csci54 – discrete math & functional programming
choice and binomials

Counting

- sum rule:
  - if A and B are disjoint, then $|A \cup B| = |A| + |B|$
  - “Difference rule”: $|A| = |A \cup B| - |B|$
- product rule:
  - the number of pairs (x,y) where $x \in A$ and $y \in B$ is $|A \times B| = |A| \cdot |B|$
  - “Quotient rule”: If for each combination C we have R redundant options, divide $|C|$ by $|R|$
- inclusion-exclusion:
  - $|A \cup B| = |A| + |B| - |A \cap B|$

Choice

- We can derive n choose k in different ways
  - $n! / (n-k)!$ (partial permutations)
    - But for each set of choices, there are $k!$ redundant orderings
  - $= n! / ((n-k)!\, k!)$
- Per wikipedia, this way was likely known to Indian mathematicians in the 6th century CE

More practice questions

- Suppose two teams A and B play a best-of-three series of games. How many different sequences of outcomes are there in which A wins the overall series? What if they play a best-of-five?
- How many 10-bit strings have at most 2 ones?
- How many solutions are there to the equation a+b+c=8 where a, b, and c are all non-negative integers?

More practice questions

- There are 141 CS majors. How many ways are there to choose a team of 3 CS majors?

- At the start of a programming contest, teams are given 10 questions to try to solve. At the start of the contest, each member of the team has to choose a problem to think about first. (More than one team member can think about the same problem.) How many ways are there for the 3 team members to choose a problem to think about first?

- Suppose that a team has calculated that they have time to code up and submit 20 different attempted answers to the 10 questions in the contest. How many different ways can they allocate their 20 submissions across the 10 problems? (The order of their submissions doesn't matter and they can submit more than once to each question.)

Choice

- In the 10th century CE, Al-Karaji noticed this phenomenon:
- 0C0 = 1
- 1C0 = 1, 1C1 = 1
- 2C0 = 1, 2C1 = 2, 2C2 = 1
- 3C0 = 1, 3C1 = 3, 3C2 = 3, 3C3 = 1
-

In Europe, known later
as “Pascal’s Triangle”

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Choice

- Laying out the numbers helps us notice a few things…
  - nCk = nC(n-k)
  - nCk = (n-1)C(k-1)+(n-1)C(k) for 0 < k < n
- To choose 2 from 3, we add up the ways to choose 1 from 2 (which we can extend with our new element) plus the ways to choose 2 from 2 which don’t include the new element
- This is sometimes called “Pascal’s Rule”

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

The Binomial Theorem

- We’ve seen polynomials before
  - ax^2 + bx + c, xy^3 + 2x - 7y + 5
- A binomial is a polynomial with two terms
  - E.g. 2x^2 - y
- There’s a rich theory of binomials and their applications, but we’ll focus on the connection to combinatorics.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

The Binomial Theorem

- Let's look at a simple binomial like $(x+y)^n$
- $(x+y)(x+y) = x^2+xy+xy+y^2 = x^2+2xy+y^2$
- $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
- Check out these coefficients!

$(x+y)^n = \sum nCk * x^{n-k}y^k$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Choice

- Pascal's Rule: for all n, 0<k<= n: nCk = (n-1)C(k-1)+(n-1)C(k)
  - By induction on n; base cases 0C0 = 1, 1C0=1,1C1=1.
  - IH: for all 0<k<=n', n'Ck = (n'-1)C(k-1) + (n'-1)C(k)
  - WTS: for all 0<k<=n'+1, (n'+1)Ck = n'C(k-1) + n'C(k)

Let k be given. By IH:
n'C(k-1) = n'! / (n'-(k-1))!(k-1)! = n'! k / (n'-(k-1))! k!   and
n'C(k) = n'! / (n'-k)! k!

WTS: (n'+1)! / (n'+1-k)! k! = n'! k / (n'-k+1)! k! + n'! / (n'-k)! k!
= n'! ((k / ((n'+1)-k)! k!) + ((n'-k+1) / (n'-k+1)(n'-k)!k!))
= n'! ((k / ((n'+1)-k)! k!) + ((n'+1-k) / ((n'+1)-k)!k!)
= n'! ((n'+1) / (n'+1-k)! k!) = (n'+1)! / ((n'+1 - k)! k!) = (n'+1) C k

Binomial Theorem

- Binomial Thm: $(x+y)^n = \sum nCk\ x^{n-k}y^k$
- By induction on n; base case n=0:0=0, n=1:$1*x^1 + 1*y^1 = x+y$
- IH: $(x+y)^{n'}=\sum n'Ck\ x^{n'-k}y^k$; WTS $(x+y)^{n'+1}=\sum(n'+1)Ck\ x^{n'+1-k}y^k$
- $x(x+y)^{n'} + y(x+y)^{n'} = \sum(n'+1)Ck\ x^{n'+1-k}y^k$
- By IH, $(x+y)(\sum n'Ck\ x^{n'-k}y^k) = x\sum n'Ck\ x^{n'-k}y^k + y\sum n'Ck\ x^{n'-k}y^k$
  $= \sum n'Ck\ x^{n'-k+1}y^k+\sum n'Ck\ x^{n'-k}y^{k+1}$.
  Adjusting the bounds on the right sum to k=1 up to n'+1:
  $= \sum n'Ck\ x^{n'-k+1}y^k+\sum n'C(k-1)\ x^{n'-(k-1)}y^{(k-1)+1}$

Binomial Theorem

- WTS $(x+y)^{n'+1}=\sum(n'+1)Ck\ x^{n'+1-k}y^k$
- Have: $(x+y)^{n'+1} = \sum n'Ck\ x^{n'-k+1}y^k+\sum n'C(k-1)\ x^{n'-(k-1)}y^{(k-1)+1}$
- For k=0 and k=n'+1, only the left or right sum has a term, but for k=1 up to n' they are both defined. So drop the first term from the left sum and the last term from the right sum, and both are in the k=1 up to n' bounds:
- $x^{n'+1} + \sum n'Ck\ x^{n'-k+1}y^k+\sum n'C(k-1)\ x^{n'-k+1}y^k + y^{n'+1}$
  $= x^{n'+1} + \sum(n'Ck + n'C(k-1))x^{n'+1-k}y^k + y^{n'+1}$
- By Pascal's Rule, n'C(k) + n'C(k-1) = (n'+1)Ck so:
  $= x^{n'+1} + \sum(n'+1)Ck\ x^{n'+1-k}y^k + y^{n'+1}$, which we can write as a sum from k=0 up to n'+1 again.

What now?

- The binomial theorem has many uses in factorization, approximating transcendental numbers like e, finding roots, …
  - e.g. to know about divisibility of an exponent by 10, you can write a number like 17 as (10+7)^x and see if you can factor a 10 out of the resulting series
- At a minimum, it's easy to do things like "compute the coefficient of the *k*th term of $(x+y)^n$" using choose
- Also shows up a lot in probability and statistics
- "Find a closed form of the sum ∑nCk", "$(x+1)^n = \sum nCk\ x^{n-k}$", etc