Lecture 37: Graphs IV

CS 62

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Single Source Shortest Path Problem

- From a starting node **s**, find the shortest path (and its length) to all other (reachable) nodes
- The collection of all shortest paths form a tree, called… the *shortest path tree*!
- If all edges have the same weight, we can use *BFS*.
- Otherwise …

Single Source Shortest Path Problem

- If all edges have weights ≥ 0 then use Dijkstra's algorithm
- Essentially BFS with priority queue
- Priorities are best known distance to a node from s
- We can keep track of parent nodes to get shortest path
- Example of a **greedy** algorithm

Dijkstra’s algorithm (1956) pseudocode

```
Q = {}; //set with unvisited vertices
for(every vertex v in V) {
   dist[v] = Infinity;
   parents[v] = null;
   Q.add(v);
}
   dist[s] = 0;
   while (!Q.isEmpty()) {
      u = vertex in Q with min dist[u];
      Q.remove(u);
      for(every edge (u,v)) {
         tentative = dist[u] + weight(u,v);
         if (tentative < dist[v]) {
            dist[v] = tentative;
            parents[v] = u;
         }
      }
   }
```

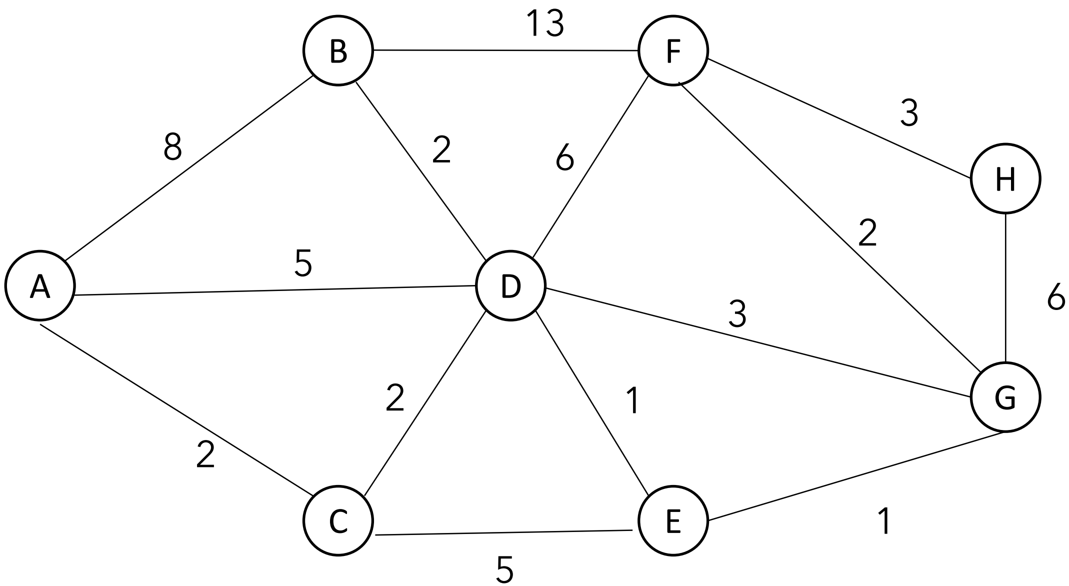
Dijkstra’s algorithm (1984) pseudocode

```
Q = new PriorityQueue();
for(every vertex v in V) {
   dist[v] = Infinity;
   parents[v] = null;
   Q.addWithPriority(v,dist[v]);
}
   dist[s] = 0;
   Q.addWithPriority(s, 0);
   while (!Q.isEmpty()) {
      u = Q.extractmin();
      Q.remove(u);
      for(every edge (u,v)) {
         tentative = dist[u] + weight(u,v);
         if (tentative < dist[v]) {
            dist[v] = tentative;
            parents[v] = u;
            Q.reducePriority(v, tentative);
         }
      }
   }
```

Run-time of Dijkstra

- Adding and removing from priority queue: $O(\log n)$
  - Each goes on and off once, so $O(n \log n)$
- `reduce_priority`: $O(\log n)$
  - Worst case, once for each edge, so $O(m \log n)$
- Total time: $O((m+n)\log n)$

Dijkstra on sample graph



Dijkstra on sample graph

	A	B	C	D	E	F	G	H
Init	$\boxed{0_A}$	∞	∞	∞	∞	∞	∞	∞
A	0_A	8_A	$\boxed{2_A}$	5_A	∞	∞	∞	∞
C	0_A	8_A	2_A	$\boxed{4_C}$	7_C	∞	∞	∞
D	0_A	6_D	2_A	4_C	$\boxed{5_D}$	10_D	7_D	∞
E	0_A	$\boxed{6_D}$	2_A	4_C	5_D	10_D	6_E	∞
B	0_A	6_D	2_A	4_C	5_D	10_D	$\boxed{6_E}$	∞
G	0_A	6_D	2_A	4_C	5_D	$\boxed{8_G}$	6_E	12_E
F	0_A	6_D	2_A	4_C	5_D	8_G	6_E	$\boxed{11_F}$
H	0_A	6_D	2_A	4_C	5_D	8_G	6_E	11_F

Follow the subscripts to find shortest path from start to any vertex

Spanning Trees

- A spanning tree T of a graph G is a subset of the edges of G such that:
  - T contains no cycles and
  - Every vertex in G is connected to every other vertex using just the edges in T
- An unconnected graph has no spanning trees.
- A connected graph will have at least one spanning tree; it may have many

Minimum Spanning Trees

- A weighted graph is a graph that has a weight associated with each edge.
- If G is a weighted graph, the cost of a tree is the sum of the costs (weights) of its edges.
- A tree T is a minimum spanning tree of G iff:
  - it is a spanning tree and
  - there is no other spanning tree whose cost is lower than that of T.

Minimum Spanning Trees

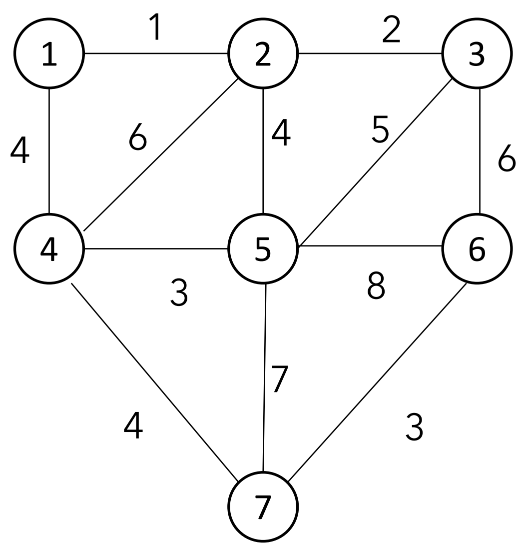
- Application:
  - The cheapest way to lay cable that connects a set of points is along a minimum spanning tree that connects those points.
- Many algorithms exist to find minimum spanning trees, most run in $O(m \log m)$ time.
- In 1995 Karger, Klein & Tarjan found a linear time randomized algorithm, but there is no known linear time deterministic algorithm

Kruskal’s Algorithm

- Create forest F with no edges, using vertices in V
- Sort the edges in the graph by their weight (smallest to largest)
- For each edge e in sorted order:
  - if e connects two different trees in F , then add e to F

Kruskal on sample graph

(1,2):1
(2,3):2
(4,5):3
(6,7):3
(1,4):4
(2,5):4
(4,7):4
(3,5):5
(2,4):6
(3,6):6
(5,7):7
(5,6):8



(1,2):1
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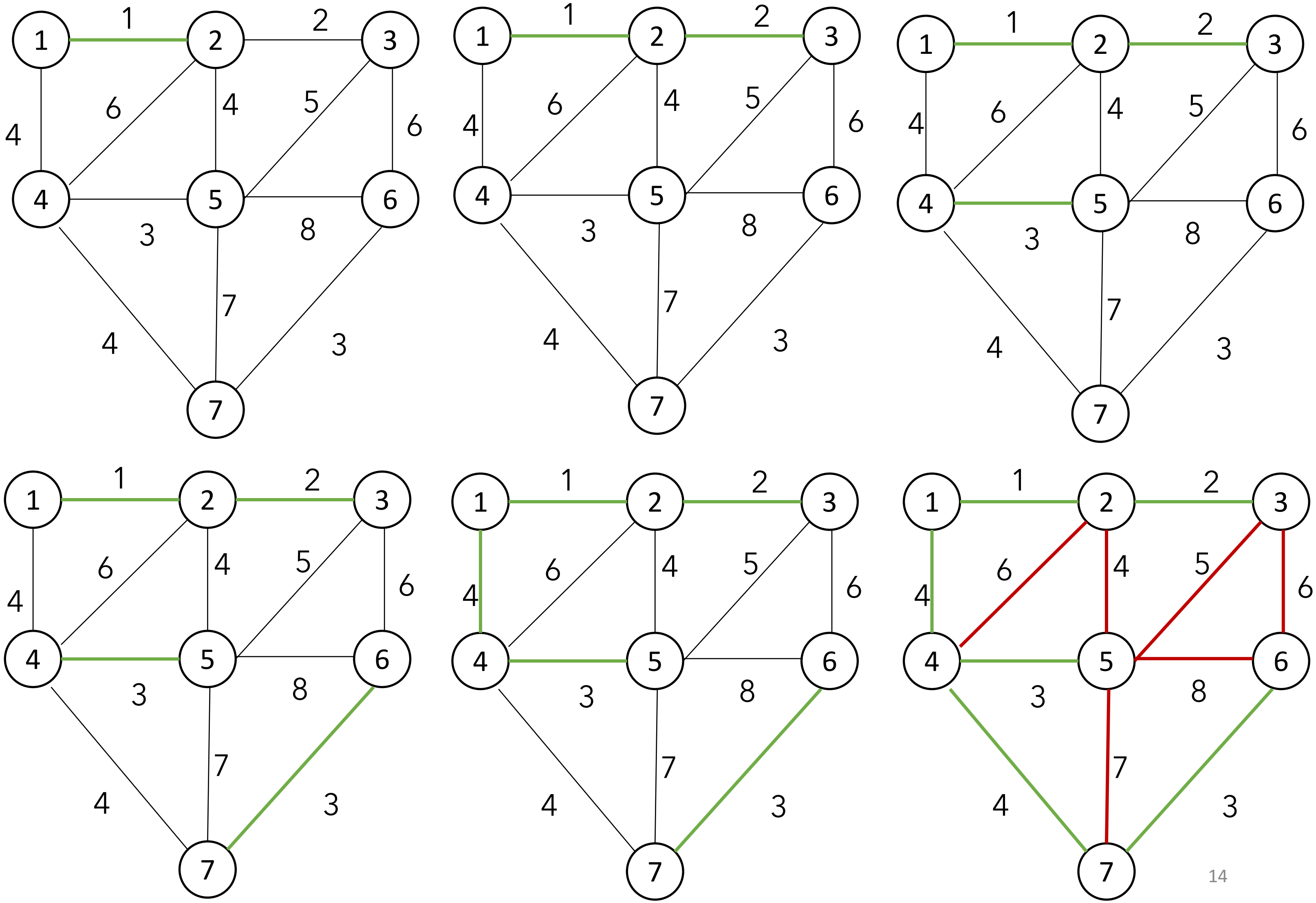
(4,7):4
(3,5):5
(2,4):6
(3,6):6
(5,7):7
(5,6):8



Kruskal’s Algorithm pseudocode

```
A = {};
for(every vertex v in V) {
    make-set(v)
    for(every edge (u, v) ordered by increasing weight) {
        if(find (u) != find (v)) {
            A.add((u, v));
            union(u, v);
    }
}
return A;
```

`make-set(v)` - makes a set from a single vertex v
`find(v)` - finds the set that v belongs to
`union(u, v)` - makes the union of the sets containing u and v

Union-find structure

Graph Algorithms

- Very important in practice!
- Sophisticated data structures
- Careful analysis of correctness and complexity
- CS 140: Algorithms