csci54 – discrete math & functional programming
counting, probability

Some probability questions

- A deck of cards consists of 52 cards, each with a rank (2-10, J, Q, K, A) and a suit (♧♢♡♤).
- If you draw a card from a perfectly-shuffled deck of cards, what is the probability that the card is a heart?
- If you draw a card from a perfectly-shuffled deck of cards, what is the probability that it is either the Queen of Hearts or the 9 of clubs?
- If you draw two cards (without replacement) from a perfectly-shuffled deck of cards, what is the probability that both cards are hearts?

Probability

- A deck of cards consists of 52 cards, each with a rank (2-10, J, Q, K, A) and a suit (♧♢♡♤).
  - If you draw a card from a perfectly-shuffled deck of cards, what is the probability that the card is a heart?

- One process:
  - define a *sample space* S, which is a set containing all possible *outcomes*
  - determine the subset of outcomes that defines the *event*
  - calculate the probability of the outcomes in the event
  - sum those probabilities

Probability – some definitions

Definition 10.1: Outcomes and sample space.
An *outcome* of a probabilistic process is the sequence of results for all randomly determined quantities. (An outcome can also be called a *realization* of the probabilistic process.) The *sample space* S is the set of all outcomes.

Definition 10.2: Probability function.
Let S be a sample space. A *probability function* $\texttt{Pr} : S \to \mathbb{R}$ describes, for each outcome $s \in S$, the fraction of the time that s occurs. (We denote probabilities using square brackets, so the probability of $s \in S$ is written $\texttt{Pr}[s]$.) We insist that the following two conditions hold of the probability function $\texttt{Pr}$:

$$\sum_{s \in S} \texttt{Pr}[s] = 1$$

$$\texttt{Pr}[s] \geq 0 \text{ for all } s \in S.$$

Definition 10.3: Event.
Let S be a sample space with probability function $\texttt{Pr}$. An *event* is a subset of S. The *probability of an event* E is the sum of the probabilities of the outcomes in E, and it is written $\texttt{Pr}[E] = \sum_{s \in E} \texttt{Pr}[s]$.

If you have equally likely outcomes, then the probability of a particular event is the number of outcomes in that event divided by the total number of possible outcomes

In practice

- A deck of cards consists of 52 cards, each with a rank (2-10, J, Q, K, A) and a suit (♧♢♡♤). For each of the following specify your sample space befeore answering the question.

- If you draw a card from a perfectly-shuffled deck of cards, what is the probability that:
  - the card is a heart?

- If you draw two cards (without replacement) from a perfectly-shuffled deck of cards, what is the probability that
  - both cards are hearts?
  - the two cards have different suits?
  - the two cards sum to 3 (i.e. you draw an Ace and a 2)

Independent vs. dependent events

Definition 10.9: Independent and dependent events.

Two events A and B are *independent* if and only if $\texttt{Pr}[A \cap B] = \texttt{Pr}[A] \cdot \texttt{Pr}[B]$. The events A and B are called *dependent* if they are not independent.

- I randomly choose a number 1, 2, ..., 10. Consider the following 3 events. Are any pair of them independent?
  - A: I choose an odd number
  - B: I choose a prime number
  - C: I choose a number (strictly) less than 5.

Conditional probability

Definition 10.11: Conditional probability.

The *conditional probability of A given B*, written $\Pr[A|B]$, is given by $\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}$.

- I randomly choose a number 1, 2, …, 10. Consider the following two events. What are the conditional probabilities Pr[A|B] and Pr[B|A]?
  - A: I choose an odd number
  - B: I choose a prime number

Theorem 10.14: The Law of Total Probability.

Let A and B be arbitrary events. Then $\Pr[A] = \Pr[A|B] \cdot \Pr[B] + \Pr[A|\overline{B}] \cdot \Pr[\overline{B}]$.

Conditional probability

Theorem 10.14: The Law of Total Probability.

Let A and B be arbitrary events. Then $\Pr[A] = \Pr[A|B] \cdot \Pr[B] + \Pr[A|\overline{B}] \cdot \Pr[\overline{B}]$.

Theorem 10.15: Bayes' Rule.

For any two events A and B, we have $\Pr[A|B] = \dfrac{\Pr[B|A] \cdot \Pr[A]}{\Pr[B]}$.

- I have two coins in an opaque bag. The coins are visually indistinguishable, but one coin is fair (Prob H = 0.5); and the other coin is biased (Prob H = 0.75). I pull one of the two coins out at random. If I flip the coin and it comes up heads, what is the probability that I'm holding the biased coin?

Example 10.30 in CDMCS