ENCRYPTION TAKE 2: PRACTICAL DETAILS

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CS54 – Fall 2022

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Admin

Assignment 4 graded

Assignment 6

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Option type

Look at option.sml

option type has two constructors:
- NONE (representing no value)
- SOME v (representing the value v)

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case statement

```
case _______ of
   pattern1 => value
 | pattern2 => value
 | pattern3 => value
…
```

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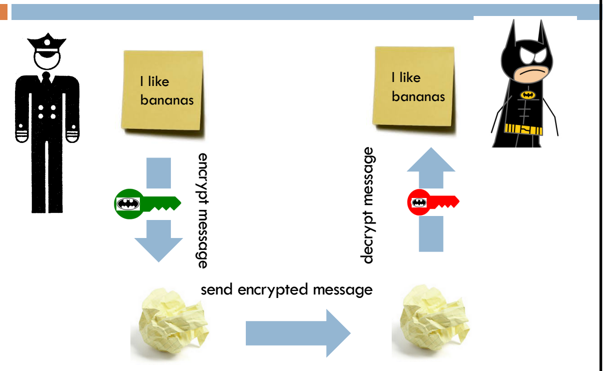
Public key encryption



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RSA public key encryption

1. Choose a bit-length k
2. Choose two primes p and q which can be represented with at most k bits
3. Let $n = pq$ and $\phi(n) = (p-1)(q-1)$
4. Find d such that $0 < d < n$ and $gcd(d,\phi(n)) = 1$
5. Find e such that $de \bmod \phi(n) = 1$
6. private key = (d,n) and public key = (e, n)
7. encrypt(m) = m^e mod n decrypt(z) = z^d mod n

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Cracking RSA

1. Choose a bit-length k
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7. encrypt(m) = m^e mod n decrypt(z) = z^d mod n

Say I maliciously intercept an encrypted message.
How could I decrypt it? (Note, you can also assume that we have the public key (e, n).)

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Cracking RSA

encrypt(m) = m^e mod n

Idea 1: undo the mod operation , i.e. mod^{-1} function

If we knew m^e and e, we could figure out m

Do you think this is possible?

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Cracking RSA

encrypt(m) = m^e mod n

Idea 1: undo the mod operation , i.e. mod^{-1} function

If we knew m^e and e, we could figure out m

Generally, no, if we don't know anything about the message.

The challenge is that the mod operator maps many, many numbers to a single value.

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Security of RSA

p: prime number
q: prime number
n = pq

$\phi(n) = (p-1)(q-1)$
d: $0 < d < n$ and $gcd(d,\phi(n)) = 1$
e: $de \bmod \phi(n) = 1$

private key (d, n) public key (e, n)

Assuming you can't break the encryption itself (i.e. you cannot decrypt an encrypted message without the private key)

How else might you try and figure out the encrypted message?

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Security of RSA

p: prime number
q: prime number
n = pq

$\phi(n) = (p-1)(q-1)$
d: $0 < d < n$ and $gcd(d,\phi(n)) = 1$
e: $de \bmod \phi(n) = 1$

private key (d, n) public key (e, n)

Assuming you can't break the encryption itself (i.e. you cannot decrypt an encrypted message without the private key)

Idea 2: Try and figure out the private key!

How would you do this?

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Security of RSA

p: prime number
q: prime number
n = pq

$\phi(n) = (p-1)(q-1)$
d: $0 < d < n$ and $gcd(d,\phi(n)) = 1$
e: $de \bmod \phi(n) = 1$

private key (d, n) public key (e, n)

Already know e and n.

If we could figure out p and q, then we could figure out the rest (i.e. d)!

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Security of RSA

p: prime number
q: prime number
n = pq

$\phi(n) = (p-1)(q-1)$
d: $0 < d < n$ and $gcd(d,\phi(n)) = 1$
e: $de \bmod \phi(n) = 1$

private key (d, n) public key (e, n)

How would you do figure out p and q?

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Security of RSA

p: prime number
q: prime number
n = pq

$\phi(n) = (p-1)(q-1)$
d: $0 < d < n$ and $gcd(d,\phi(n)) = 1$
e: $de \bmod \phi(n) = 1$

private key (d, n) public key (e, n)

For every prime p (2, 3, 5, 7 …):
- If n mod p = 0 then $\underbrace{q = n / p}$

Why do we know that this *must* be p and q?

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Security of RSA

p: prime number
q: prime number
n = pq

$\phi(n) = (p-1)(q-1)$
d: $0 < d < n$ and $gcd(d,\phi(n)) = 1$
e: $de \bmod \phi(n) = 1$

private key (d, n) public key (e, n)

For every prime p (2, 3, 5, 7 …):
- If n mod p = 0 then $\underbrace{q = n / p}$

Since p and q are both prime, there are no other numbers that divide them evenly, therefore no other numbers divide n evenly

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Security of RSA

p: prime number
q: prime number
n = pq

$\phi(n) = (p-1)(q-1)$
d: $0 < d < n$ and $gcd(d,\phi(n)) = 1$
e: $de \bmod \phi(n) = 1$

private key (d, n) public key (e, n)

For *every number* p (2, 3, 4, 5, 6, 7 …):
- If n mod p = 0 then q = n / p

How long does this take?
I.e. how many p do we need to check in the worst case assuming n has k bits?

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Security of RSA

p: prime number
q: prime number
n = pq

$\phi(n) = (p-1)(q-1)$
d: $0 < d < n$ and $gcd(d,\phi(n)) = 1$
e: $de \bmod \phi(n) = 1$

private key (d, n) public key (e, n)

For every number p (2, 3, 4, 5, 6, 7 …):
- If n mod p = 0 then q = n / p

- p is at most k bits
- With k bits we can represent numbers up to 2^k
- If we assumed that p was picked randomly from these numbers, then on average we'd have to check 2^{k-1} numbers (half of them)
- For large k (e.g. 1024) this is a very big number!

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Security of RSA

p: prime number
q: prime number
n = pq

$\phi(n) = (p-1)(q-1)$
d: $0 < d < n$ and $gcd(d,\phi(n)) = 1$
e: $de \bmod \phi(n) = 1$

private key (d, n) public key (e, n)

For every number p (2, 3, 4, 5, 6, 7 …):
- If n mod p = 0 then q = n / p

Currently, there are no known "efficient" methods for factoring a number into it's primes.
This is the key to why RSA works!

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Implementing RSA

1. Choose a bit-length *k*

For generating the keys, this is the only input the algorithm has

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Implementing RSA

2. Choose two primes *p* and *q* which can be represented with at most *k* bits

Ideas?

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Finding primes

2. Choose two primes *p* and *q* which can be represented with at most *k* bits

Idea: pick a random number and see if it's prime

How do we check if a number is prime?

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Finding primes

2. Choose two primes *p* and *q* which can be represented with at most *k* bits

Idea: pick a random number and see if it's prime

```
isPrime(num):
    for i = 2 … sqrt(num):
        if num % i == 0:
            return false
    return true
```

If the number is k bits, how many numbers (worst case) might we need to examine?

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Finding primes

2. Choose two primes *p* and *q* which can be represented with at most *k* bits

Idea: pick a random number and see if it's prime

- Again: with k bits we can represent numbers up to 2^k
- Counting up to sqrt = $(2^k)^{1/2} = 2^{k/2}$

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Finding primes

Primality test for *num*:

- pick a random number *a*
- perform *test*(*num*, *a*)
  - if test fails, *num* is not prime
  - if test passes, 50% chance that *num* is prime

Does this help us?

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Finding primes

Primality test for *num*:

- pick a random number *a*
- perform *test*(*num*, *a*)
  - if test fails: return false
  - if test passes: return true

If num is not prime, what is the probability (chance) that we incorrectly say num is a prime?

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Finding primes

Primality test for *num*:

- pick a random number *a*
- perform *test*(*num*, *a*)
  - if test fails: return false
  - if test passes: return true

0.5 (50%)

Can we do any better?

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Finding primes

Primality test for *num*:

- Repeat **2** times:
  - pick a random number *a*
  - perform *test*(*num*, *a*)
    - if test fails: return false
- return true

If num is not prime, what is the probability that we incorrectly say num is a prime?

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Finding primes

Primality test for *num*:

- Repeat **2** times:
  - pick a random number *a*
  - perform *test*(*num*, *a*)
    - if test fails: return false
- return true

p(0.25)

- Half the time we catch it on the first test
- Of the remaining half, again, half (i.e. a quarter total) we catch it on the second test
- ¼ we don't catch it

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Finding primes

Primality test for *num*:

- Repeat **3** times:
  - pick a random number *a*
  - perform *test*(*num*, *a*)
    - if test fails: return false
- return true

If num is not prime, what is the probability that we incorrectly say num is a prime?

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Finding primes

Primality test for *num*:

- Repeat **3** times:
  - pick a random number *a*
  - perform *test*(*num*, *a*)
    - if test fails: return false
- return true

p(1/8)

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Finding primes

Primality test for *num*:

- Repeat **m** times:
  - pick a random number *a*
  - perform *test*(*num*, *a*)
    - if test fails: return false
- return true

If num is not prime, what is the probability that we incorrectly say num is a prime?

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Finding primes

Primality test for *num*:

- Repeat **m** times:
  - pick a random number *a*
  - perform *test*(*num*, *a*)
    - if test fails: return false
- return true

p($1/2^m$)

For example, m = 20: p($1/2^{20}$) = p(1/1,000,000)

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Finding primes

Primality test for *num*:

- Repeat m times:
  - pick a random number *a*
  - perform *test*(*num*, *a*)
    - if test fails: return false
- return true

Fermat's little theorem: If *p* is a prime number, then for all integers *a*:

$a \equiv a^p \ (mod\ p)$

How does this help us?

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Finding primes

Fermat's little theorem: If *p* is a prime number, then for all integers *a*:

$a \equiv a^p \ (mod\ p)$

test(num, a):

- generate a random number a < p
- check if a^p mod p = a

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Implementing RSA

1. Choose a bit-length *k*
2. Choose two primes *p* and *q* which can be represented with at most *k* bits
3. Let $n = pq$ and $\phi(n) = (p-1)(q-1)$

How do we do this?

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Implementing RSA

4. Find *d* such that $0 < d < n$ and $gcd(d,\phi(n)) = 1$
5. Find e such that $de \bmod \phi(n) = 1$

How do we do these steps?

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Greatest Common Divisor

A useful property:

If two numbers are relatively prime (i.e. gcd(a,b) = 1), then there exists a c such that

a*c mod b = 1

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Greatest Common Divisor

A more useful property:

two numbers are relatively prime (i.e. gcd(a,b) = 1) *iff* there exists a c such that a*c mod b = 1

What does iff mean?

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Greatest Common Divisor

A more useful property:

1. If two numbers are relatively prime (i.e. gcd(a,b) = 1), then there exists a c such that a*c mod b = 1

2. If there exists a c such that a*c mod b = 1, then the two numbers are relatively prime (i.e. gcd(a,b) = 1)

We're going to leverage this second part

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Implementing RSA

4. Find d such that $0 < d < n$ and $gcd(d,\phi(n)) = 1$

5. Find e such that $de \bmod \phi(n) = 1$

If there exists a c such that a*c mod b = 1, then the two numbers are relatively prime (i.e. gcd(a,b) = 1)

To find d and e:
- pick a random d, $0 < d < n$
- *try* and find an e such that $de \bmod \phi(n) = 1$
  - if none exists, try another d
  - if one exists, we're done!

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Modular multiplicative inverse

From Wikipedia, the free encyclopedia

This article **needs additional citations for verification**. Please help improve this article by adding citations to reliable sources. Unsourced material may be challenged and removed. *(March 2007)*

In modular arithmetic, the **modular multiplicative inverse** of an integer *a* modulo *m* is an integer *x* such that

$$a\,x \equiv 1 \pmod{m}.$$

That is, it is the multiplicative inverse in the ring of integers modulo *m*, denoted $\mathbb{Z}_m$.

Once defined, *x* may be noted a^{-1}, where the fact that the inversion is m-modular is implicit.

The multiplicative inverse of *a* modulo *m* exists if and only if *a* and *m* are coprime (i.e., if gcd(*a*, *m*) = 1).[1] If the modular multiplicative inverse of *a* modulo *m* exists, the operation of division by *a* modulo *m* can be defined as multiplying by the inverse of *a*, which is in essence the same concept as division in the field of reals.

Known problem with known solutions

For the assignment, I've provided you with a function: *inversemod*

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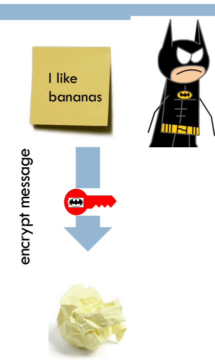
Signing documents

If a message is encrypted with the *private key* how can it be decrypted?

Hint:
- $(m^e)^d = m^{ed} = m \pmod{n}$
- encrypt(m, (e, n)) = m^e mod n
- decrypt(z, (d, n)) = z^d mod n



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Signing documents

- $(m^e)^d = m^{ed} = m \pmod{n}$
- encrypt(m, (e, n)) = m^e mod n
- decrypt(z, (d, n)) = z^d mod n

encrypt(m, (d,n)) = m^d mod n

decrypt(m^d mod n , (e, n)) = $(m^d)^e$ mod n

= m^{de} mod n

= m^{ed} mod n

= m (if m < n)

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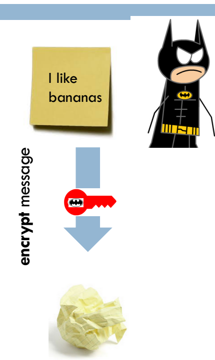
Signing documents

What does this do for us?



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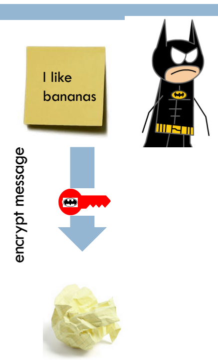
Signing documents

If the message can be decrypted with the public key then the sender must have had the private key

This is a way to digitally sign a document!



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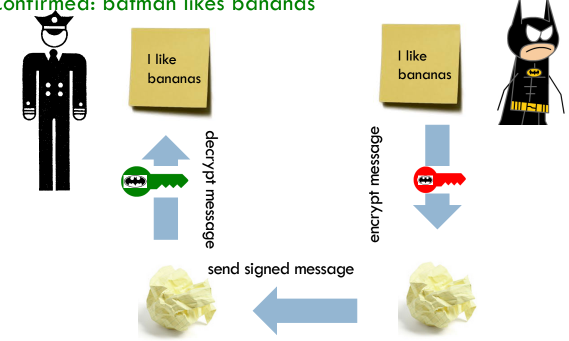
Signing documents

Confirmed: batman likes bananas



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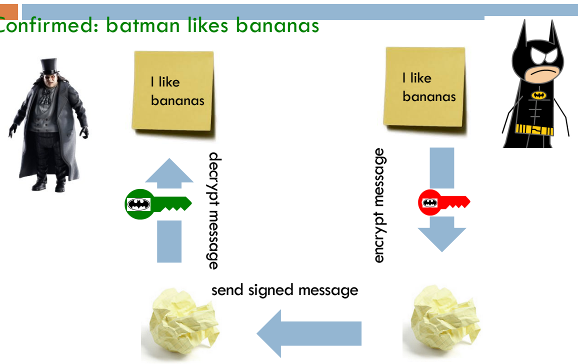
Signing documents

Confirmed: batman likes bananas



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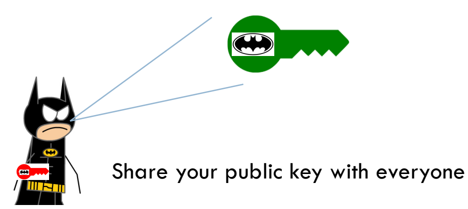
Public key encryption



How does this happen?

Anything we have to be careful of?

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