csci54 – discrete math & functional programming
counting, probability

Some example questions

- There are 141 CS majors. How many ways are there to choose a team of 3 CS majors?
- In a programming contest, teams of 3 try to solve 10 questions. To start, each team member must pick a problem to work on (with possible collaboration). How many ways are there for a team to choose who to solve what problem first?
- Say a team calculates that they can code and submit 20 attempts across all 10 questions. How many different ways can they allocate these submission attempts across the questions? (We don’t care the submission order, and you can submit to a problem multiple times)

Choosing k of n elements

- Can you choose an element more than once? (is repetition allowed)
- Does the order matter?
- order matters, without repetition: $P(n,k) = (n!)/(n-k)!$
- order matters, with repetition: n^k
- order doesn't matter, without repetition: $C(n,k) = (n!)/((n-k)!k!)$
- order doesn’t matter, with repetition: $C(n-1+r, r)$

More practice questions

- (State any assumptions!)
- Suppose I have 20 different colored markers and 8 students. How many ways are there to give the markers to the students if each student needs exactly 1 marker? If each student needs exactly 2 markers?
- Suppose I have 5 identical "I ♥ CS54" stickers all of which I plan to give away to the 8 students. How many ways are there to distribute the stickers if each student gets at most 1 sticker? If each student can get more than 1 sticker?

Proof Techniques

- Remember that we want to drive our proofs *syntactically*
- In your group, make a table with these columns:

- Connective | How to use it | How to prove it

- And with one row for each logical connective; you can write e.g. “forall x, P(x)” or “P /\ Q” or whatever, where P or Q stand in for any other formula or predicate
- (forall, exists, implies, iff, and, or, not, …)

Proof Techniques

- 1. For all x, (P(x) → Q(x) ↔ ¬P(x) ∨ Q(x))
- 2. (Exists x, P(x) ∧ Q(x)) → (exists y, P(y))
- 3. ((for all x, Q(x)) → (for all x, Q(x) → R(x)) → (exists x, R(x) → P(x)) → (for all x, Q(x)) → (exists x, P(x))
  - Do this one slowly and carefully!

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Element-Wise Proof

- Remember that we want to get out of set notation and into logical notation.
- $x \in S \cup T \leftrightarrow x \in S \vee x \in T$
- $x \in S \cap T \leftrightarrow x \in S \wedge x \in T$
- $S \subseteq T \leftrightarrow \forall x, x \in S \rightarrow x \in T$
- … and so on. Finish listing out the axioms!

Probability

- A deck of cards consists of 52 cards, each with a rank (2-10, J, Q, K, A) and a suit (♧ ♢ ♡ ♤).
  - If you draw a card from a perfectly-shuffled deck of cards, what is the probability that the card is a heart?

- One process:
  - define a *sample space* S, which is a set containing all possible *outcomes*
  - determine the subset of outcomes that defines the *event*
  - calculate the probability of the outcomes in the event
  - sum those probabilities

Probability – some definitions

Definition 10.1: Outcomes and sample space.
An *outcome* of a probabilistic process is the sequence of results for all randomly determined quantities. (An outcome can also be called a *realization* of the probabilistic process.) The *sample space* S is the set of all outcomes.

Definition 10.2: Probability function.
Let S be a sample space. A *probability function* $\texttt{Pr} : S \to \mathbb{R}$ describes, for each outcome $s \in S$, the fraction of the time that s occurs. (We denote probabilities using square brackets, so the probability of $s \in S$ is written $\texttt{Pr}[s]$.) We insist that the following two conditions hold of the probability function $\texttt{Pr}$:

$$\sum_{s \in S} \texttt{Pr}[s] = 1$$

$$\texttt{Pr}[s] \geq 0 \text{ for all } s \in S.$$

Definition 10.3: Event.
Let S be a sample space with probability function $\texttt{Pr}$. An *event* is a subset of S. The *probability of an event* E is the sum of the probabilities of the outcomes in E, and it is written $\texttt{Pr}[E] = \sum_{s \in E} \texttt{Pr}[s]$.

If you have equally likely outcomes, then the probability of a particular event is the number of outcomes in that event divided by the total number of possible outcomes

Equally likely outcomes

- A deck of cards consists of 52 cards, each with a rank (2-10, J, Q, K, A) and a suit (♣♢♡♠). For each of the following, define the event as a subset of a set of equally likely outcomes. Then compute the probability of the event.
  - If you draw a card from a perfectly-shuffled deck of cards, what is the probability that the card is a heart?
  - If you draw a card from a perfectly-shuffled deck of cards, what is the probability that it is either the Queen of Hearts or the 9 of clubs?
  - If you draw two cards from a perfectly-shuffled deck of cards, what is the probability that both cards are hearts?