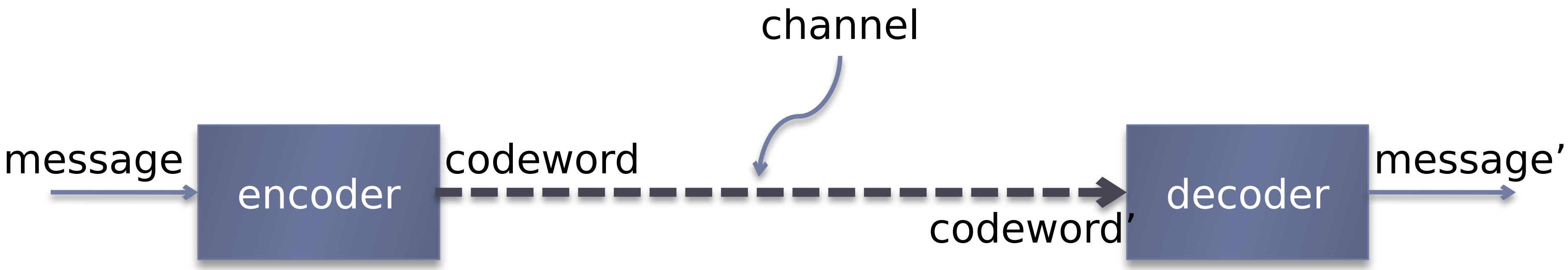
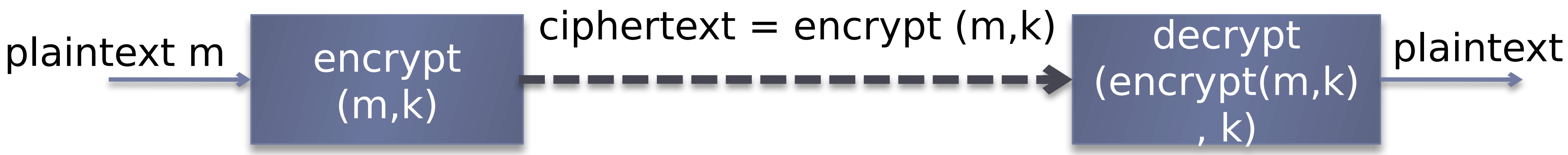
csci54 – discrete math & functional programming
RSA

Transmitting information - cryptography



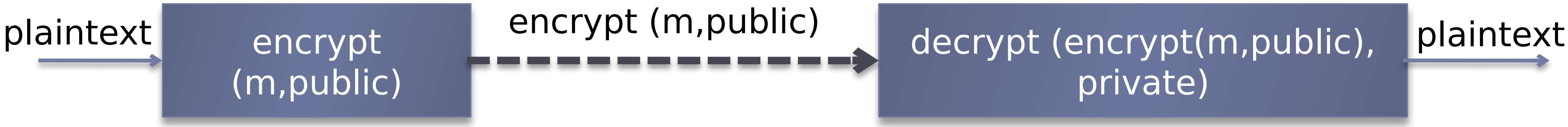
- goal is to keep someone with access to the channel from finding out information about the message.
- assumptions (for now)
  - message = message'
  - codeword = codeword'
- why?
- how?

Private key cryptography

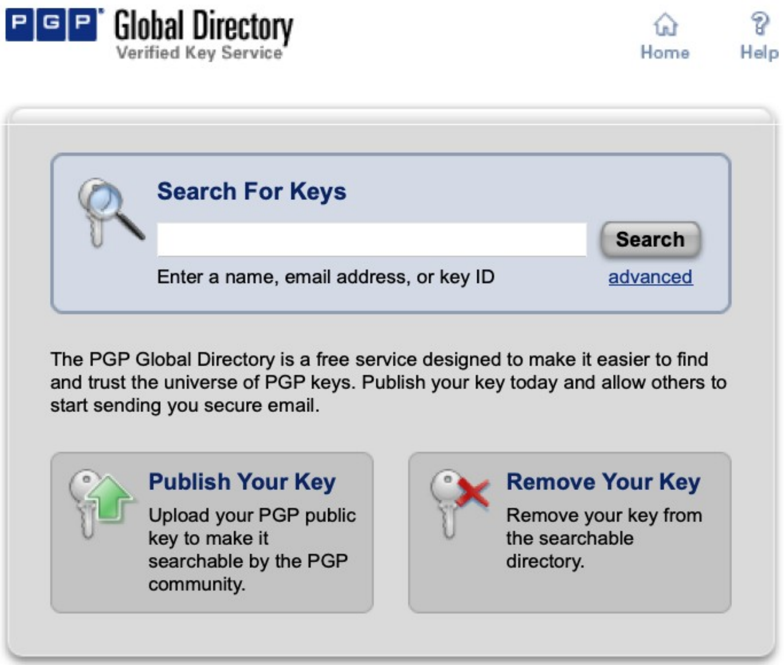


- Symmetric-key algorithms
- The communicating parties share a piece of secret information (the key k)

Public key cryptography



- asymmetric-key algorithm
- Everyone who wants to receive messages generates a public/private key pair and publishes their public key.
- To send a message to someone, you encrypt it with their public key.
- When you receive a message you decrypt it with your private key.



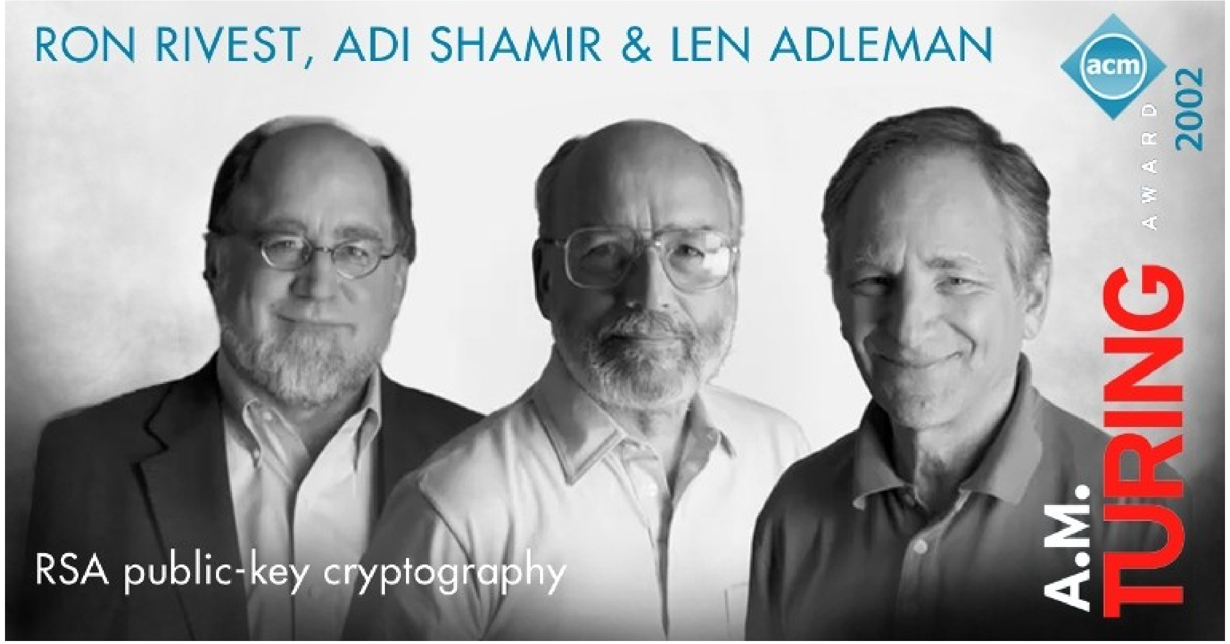
https://keyserver.pgp.com/vkd/GetWelcomeScr

RSA algorithm

- A very widely used public key encryption algorithm

- Three algorithmic components
  - key generation
  - encryption
  - decryption

- Our plan
  - What is the algorithm?
  - Why does it work?
  - How to implement it efficiently?



Greatest common divisor (gcd)

- gcd(a,b) is the largest positive integer that divides both a and b without a remainder.

- Practice:
  - gcd(14, 63)
  - gcd(23, 5)
  - gcd(100, 9)

- if gcd (a,b) = 1 then:
  - a and b have no factors in common
  - we say that a and b are relatively prime
  - there exists an integer x such that ax = 1 (mod b)

RSA algorithm: key generation

1. Choose a bit-length k
2. Choose two primes p and q which can be represented with k bits
3. Let $n = pq$. This means $\phi(n) = (p-1)(q-1)$
4. Find e such that $0 < e < n$ and $\gcd(e,\phi(n)) = 1$
5. Find d such that $(d*e) \bmod \phi(n) = 1$

RSA encryption: example (part 1)

p: prime number
q: prime number
$n = pq$

$\phi(n) = (p-1)(q-1)$
e: $0 < e < n$ and $\gcd(e, \phi(n)) = 1$
d: $(d*e) \bmod \phi(n) = 1$

$p = 3$
$q = 13$
$n =$
$\phi(n) =$
$e =$
$d =$

RSA algorithm: encryption, decryption

- You now have your
  - public key: (e,n)
  - private key: (d,n)

- If someone wants to send you a message (number) m, they:
- compute and send: $encrypt(m) = m^e \bmod n$

- When you get a message z, you:
- compute and read: $decrypt(z) = z^d \bmod n$

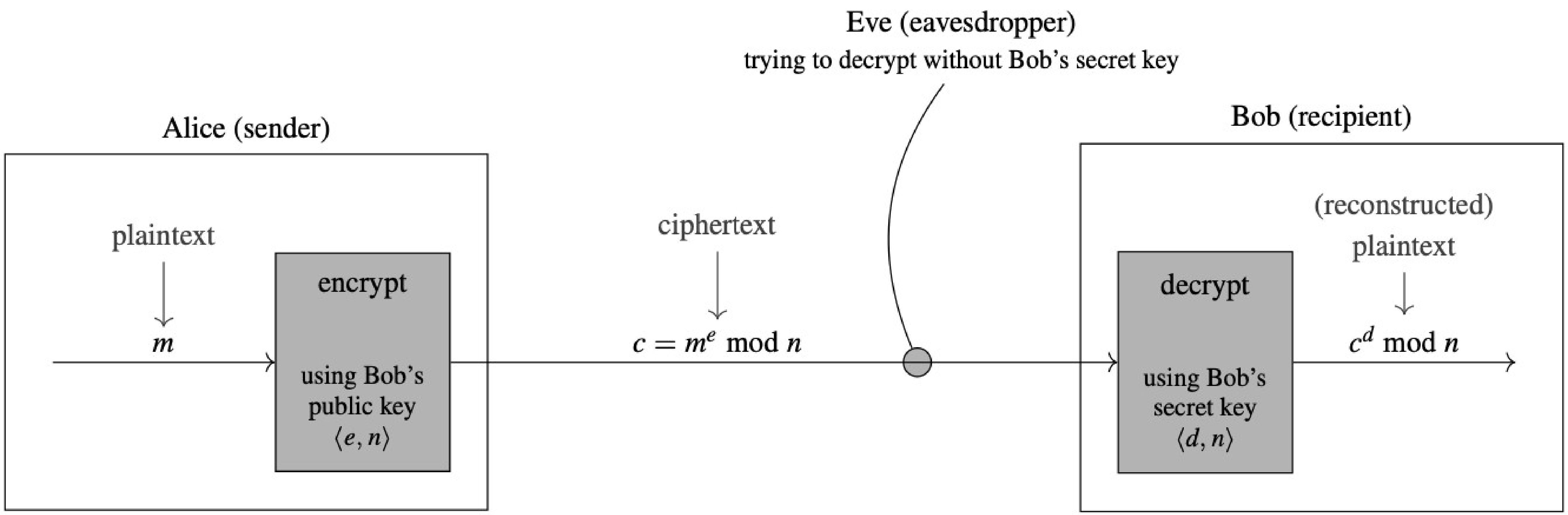


Figure 7.27 A schematic of the RSA cryptosystem, where $n = pq$ and $de \equiv_{(p-1)(q-1)} 1$, for prime numbers p and q.

RSA encryption: example (part 2)

p: prime number
q: prime number
n = pq

$\phi(n) = (p-1)(q-1)$
e: $0 < e < n$ and $\gcd(e,\phi(n)) = 1$
d: $(d*e) \bmod \phi(n) = 1$

$p = 3$
$q = 13$
$n = 39$
$\phi(n) = 24$
$e = 5$
$d = 29$

What is the public key?

What is the private key?

What do you get if you encrypt 10?

RSA encryption: an example

p: prime number
q: prime number
$n = pq$

$\phi(n) = (p-1)(q-1)$
e: $0 < e < n$ and $\gcd(e,\phi(n)) = 1$
d: $(d*e) \bmod \phi(n) = 1$

$p = 3$
$q = 13$
$n = 39$
$\phi(n) = 24$
$e = 5$
$d = 29$

What is the public key?
(5, 39)
What is the private key?
(29, 39)
What do you get if you encrypt 10?
$10^5 \bmod 39 = 4$

Why does the RSA algorithm work?

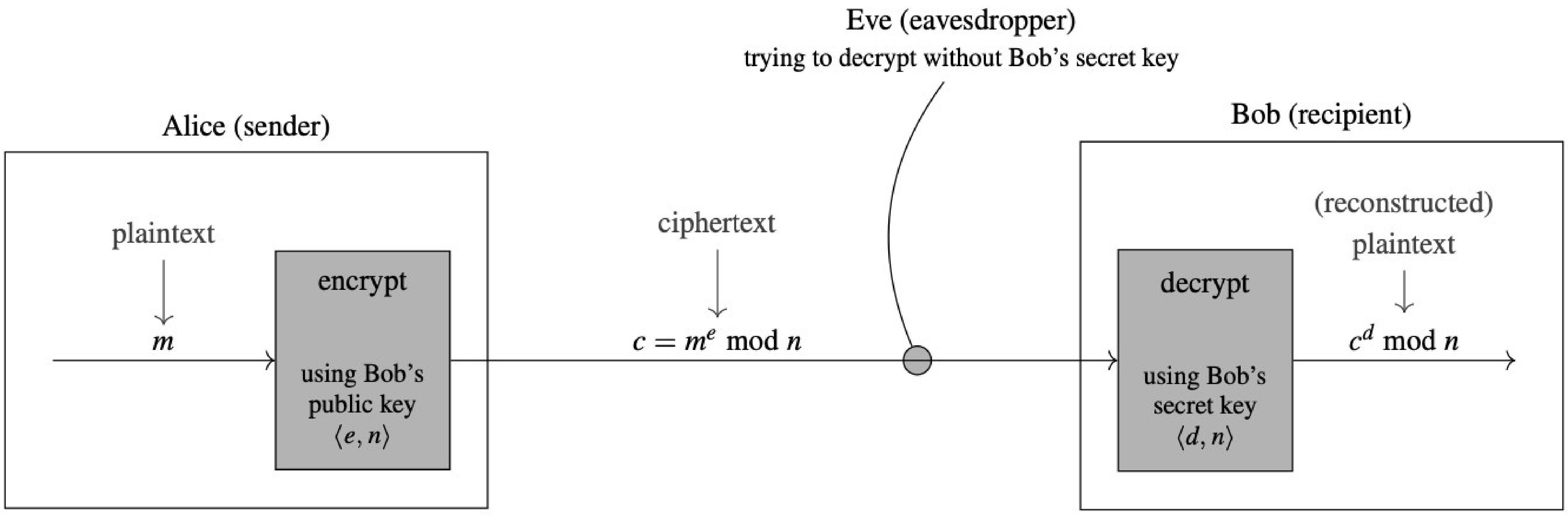


Figure 7.27 A schematic of the RSA cryptosystem, where $n = pq$ and $de \equiv_{(p-1)(q-1)} 1$, for prime numbers p and q.

RSA: correctness

- Claim: decrypt(encrypt(m)) = m
- Proof:

  decrypt(encrypt(m)) = ...

p: prime number
q: prime number
n = pq

ϕ(n) = (p-1)(q-1)
e: gcd(e,ϕ(n)) = 1
d: (d*e) mod ϕ(n) = 1

encrypt(m) = m^e mod n
decrypt(z) = z^d mod n

RSA: correctness

- Claim: decrypt(encrypt(m)) = m
- Proof:

decrypt(encrypt(m)) $= \text{decrypt}(m^e \bmod n)$

$= (m^e \bmod n)^d \bmod n$

$= (m^e)^d \bmod n$

$= (m^{ed}) \bmod n$

$= (m^{k\phi(n)+1}) \bmod n$

$= (m m^{k\phi(n)}) \bmod n$

$= (m \bmod n) * (m^{k\phi(n)} \bmod n)$

... now what?

p: prime number
q: prime number
n = pq

$\phi(n) = (p-1)(q-1)$
e: $\gcd(e,\phi(n)) = 1$
d: $(d*e) \bmod \phi(n) = 1$

encrypt(m) = $m^e \bmod n$
decrypt(z) = $z^d \bmod n$

Fermat and Euler

- Fermat's Little Theorem:
  - If p is prime and gcd(a,p) = 1, then $a^{p-1} = 1 \bmod p$
  - Equivalently, $a^p = a \bmod p$
- Euler:
  - Euler's totient function: $\phi(n) = | \{ x : x < n \text{ and } \gcd(n,x) = 1\} |$
    - What is $\phi(n)$ if n is prime?
  - Theorem: If gcd(a,n) = 1, then $a^{\phi(n)} = 1 \bmod n$

RSA: correctness

- Claim: decrypt(encrypt(m)) = m
- Proof:

decrypt(encrypt(m)) = decrypt(m^e mod n)

= (m^e mod n)^d mod n

= $(m^e)^d$ mod n

= (m^{ed}) mod n

= ($m^{k\phi(n)+1}$) mod n

= ($mm^{k\phi(n)}$) mod n

= (m mod n) * ($m^{k\phi(n)}$ mod n)

p: prime number
q: prime number
n = pq

$\phi(n) = (p-1)(q-1)$
e: gcd(e,$\phi(n)$) = 1
d: (d*e) mod $\phi(n)$ = 1

encrypt(m) = m^e mod n
decrypt(z) = z^d mod n

Euler: If gcd(a,n) = 1, then $a^{\phi(n)}$=1 mod n

RSA: correctness

- Claim: decrypt(encrypt(m)) = m
- Proof:

decrypt(encrypt(m)) = decrypt(m^e mod n)

= $(m^e \bmod n)^d \bmod n$

= $(m^e)^d \bmod n$

= $(m^{ed}) \bmod n$

= $(m^{k\phi(n)+1}) \bmod n$

= $(m m^{k\phi(n)}) \bmod n$

= (m mod n) * ($m^{k\phi(n)}$ mo

= (m mod n) * ($(m^{\phi(n)})^k$ mod n)

= (m mod n) , as long as gcd(m,n) = 1

= m, as long as m < n

p: prime number
q: prime number
n = pq

$\phi(n) = (p-1)(q-1)$
e: $\gcd(e,\phi(n)) = 1$
d: $(d*e) \bmod \phi(n) = 1$

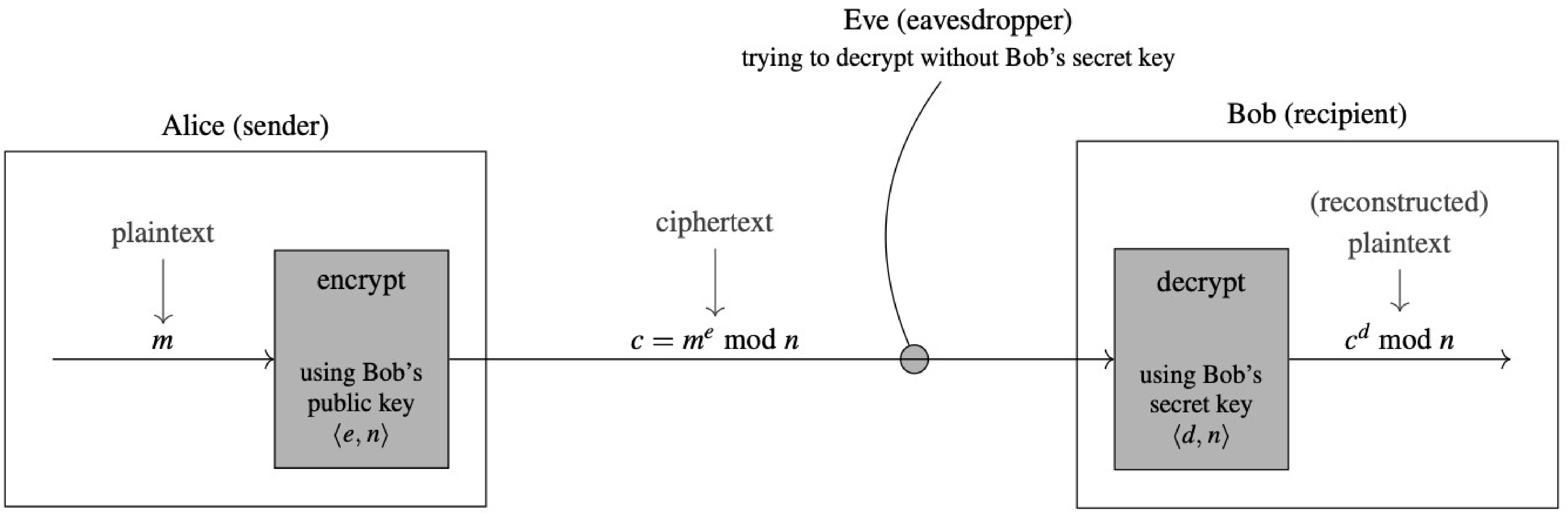
encrypt(m) = m^e mod n
decrypt(z) = z^d mod n

Euler: If gcd(a,n) = 1, then $a^{\phi(n)} = 1 \bmod n$

RSA in practice

- What if the message isn't a number?
  - Everything is a number
- What if the message isn't a number less than n?
  - Divide it into chunks
- Would you ever flip? Encrypt with private key and decrypt with public key?
  - Digital signature

Why is RSA algorithm good?



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How secure is this?

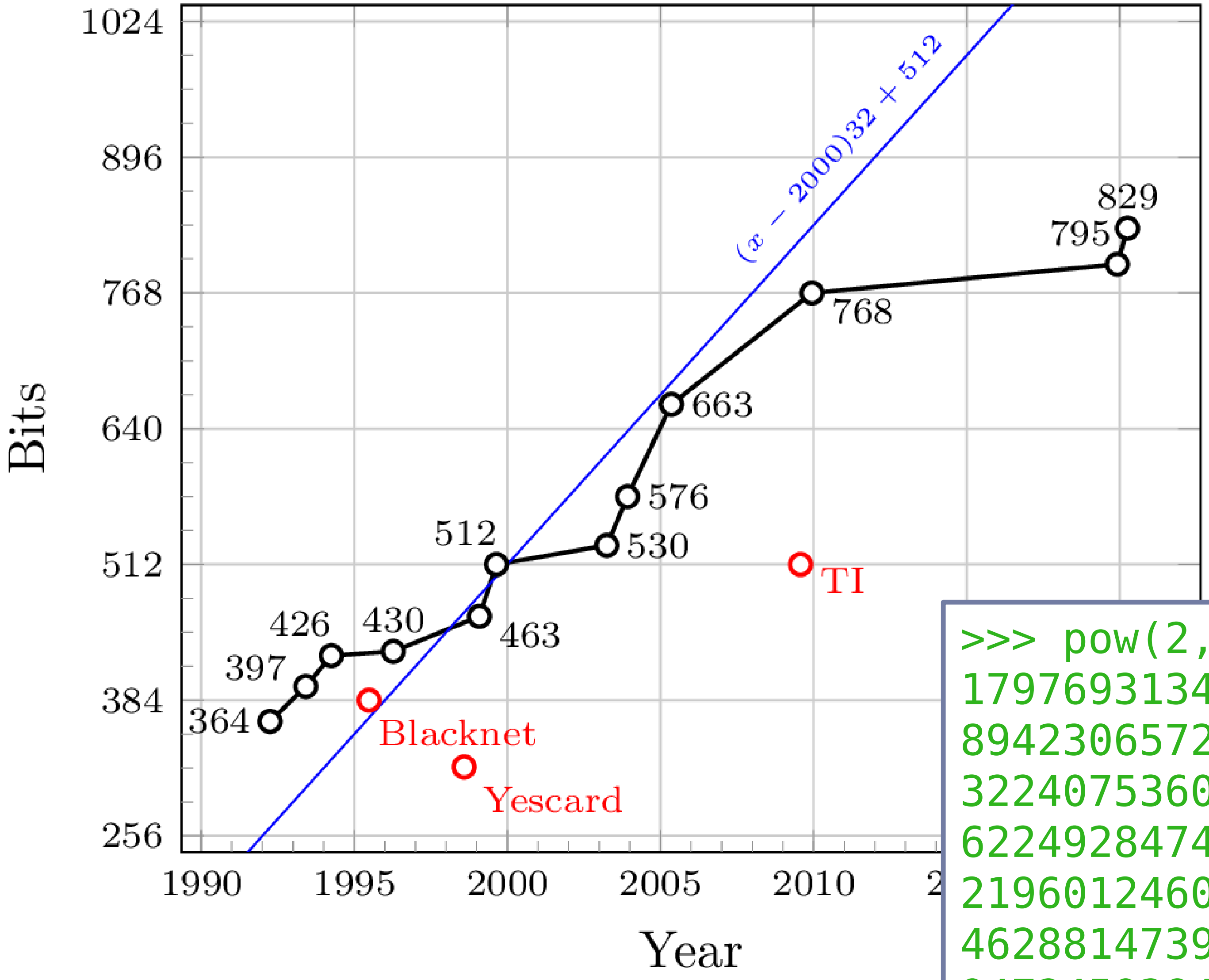
Security of RSA

- Given encrypt(m), can you figure out m?
  - given m^e mod n can you figure out m?
  - issue is that many, many messages m will map to the same encrypted value.
- Given (e,n), can you figure out (d,n)?
  - know: (d*e) mod ϕ(n) = 1
  - but you don't know ϕ(n) and there isn't a good way to get it unless you can figure out p and q from n
  - how expensive is this?

p: prime number
q: prime number
n = pq

ϕ(n) = (p-1)(q-1)
e: gcd(e,ϕ(n)) = 1
d: (d*e) mod ϕ(n) = 1

encrypt(m) = m^e mod n
decrypt(z) = z^d mod n




```
>>> pow(2,1024)
179769313486231590772930519078902473361797697
894230657273430081157732675805500963132708477
322407536021120113879871393357658789768814416
622492847430639474124377767893424865485276302
219601246094119453082952085005768838150682342
462881473913110540827237163350510684586298239
947245938479716304835356329624224137216
```

https://crypto.stackexchange.com/questions/1978/how-big-an-rsa-key-is-considered-secure-today