion.
- Number of compares for search/insert is equal to depth of node +1.







BSTs mathematical analysis

- If n distinct keys are inserted into a BST in random order, the expected number of compares of search/insert is $2 \ln n$ (or $1.39 \log n$).
- If n distinct keys are inserted into a BST in random order, the expected height of tree is $4.311 \ln n$
  [Reed, 2003].
- Worst case height is n but highly unlikely.
  - Keys would have to come (reversely) sorted!

Correspondence between BSTs and quicksort partitioning

- If array has no duplicate keys 1-1 correspondence.
- In quicksort, pivot separates array in elements that are smaller in its left subarray and larger in its right subarray.
- In BST, root separates tree in elements that are smaller in its left subtree and larger in its right subtree.
- This is why the mathematical analysis for BSTs was the same with quicksort's partitioning (the expected number of compares of search/insert is $2 \ln n$ as is the number of compares in quicksort).

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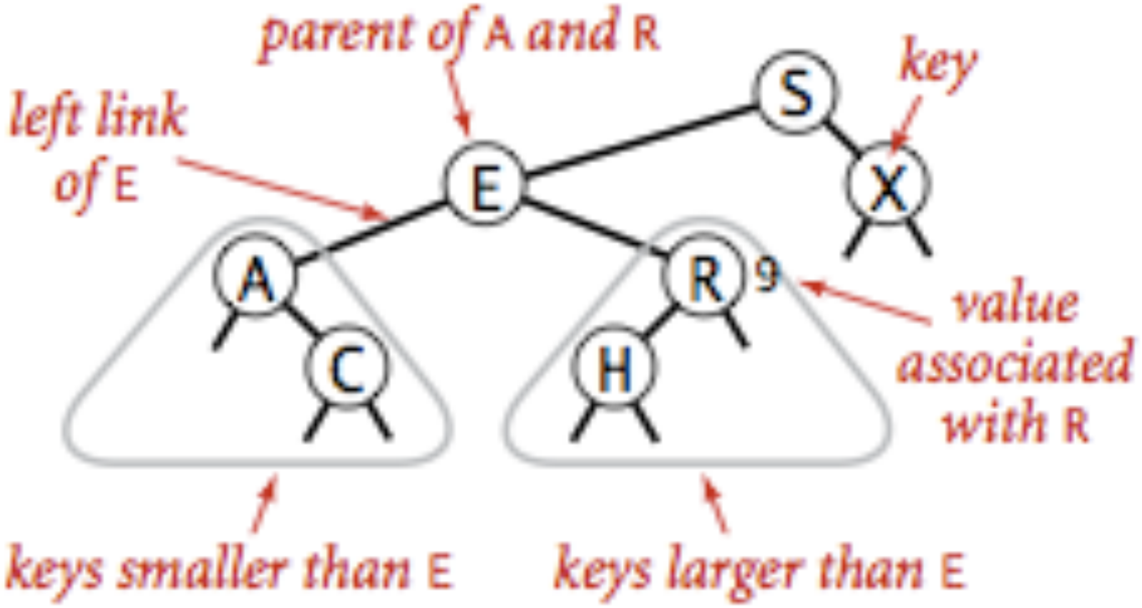
Minimum and maximum

- Minimum: go all the way left until you find a node with no left child.
- Maximum: go all the way to the right until you find a node with no right child.

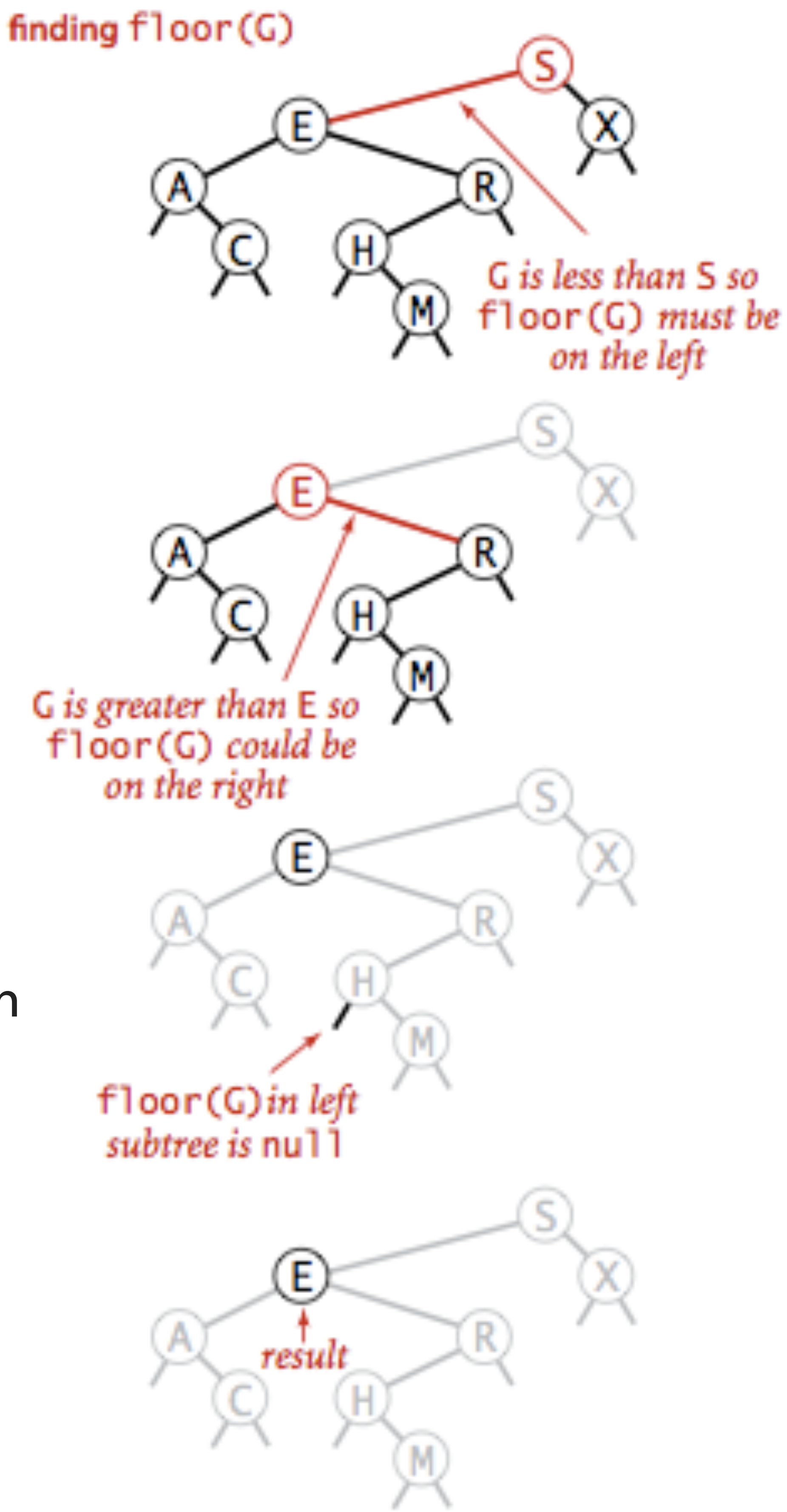
```
public Key min() {
     return min(root).key;
  }

  private Node min(Node x) {
      if (x.left == null)
          return x;
      else
          return min(x.left);
  }
```



Floor

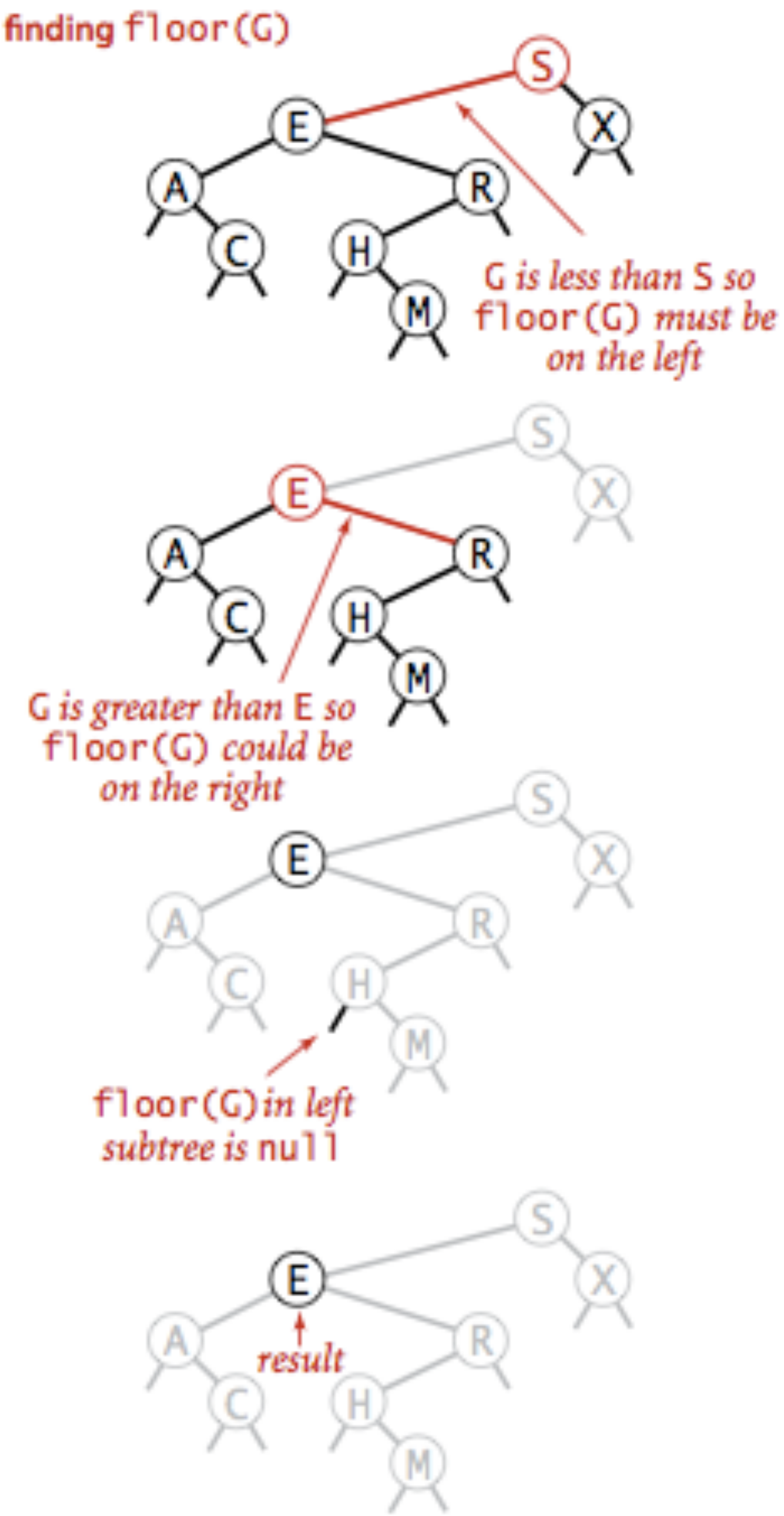
- Floor: Largest key in BST <= query key k.
- Case 1: [k equals the key in node]
  - Floor of k is k.
- Case 2: [k is less than key in node]
  - Floor of k is in left subtree.
- Case 3: [k is greater than key in node]
  - Floor of k is in right subtree if there is any key <=k in right subtree.
  - Else, floor is the key in node.
- Same idea for ceiling (smallest key in BST>=query key)



Floor

```
public Key floor(Key key) {
      Node x = floor(root, key);
      if (x == null)
          return null;
      else
          return x.key;
}
```

```
private Node floor(Node x, Key key) {
       if (x == null)
           return null;
       int cmp = key.compareTo(x.key);
       if (cmp == 0)
           return x;
       if (cmp <  0)
           return floor(x.left, key);
       Node t = floor(x.right, key);
       if (t != null)
            return t;
       else
            return x;
    }
```



finding floor(G)

G is less than S so floor(G) must be on the left

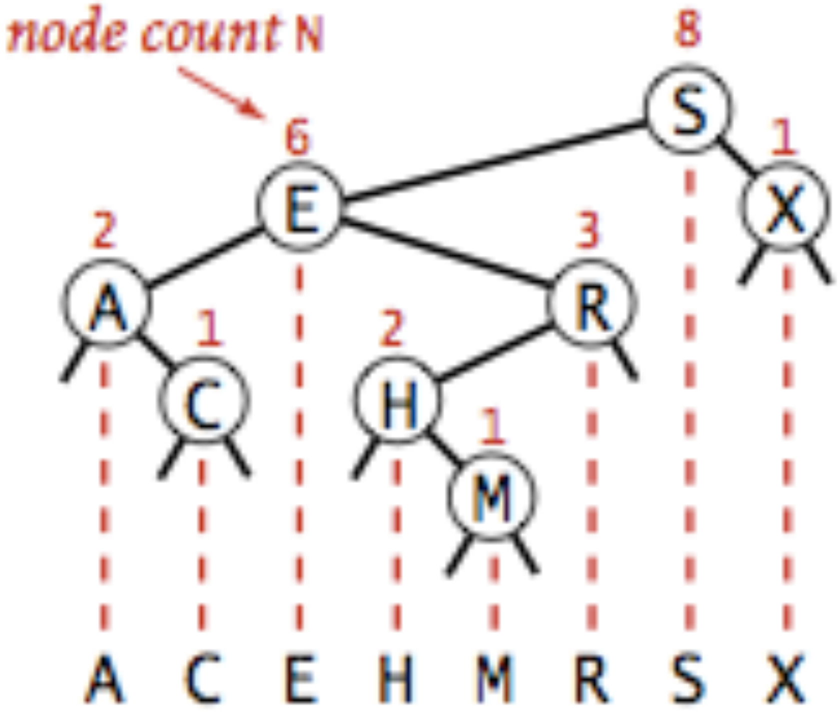
G is greater than E so floor(G) could be on the right

floor(G) in left subtree is null

result

Rank

- Rank: How many keys < query key k.
- k<key: Recur on left subtree.
- k == key: Everything in left subtree.
- k > key: Everything in left subtree + 1 + recur on right.



Rank

- Rank: How many keys < query key k.

```
public int rank(Key key) {
   return rank(key, root);
}

// Number of keys in the subtree less than key.
private int rank(Key key, Node x) {
    if (x == null)
        return 0;
    int cmp = key.compareTo(x.key);
    if (cmp < 0)
        return rank(key, x.left);
    else if (cmp > 0)
        return 1 + size(x.left) + rank(key, x.right);
    else
        return size(x.left);
}
```

Order of growth for ordered symbol table operations

	Sequential search	**Binary search**	**BST**
search	n	$\log n$	h
insert	n	n	h
min/max	n	1	h
floor/ceiling	n	$\log n$	h
rank	n	$\log n$	h
select	n	1	h

- Worst case search and insert are $O(n)$ for BSTs. Not great!

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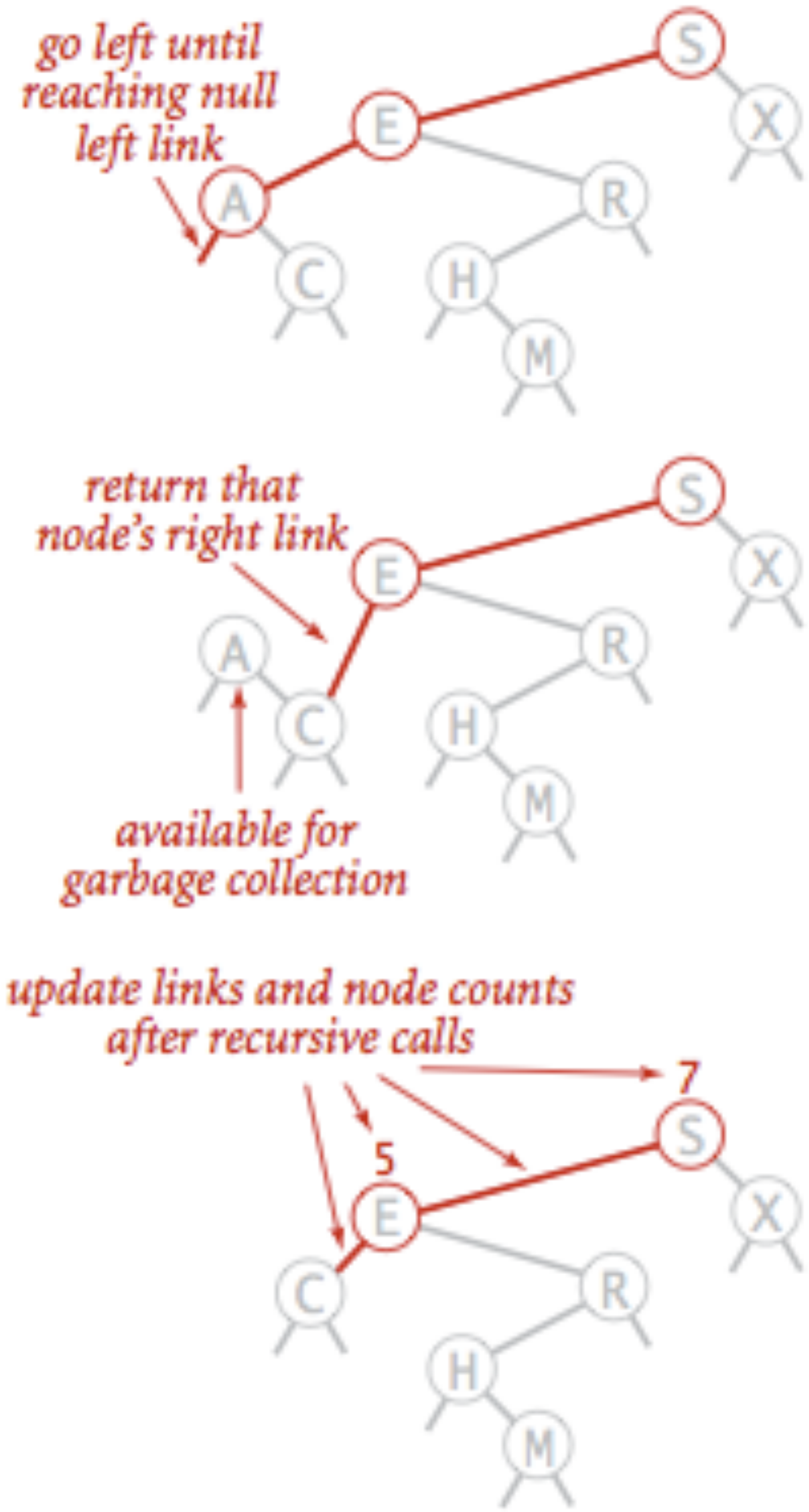
Delete minimum key

- Go left until finding a node with null left subtree.
- Replace the link to that node with its right subtree.
- Update subtree counts.

```
 public void deleteMin() {
     root = deleteMin(root);
}

 private Node deleteMin(Node x) {
     if (x.left == null)
        return x.right;
     x.left = deleteMin(x.left);
     x.size = size(x.left) + size(x.right) + 1;
     return x;
 }
```

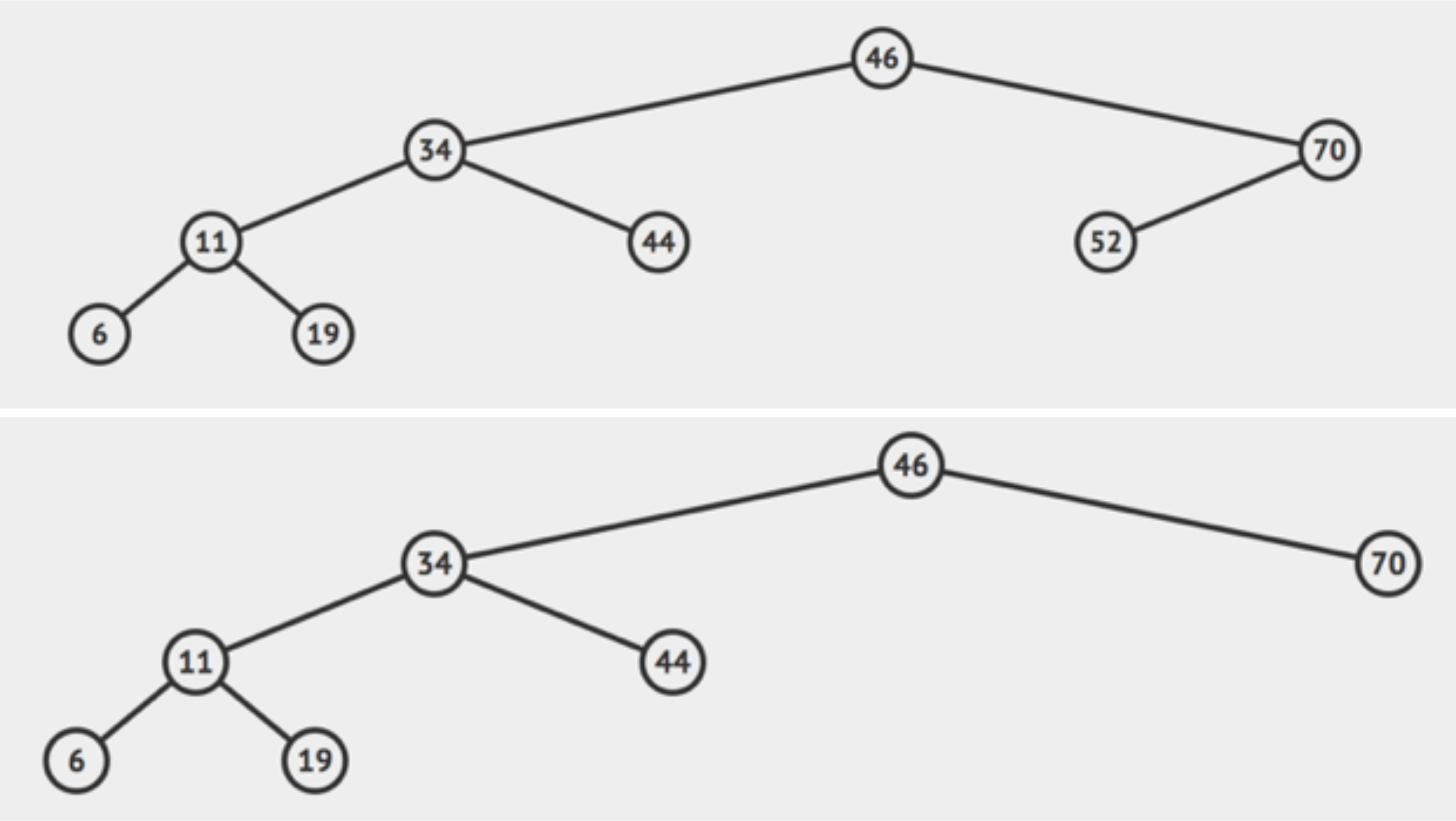
- Symmetric for delete maximum



Deleting the minimum in a BST

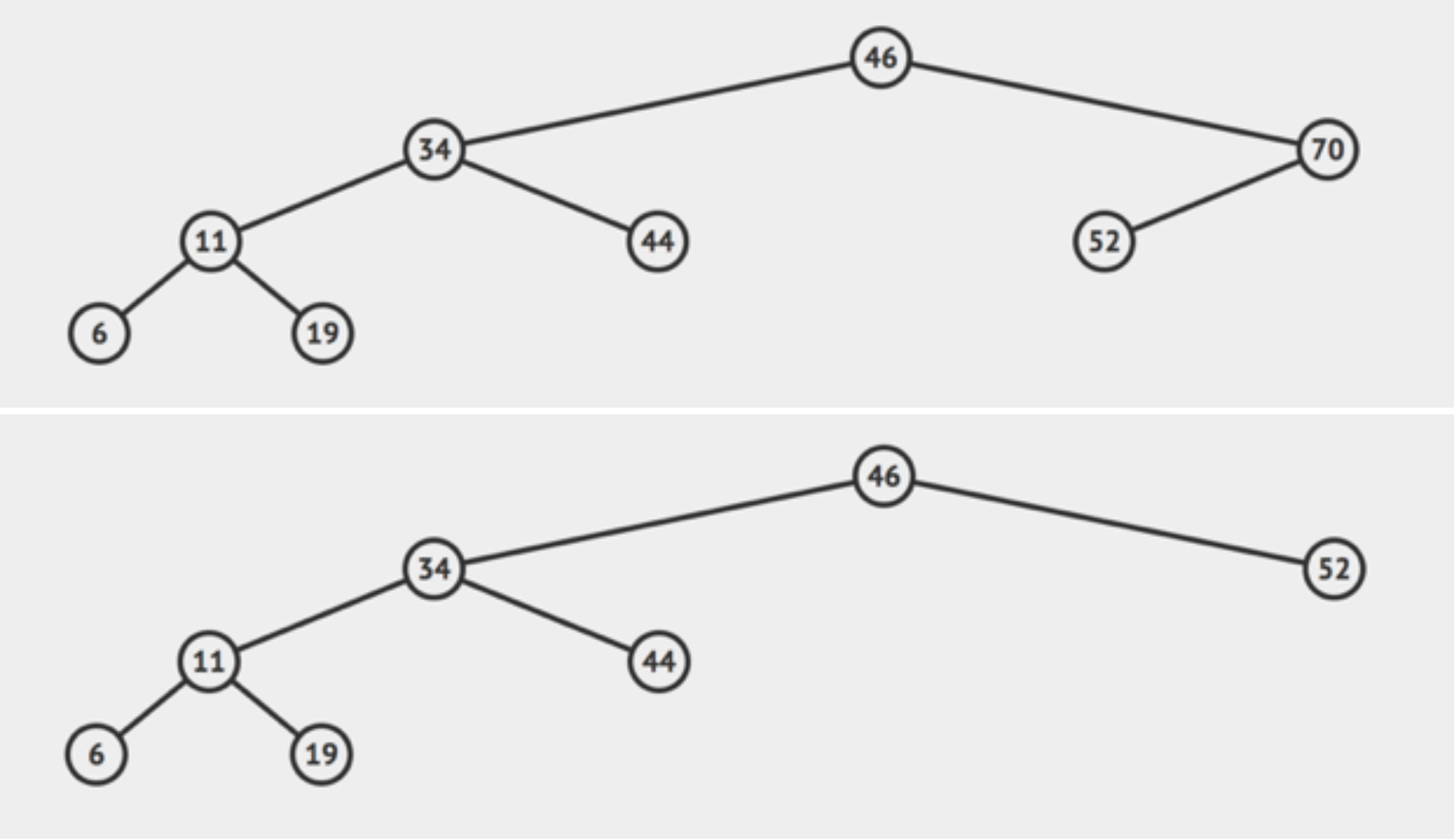
Hibbard deletion: Delete node which is a leaf

- Delete node by setting parent link to null.
- Example: delete 52 locates a node which is a leaf.



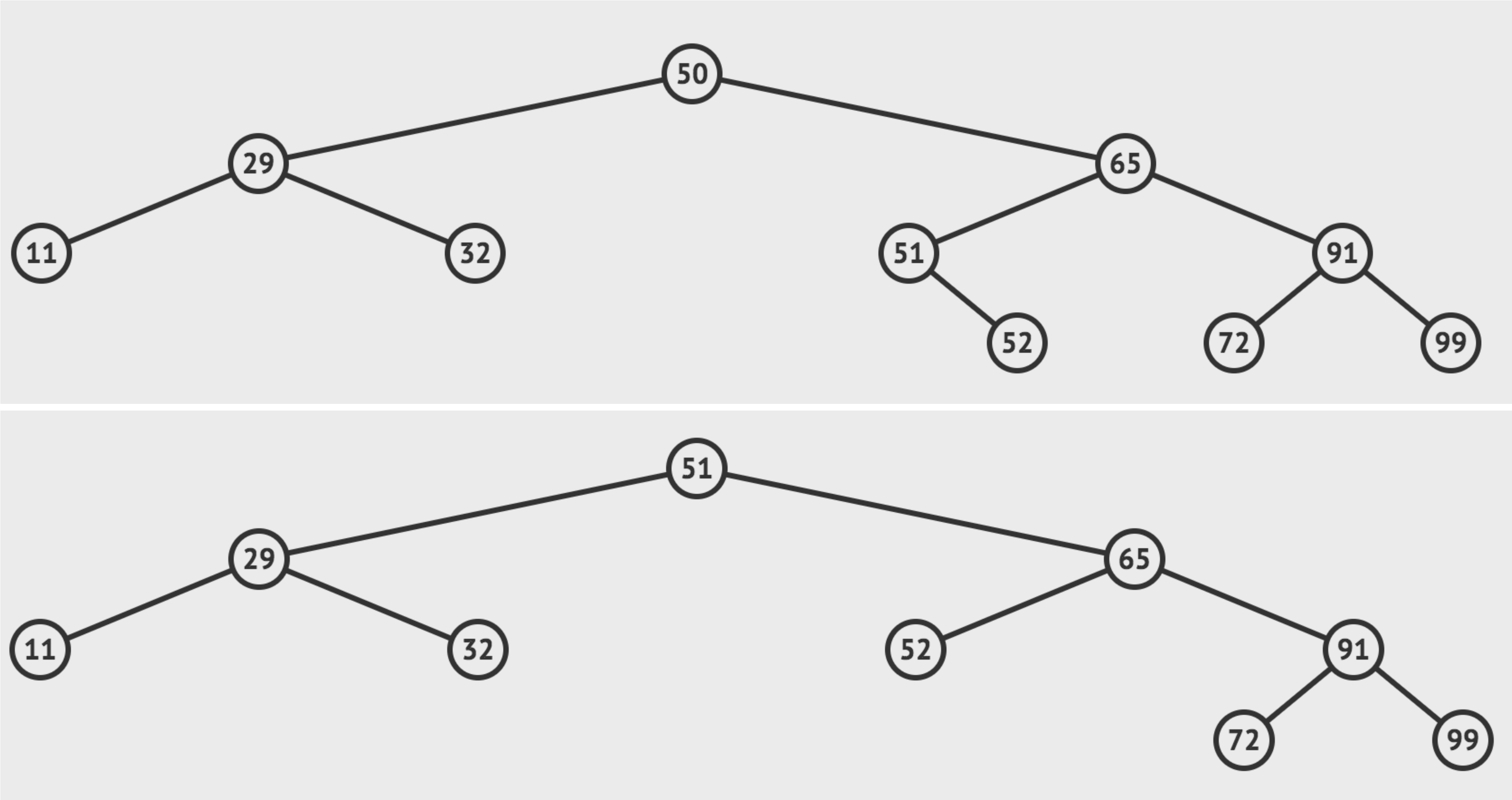
Hibbard deletion: Delete node with one child

- Delete node by replacing parent link.
- Example: delete 70 locates a node which has one child.



Hibbard deletion: Delete node with two children

- Delete node and replace it with successor (node with smallest of the larger keys)
- Example: delete 50 locates a node which has two children. Successor is 51.



https://visualgo.net/en/bst

DELETION IN BST

```
public void delete(Key key) {
    root = delete(root, key);
}

 private Node delete(Node x, Key key) {
     if (x == null) return null;

     int cmp = key.compareTo(x.key);
     if (cmp < 0)
         x.left  = delete(x.left,  key);
     else if (cmp > 0)
         x.right = delete(x.right, key);
     else {
         if (x.right == null)
             return x.left;
         if (x.left  == null)
             return x.right;
         Node t = x; //replace with successor
         x = min(t.right);
         x.right = deleteMin(t.right);
         x.left = t.left;
     }
     x.size = size(x.left) + size(x.right) + 1;
     return x;
 }
```

Hibbard deletion

- Unsatisfactory solution. If we were to perform many insertions and deletions the BST ends up being not symmetric and skewed to the left.
  - The cost is $\sqrt{n}$ (extremely complicated analysis).
  - No one has proven that alternating between predecessor and successor will fix this.
- Hibbard devised the algorithm in 1962. Still no algorithm for efficient deletion in BST.
- Overall, BSTs can have $O(n)$ worst-case for search, insert, and delete. We want to do better (see future lectures).

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Readings:

- Textbook: Chapter 3.2 (Pages 396-414)
- Website:
    - https://algs4.cs.princeton.edu/32bst/

Practice Problems:

- 3.2.1-3.2.13