# CS062

## DATA STRUCTURES AND ADVANCED PROGRAMMING

## 14: Analysis of Algorithms II


**Alexandra Papoutsaki**
LECTURES

**Mark Kampe**
LABS

# Lecture 14: Analysis of Algorithms II

- Theory of Algorithms
- Running Time of Linked List operations
- Running Time of Linked Stack operations
- Running Time of Linked Queue operations
- Running Time of ArrayList operations
- Memory Consumption of Stacks

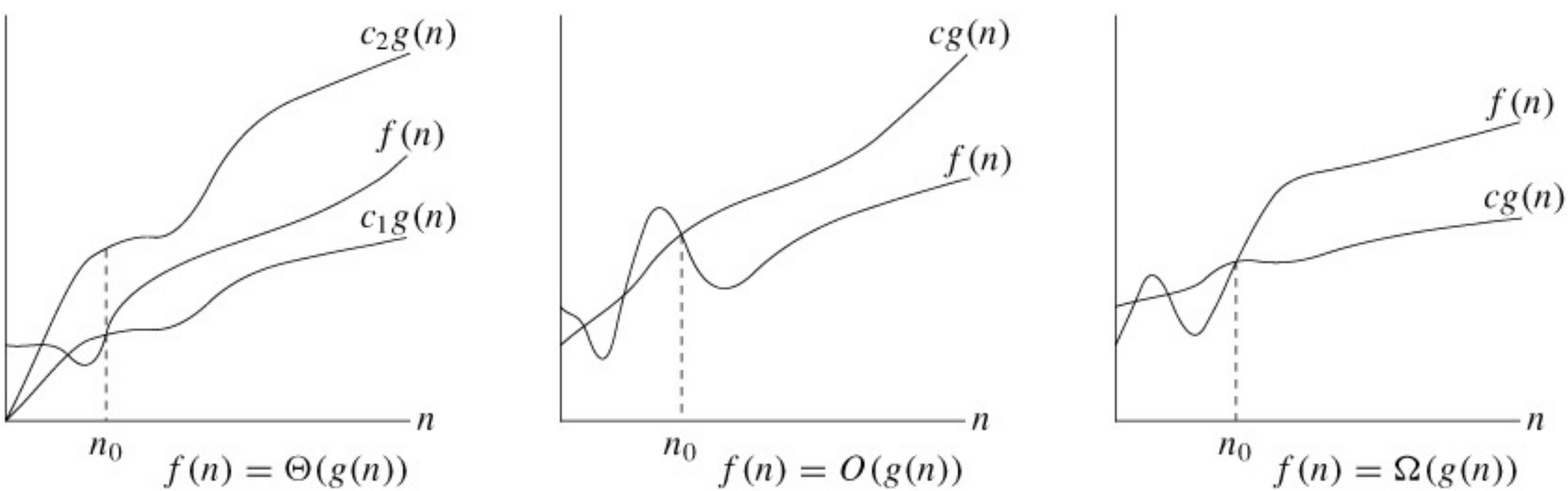
Some slides adopted from Algorithms 4th Edition or COS226

# Type of analyses

- Best case: lower bound on cost.
  - What the goal of all inputs should be.
  - Often not realistic, only applies to “easiest” input.
- Worst case: upper bound on cost.
  - Guarantee on all inputs.
  - Calculated based on the “hardest” input.
- Average case: expected cost for random input.
  - A way to predict performance.
  - Not straightforward how we model random input.

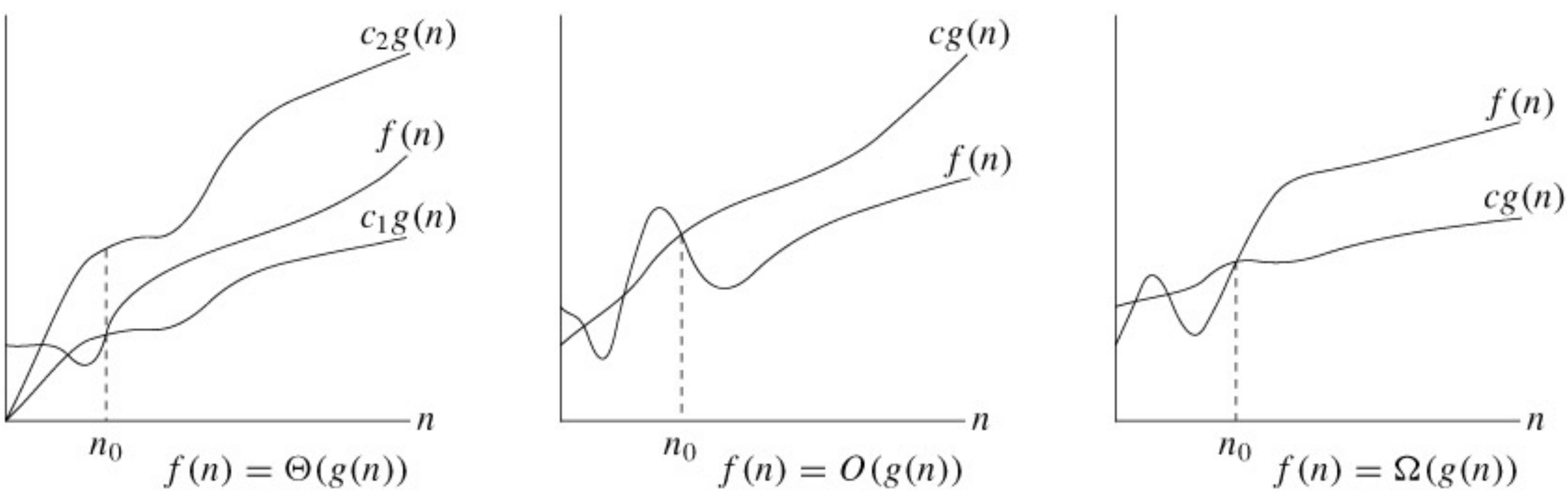
# Asymptotic Notations

- Θ notation: bounds function from above and below.
- O notation: bounds function from above.
- Ω notation: bounds function from below.



# Big O - asymptotic upper bound

- For a given function $g(n)$, $O(g(n))$ is the set of functions $\{f(n)$: there exist positive constants $c$ and $n_0$ such that $0 \leq f(n) \leq cg(n)$, for all $n > n_0\}$



## Asymptotic analysis simplifies analyzing worst-case performance

- We will be dropping constants. For example:
  - $3n^3 + 2n + 7 = O(n^3)$
  - $2^n + n^2 = O(2^n)$
  - $1000 = O(1)$
- Yes, $3n^3 + 2n + 7 = O(n^6)$, but that's a rather useless bound.
- Sorting them by increasing rate of growth:
  - $O(1), O(\log n), O(n), O(n \log n), O(n^2), O(n^3), O(2^n), O(n!)$

## How to interpret Big O

- $O(1)$ or "order one”: running time does not change as size of the problem changes, that is running time stays constant and independent of problem size.
- $O(\log n)$ or "order log n”: running time increases as problem size grows. Whenever problem size doubles, running time increases by a constant.
- $O(n)$ or "order n”: time increases proportionally to the the rate of growth of the size of the problem, that is in a linear rate. Double the problem size, you get double running time.
- $O(n^2)$ or "order n squared”: Double the problem size you get quadruple running time.

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## add() in singly linked lists is $O(1)$ for worst case

```
public void add(Item item) {
    // Save the old node
    Node oldfirst = first;

    // Make a new node and assign it to head. Fix pointers.
    first = new Node();
    first.item = item;
    first.next = oldfirst;

    n++; // increase number of nodes in singly linked list.
}
```

## get() in singly linked lists is $O(n)$ for worst case

```
public Item get(int index) {
    rangeCheck(index);

    Node finger = first;
    // search for index-th element or end of list
    while (index > 0) {
        finger = finger.next;
        index--;
    }
    return finger.item;
}
```

## `add(int index, Item item)` in singly linked lists is $O(n)$ for worst case

```
public void add(int index, Item item) {
        rangeCheck(index);

        if (index == 0) {
            add(item);
        } else {

            Node previous = null;
            Node finger = first;
            // search for index-th position
            while (index > 0) {
                previous = finger;
                finger = finger.next;
                index--;
            }
            // create new value to insert in correct position.
            Node current = new Node();
            current.next = finger;
            current.item = item;
            // make previous value point to new value.
            previous.next = current;

            n++;
        }
    }
```

## `remove()` in singly linked lists is $O(1)$ for worst case

```
public Item remove() {
    Node temp = first;
    // Fix pointers.
    first = first.next;

    n--;

    return temp.item;
}
```

## `remove(int index)` in singly linked lists is $O(n)$ for worst case

```
public Item remove(int index) {
    rangeCheck(index);

    if (index == 0) {
        return remove();
    } else {
        Node previous = null;
        Node finger = first;
        // search for value indexed, keep track of previous
        while (index > 0) {
            previous = finger;
            finger = finger.next;
            index--;
        }
        previous.next = finger.next;

        n--;
        // finger's value is old value, return it
        return finger.item;
    }

}
```

# addFirst() in doubly linked lists is $O(1)$ for worst case

```
public void addFirst(Item item) {
      // Save the old node
      Node oldfirst = first;

      // Make a new node and assign it to head. Fix pointers.
      first = new Node();
      first.item = item;
      first.next = oldfirst;
      first.prev = null;

      // if first node to be added, adjust tail to it.
      if (last == null)
         last = first;
      else
         oldfirst.prev = first;

      n++; // increase number of nodes in doubly linked list.
   }
```

# addLast() in doubly linked lists is $O(1)$ for worst case

```
public void addLast(Item item) {
        // Save the old node
        Node oldlast = last;

        // Make a new node and assign it to tail. Fix pointers.
        last = new Node();
        last.item = item;
        last.next = null;
        last.prev = oldlast;

        // if first node to be added, adjust head to it.
        if (first == null)
            first = last;
        else
            oldlast.next = last;

        n++;
    }
```

`add(int index, Item item)` in doubly linked lists is $O(n)$ for worst case

```
public void add(int index, Item item) {
        rangeCheck(index);

        if (index == 0) {
            addFirst(item);
        } else if (index == size()) {
            addLast(item);
        } else {

            Node previous = null;
            Node finger = first;
            // search for index-th position
            while (index > 0) {
                previous = finger;
                finger = finger.next;
                index--;
            }
            // create new value to insert in correct position
            Node current = new Node();
            current.item = item;
            current.next = finger;
            current.prev = previous;
            previous.next = current;
            finger.prev = current;

            n++;
        }
    }
```

## `removeFirst()` in doubly linked lists is $O(1)$ for worst case

```
public Item removeFirst() {
      Node oldFirst = first;
      // Fix pointers.
      first = first.next;
      // at least 1 nodes left.
      if (first != null) {
         first.prev = null;
      } else {
         last = null; // remove final node.
      }
      oldFirst.next = null;

      n--;

      return oldFirst.item;
   }
```

## `removeLast()` in doubly linked lists is $O(1)$ for worst case

```
public Item removeLast() {

    Node temp = last;
    last = last.prev;

    // if there was only one node in the doubly linked list.
    if (last == null) {
        first = null;
    } else {
        last.next = null;
    }
    n--;
    return temp.item;
}
```

## `remove(int index)` in doubly linked lists is $O(n)$ for worst case

```
public Item remove(int index) {
    rangeCheck(index);

    if (index == 0) {
        return removeFirst();
    } else if (index == size() - 1) {
        return removeLast();
    } else {
        Node previous = null;
        Node finger = first;
        // search for value indexed, keep track of previous
        while (index > 0) {
            previous = finger;
            finger = finger.next;
            index--;
        }
        previous.next = finger.next;
        finger.next.prev = previous;

        n--;
        // finger's value is old value, return it
        return finger.item;
    }

}
```

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## `push(Item item)` in linked stack is $O(1)$ for worst case

```java
public void push(Item item) {
    Node oldfirst = first;
    first = new Node();
    first.item = item;
    first.next = oldfirst;
    n++;
}
```

- Same time complexity both for singly and doubly linked list

## pop() in linked stack is $O(1)$ for worst case

```
public Item pop() {
     if (isEmpty()) throw new NoSuchElementException("Stack underflow");
     Item item = first.item;
     first = first.next;
     n--;
    return item;
}
```

- Same time complexity both for singly and doubly linked list

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`enqueue(Item item)` in (doubly) linked queue is $O(1)$ for worst case

```
public void enqueue(Item item) {
     Node oldlast = last;
     last = new Node();
     last.item = item;
     last.next = null;
     if (isEmpty())
         first = last;
     else
         oldlast.next = last;
     n++;
}
```

`dequeue(Item item)` in (doubly) linked queue is $O(1)$ for worst case

```
public Item dequeue() {
    if (isEmpty())
       throw new NoSuchElementException("Queue underflow");
    Item item = first.item;
    first = first.next;
    n--;
    if (isEmpty())
        last = null;
   return item;
}
```

## Queues as singly linked lists

- $O(n)$ if only head pointer and have to enqueue at the tail.
- $O(1)$ if we have a tail pointer.
  - Simple modification in code, big gains!
  - Version that textbook follows.

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## Worst-case performance of `add()` is $O(n)$

- Cost model: 1 for insertion, $n$ for copying $n$ items to a new array.
- Worst-case: If arraylist is full, `add()` will need to call `resize` to create a new array of double the size, copy all items, insert new one.
- Total cost: $n + 1 = O(n)$.

- Realistically, this won't be happening often and worst-case analysis can be too strict. We will use amortized time analysis instead.

# Amortized analysis

- Amortized cost per operation: for a sequence of $n$ operations, it is the total cost of operations divided by $n$.
  - Simplest form of amortized analysis called aggregate method. More complicated methods exist, such as accounting (banking) and potential (physicist's).

# Amortized analysis for $n$ add( ) operations

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Insertion Cost | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Copying Cost | 0 | 1 | 2 | 0 | 4 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 |
| Total Cost | 1 | 2 | 3 | 1 | 5 | 1 | 1 | 1 | 9 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 17 |

- As the arraylist increases, doubling happens *half as often* but *costs twice as much.*
- $O$(total cost)= $\sum$(“cost of insertions”) + $\sum$(“cost of copying”)
- $\sum$(“cost of insertions”) $= n.$
- $\sum$(“cost of copying”) = $1 + 2 + 2^2 + \ldots 2^{\lfloor \log 2^n \rfloor} \leq 2n.$
- $O$(total cost) $\leq 3n$, therefore amortized cost is $\leq \dfrac{3n}{n} = 3 = O(1)$, but “lumpy”.

Amortized analysis for $n$ `add()` operations when increasing arraylist by 1.

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Insertion Cost | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| Copying Cost | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| Total Cost | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |

- $\sum$(“cost of insertions”) $= n$.
- $\sum$(“cost of copying”) = $1 + 2 + 3 + \ldots + n - 1 = n(n-1)/2$.
- $O$(total cost) $= n + n(n-1)/2 = n(n+1)/2$, therefore amortized cost is $(n+1)/2$ or $O(n)$.
- Same idea when increasing arraylist size by a constant.

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A (linked) stack with $n$ items uses ~$40n$ bytes

- 16 bytes (object overhead)
- 8 bytes (inner class overhead)
- 8 bytes (reference to an Item)
- 8 bytes (reference to next node)
- Total: 40 bytes per stack Node

- This analysis does not take into consideration the size of the Item objects.

## Readings:

- Textbook:
  - Chapter 1.4 (pages 197–199)
- Website:
  - Analysis of Algorithms: https://algs4.cs.princeton.edu/14analysis/

## Practice Problems:

- 1.4.1, 1.4.5 - 1.4.7, 1.4.32, 1.4.35-1.4.36.