csci54 – discrete math & functional programming
propositional logic continued, predicate logic

last time

- introduction to propositional logic:
  - Boole
  - proposition
  - well-formed propositional logic formulas (wff)

$$\phi ::= T|F|(\neg\phi)|(\phi \wedge \phi)|(\phi \vee \phi)|(\phi \Rightarrow \phi)$$

  - truth tables for operators
  - tautology/satisfiable/contingency (falsifiable)/contradiction
  - implication
  - logical equivalence

converse, inverse, contrapositive

Given an implication $p \Rightarrow q$, we can derive three other implications:

- converse: $q \rightarrow p$
- inverse: $\neg p \rightarrow \neg q$
- contrapositive: $\neg q \rightarrow \neg p$

- Which, if any, of the converse, inverse, and contrapositive is logically equivalent to the original implication?

consider the following statements . . .

- If 2 is an even number then 3 is an odd number.
- If x is an even number, then x+1 is an odd number.

- How would you express these two statements in propositional logic?

predicate logic

- A predicate P is function that assigns the value True or False to each element of a set U.
  - The set U is called the universe or domain of discourse
  - P is a predicate over U

- Examples:
  - the predicate "is an even number" over the positive integers.
  - the predicate "last name has at least 6 characters" over the set of people currently in this room.

- Once you specify the element of U, then you have a proposition with a truth value.

quantifiers

- quantifiers are another way to form propositions from a predicate

Definition 3.21: Universal quantifier [for all, $\forall$].

Let P be a predicate over S. The proposition $\forall x \in S : P(x)$ is true if, for *every* possible $x \in S$, we have that $P(x)$ is true.

Definition 3.22: Existential quantifier [there exists, $\exists$].

Let P be a predicate over S. The proposition $\exists x \in S : P(x)$ is true if, for *at least one* possible $x \in S$, we have that $P(x)$ is true.

quantifiers - example

- Imagine these predicates
  - "rested(n)" means "n got at least 8 hours of sleep in the past 24 hours"
  - "bornMA(n)" means "n was born in Massachusetts"

- Which, if any, of the following propositions is true? Justify your answer.
  - $\forall$ n in this room : rested(n)
  - $\forall$ n in this room : (rested(n) $\wedge$ bornMA(n))
  - $\exists$ n currently enrolled at Pomona College : (rested(n) $\vee$ bornMA(n))
  - $\exists$ n currently enrolled at Pomona College : (rested(n) $\wedge$ bornMA(n))

free and bound variables (an aside)

- In an expression variables can be free/unbound or bound
  - With a free variable the value is not fixed by the expression
  - With a bound variable the value is defined within the expression

$$\forall x \in \mathbb{Z} : x^2 \geq y$$

- An expression of predicate logic with no free variables is called fully quantified

theorems in predicate logic

- A fully quantified expression of predicate logic is a theorem if and only if it is true for every possible meaning of each of its predicates.

- Is the following a theorem?

$$[\forall x \in S : P(x)] \vee [\forall x \in S : \neg P(x)]$$

- What is an example of a predicate for which the statement is false? is true?

practice question

- Exactly one of the following two propositions is a theorem. Which one?

(1) $[\forall x \in S : P(x) \vee Q(x)] \Leftrightarrow [\forall x \in S : P(x)] \vee [\forall x \in S : Q(x)]$

(2) $[\exists x \in S : P(x) \vee Q(x)] \Leftrightarrow [\exists x \in S : P(x)] \vee [\exists x \in S : Q(x)]$

- Justify your answer

example 3.44 in CDMCS