csci54 – discrete math & functional programming
relations

a last note on (sets of) sets

- partition
  - set of subsets such that every element of S is in exactly one subset
- power set

$$U = \mathbb{Z}^+$$
$$A = \{n : n \geq 6\}$$
$$B = \{1, 2, 4, 5, 7, 8\}$$

- Give two distinct partitions of A
- Give 5 elements of P(B), each with a different cardinality

recap of functions

- a function from a set A to a set B maps each element of A to an element of B.
- terminology: domain, co-domain, range/image
- a function f : A → B can be one-to-one or onto (or both or neither).
  - one-to-one:
    - if f(x)=f(y) then x=y
    - if x != y then f(x) != f(y).
  - onto: forall b in B, exists a in A : f(a)=b
- a function that is both one-to-one and onto is a bijection.

Relations

- (Binary) relations are a generalization of functions
  - (binary) relations express the idea of two items being related to one another
- For example, consider the relation "was born before" on AxB, where A is the set of people in this room and B is the set of dates {1/1/2002, 1/1/2003, 1/1/2004, 1/1/2005}.
  - What are some elements in the relation?
- More formally, a (binary) relation on AxB is a subset of AxB
- Note that a function can be thought of as a relation on where every element of A shows up exactly once as the first element of a pair.

Definitions

Definition 8.1: (Binary) relation.

A *(binary) relation on* $A \times B$ is a subset of $A \times B$.

Often we'll be interested in a relation on $A \times A$, where the two sets are the same. If there is no danger of confusion, we may refer to a subset of $A \times A$ as simply a *relation on* A.

- Example: what are the elements of the "|" (divides) relation on the set {1,2,3,4,5,6,7,8}, where (d,n) is in | if and only if n mod d = 0?

Classifying relations

- reflexive:

  forall x in S, (x,x) in R

- symmetric:

  forall x,y S, if (x,y) in R, then (y,x) in R

- transitive:

  forall x,y,z S, if (x,y) in R and (y,z) in R, then (x,z) in R

- Consider the relation R on the set {1,2,3,4,5,6,7,8}, where (d,n) in R if and only if n mod d =

Classifying relations

$$R_1 = \{\langle n, m\rangle : m \bmod n = 0\}$$

$$R_2 = \{\langle n, m\rangle : n > m\}$$

$$R_3 = \{\langle n, m\rangle : n \leq m\}$$

$$R_4 = \left\{\langle n, m\rangle : n^2 = m\right\}$$

$$R_5 = \{\langle n, m\rangle : n \bmod 5 = m \bmod 5\}$$

- Is each relation reflexive? symmetric? transitive? Why or why not?

Example

Let H be the set of all students enrolled at Pomona College this semester.

Let R = {(x,y) in (H,H) | x and y share a primary academic advisor}

1. What is an element that is in the relation?
2. What is an element that is not in the relation?
3. Is the relation symmetric?
4. Is the relation reflexive?
5. Is the relation transitive?

Equivalence relations and equivalence classes

- A relation that is reflexive, symmetric, and transitive is called an equivalence relation
- An equivalence relation partitions the set into equivalence classes.

Definition 8.14: Equivalence class.

Let $R \subseteq A \times A$ be an equivalence relation. The *equivalence class* of $a \in A$ is defined as the set $\{b \in A : \langle a, b\rangle \in R\}$ of elements related to A under R. The equivalence class of $a \in A$ under R is denoted by $[a]_R$—or, when R is clear from context, just as $[a]$.

Let H be the set of all students enrolled at Pomona College this semester.

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Classifying relations

$$R_1 = \{\langle n, m\rangle : m \bmod n = 0\}$$
$$R_2 = \{\langle n, m\rangle : n > m\}$$
$$R_3 = \{\langle n, m\rangle : n \leq m\}$$
$$R_4 = \left\{\langle n, m\rangle : n^2 = m\right\}$$
$$R_5 = \{\langle n, m\rangle : n \bmod 5 = m \bmod 5\}$$

- If the relation is an equivalence relation, what are the equivalence classes?

Closures

Definition 8.11: Reflexive, symmetric, and transitive closures.

Let $R \subseteq A \times A$ be a relation. Then:

The *reflexive closure of* R is the smallest relation $R' \supseteq R$ such that R' is reflexive.

The *symmetric closure of* R is the smallest relation $R'' \supseteq R$ such that R'' is symmetric.

The *transitive closure of* R is the smallest relation $R^+ \supseteq R$ such that R^+ is transitive.

- Consider the relation $R = \{\langle 1, 5\rangle, \langle 2, 2\rangle, \langle 2, 4\rangle, \langle 4, 1\rangle, \langle 4, 2\rangle\}$ on $\{1, 2, 3, 4, 5\}$
  - Is it reflexive? If not, what is the reflexive closure?
  - Is it symmetric? If not, what is the symmetric closure?
  - Is it transitive? If not, what is the transitive closure?