



Mathematical Foundations

Strong induction

CS51 – Spring 2026

Last time, we talked about the proof technique of mathematical induction and how it allows us to prove properties for all non-negative integers.

Proof by induction

- From last time:
 - To prove that a property $P(n)$ holds for all non-negative integers n , we prove that:
 - $P(0)$ is true,
 - for every $n \geq 0$, if $P(n)$ is true, then $P(n + 1)$ is true, too.
 - The proof of $P(0)$ is called the **base case**.
 - The proof that $P(n) \Rightarrow P(n + 1)$ is called the **inductive case**.
 - Practically, the principle of mathematical induction says the following: to prove that a statement $P(n)$ is true for all non-negative integers n , we can prove that P “starts being true” (the base case) and that P “never stops being true” (the inductive case).

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As a reminder, the structure of a proof by induction assumes that we have a property $P(n)$ that holds true for all non-negative integers n . We start by proving that $P(0)$ is true, what is known as the base case. And then we assume that for every $n \geq 0$, if the property is true for n then we show that it is true for $P(n+1)$. This is known as the inductive case. The intuition behind mathematical induction is that P starts being true and never stops being true, which corresponds to the base and inductive case, respectively.

Proof by strong induction

- To prove that a property $P(n)$ holds for all non-negative integers n , we prove that:
 - $P(0)$ is true,
 - for every $n \geq 0$, if $P(0) \wedge P(1) \wedge \dots \wedge P(n)$ is true, then $P(n + 1)$ is true, too.
- The proof of $P(0)$ is called the **base case**.
- The proof that $P(0) \wedge P(1) \wedge \dots \wedge P(n) \Rightarrow P(n + 1)$ is called the **inductive case**.
- Practically, what we saw last time is known as **weak induction**. In weak induction: we show that if P is true this time, then it will be true next time. We only assume $P(n)$.
- In **strong induction**, we show that if P has been true up until now, then it will be true next time. We assume all of previous steps $P(0) \wedge P(1) \wedge \dots \wedge P(n)$ to show $P(n + 1)$.
- Stronger doesn't mean better, just that we have a stronger inductive hypothesis.

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Today, we will see a variation of this proof technique known as strong induction. Strong induction again allows us to prove that a property $P(n)$ holds for all non-negative integers n . The base case is the same, we just prove that $P(0)$ is true. What changes is the inductive case: we don't just assume that $P(n)$ is true for every $n \geq 0$ and prove that $P(n+1)$ is also true. Instead, we assume that P is true for all numbers UP to n , that is $P(0)$ AND $P(1)$ AND $P(2)$ AND... AND $P(n)$ are all true and then we show that $P(n+1)$ is also true. Our approach last time is known as weak induction and only assumes $P(n)$. In strong induction, we show that If P has been true for all integers up until now, then it's still true this time.

An aside: floor and ceiling

- The **floor** of a number x , written as $\lfloor x \rfloor$ is the largest integer that is less than or equal to x .
 - For example, $\lfloor 4.7 \rfloor = 4$.
- The **ceiling** of a number x , written as $\lceil x \rceil$ is the smallest integer that is greater than or equal to x .
 - For example, $\lceil 4.7 \rceil = 5$.

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Before we work on our first strong induction proof, let's go over some notation that will be useful (similar to how sums were handy last time).

The floor of a number x is the largest integer that is less than or equal to x .

The ceiling is the opposite idea, the smallest integer that is greater or equal to x .

For any integers k and n (n has to strictly be greater than 0), the integer $n \bmod k$ is the remainder when we divide n by k , that is $n - k \cdot \text{floor of } (n/k)$.

A reminder: moduli and division

- For any integers $k > 0$ and n , the integer $n \bmod k$ (also denoted as $n \% k$) is the remainder when we divide n by k . The value of $n \bmod k$ is $n - k \lfloor \frac{n}{k} \rfloor$.
 - For example, $8 \bmod 3 = 2$ because $8 = 3 \times 2 + 2$ or by definition $8 - 3 \lfloor \frac{8}{3} \rfloor = 8 - 3 \times 2 = 2$.
- For any integer $k > 0$ and n , we say that k evenly divides n (or n is evenly divisible by k), written $k|n$, if $\frac{n}{k}$ is an integer. Note that $k|n$ is equivalent to $n \bmod k = 0$.
 - For example, $5|10$ but $5 \nmid 9$.
- A positive integer $n \geq 1$ is **prime** if the only positive integers that evenly divide n are 1 and n itself. If a number $n > 1$ is not prime it is called **composite**.

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For any integers k and n (n has to strictly be greater than 0), the integer $n \bmod k$ is the remainder when we divide n by k , that is $n - k \times \text{floor of } (n/k)$. We say that k evenly divides n or n is divisible by k and denote it as $k|n$ if n/k is an integer, that is $n \bmod k = 0$. evenly

This helps explain why a number is prime if the only positive integers that evenly divide n are 1 and n itself. Otherwise a number is called a composite.

Example: divisibility by a prime

- Using strong induction, we will prove the claim that any integer $n \geq 2$ is evenly divisible by a prime number.

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Now that we established this vocabulary, we will use strong induction to prove the claim that any integer greater or equal to 2 is evenly divisible by a prime number.

Example: divisibility by a prime

- Using strong induction, we will prove the claim that any integer $n \geq 2$ is evenly divisible by a prime number.
- 1. Let $P(n)$ denote the property that n is evenly divisible by a prime number. We will use strong induction on n to show that $P(n)$ holds for all $n \geq 2$.

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Now that we established this vocabulary, we will use strong induction to prove the claim that any integer greater or equal to 2 is evenly divisible by a prime number.

As a first step, we need to denote by $P(n)$ the property that evenly divisible by a prime number. We have to state that we will use strong induction on n to show that $P(n)$ holds for all n greater or equal to 2.

Example: divisibility by a prime

- Using strong induction, we will prove the claim that any integer $n \geq 2$ is evenly divisible by a prime number.
- 1. Let $P(n)$ denote the property that n is evenly divisible by a prime number. We will use strong induction on n to show that $P(n)$ holds for all $n \geq 2$.
- 2. Base case ($n = 2$): The property holds because 2 is a prime number and $2|2$.

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Our base case starts with $n=2$ (not equal to 0, since our property starts at $n=2!$). $P(2)$ holds because 2 is a prime number and 2 evenly divides 2.

Example: divisibility by a prime

- Using strong induction, we will prove the claim that any integer $n \geq 2$ is evenly divisible by a prime number.
- 1. Let $P(n)$ denote the property that n is evenly divisible by a prime number. We will use strong induction on n to show that $P(n)$ holds for all $n \geq 2$.
- 2. Base case ($n = 2$): The property holds because 2 is a prime number and $2|2$.
- 3. Inductive case ($n \geq 2$): Assume the inductive hypothesis $P(2) \wedge P(3) \wedge \dots \wedge P(n)$, that is assume for any integer $n \geq 2$, we have that 2, 3, 4, ..., n are evenly divisible by a prime number.

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The inductive case assumes that for any integer n that is all $P(2)$ AND $P(3)$ AND...AND $P(n)$ are true, that is we assume that 2, 3, 4, ..., n are evenly divisible by a prime number. Given this inductive hypothesis, we must prove that $P(n+1)$ is also evenly divisible by a prime number.

Example: divisibility by a prime

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- 1. Let $P(n)$ denote the property that n is evenly divisible by a prime number. We will use strong induction on n to show that $P(n)$ holds for all $n \geq 2$.
- 2. Base case ($n = 2$): The property holds because 2 is a prime number and $2|2$.
- 3. Inductive case ($n \geq 2$): Assume the inductive hypothesis $P(2) \wedge P(3) \wedge \dots \wedge P(n)$, that is assume for any integer $n \geq 2$, we have that 2, 3, 4, ..., n are evenly divisible by a prime number.
- We must prove $P(n + 1)$, that is we must prove $n + 1$ is also evenly divisible by a prime number.

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Given this inductive hypothesis, we must prove that $P(n+1)$ is also evenly divisible by a prime number.

Example: divisibility by a prime

- $n + 1$ is either prime or composite.

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What do we know about $n+1$? It can either be prime or composite.

Example: divisibility by a prime

- $n + 1$ is either prime or composite.
- If $n + 1$ is prime, then $n + 1$ is evenly divisible by a prime number, namely itself.

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What do we know about $n+1$? It can either be prime or composite. If it is prime, then it is evenly divisible by a prime number, namely itself.

Example: divisibility by a prime

- $n + 1$ is either prime or composite.
- If $n + 1$ is prime, then $n + 1$ is evenly divisible by a prime number, namely itself.
- If $n + 1$ is not prime, then there exists an integer a , $2 \leq a \leq n$ so that $a|(n + 1)$.

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What do we know about $n+1$? It can either be prime or composite. If it is prime, then it is evenly divisible by a prime number, namely itself. The alternative is that it is not a prime number, that is, there is a number a between 2 and n that divides $n+1$.

Example: divisibility by a prime

- $n + 1$ is either prime or composite.
- If $n + 1$ is prime, then $n + 1$ is evenly divisible by a prime number, namely itself.
- If $n + 1$ is not prime, then there exists an integer a , $2 \leq a \leq n$ so that $a|(n + 1)$.
 - By the inductive hypothesis, a is evenly divisible by a prime number p , therefore $n + 1$ is also divisible by p .

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What do we know about $n+1$? It can either be prime or composite. If it is prime, then it is evenly divisible by a prime number, namely itself. The alternative is that it is not a prime number, that is, there is a number a between 2 and n that divides $n+1$. But so far, we have assumed that all numbers up to n are evenly divisible by a prime number. That means that a is evenly divisible by a prime number, let's call it p . That means that $n+1$ is also divisible by p !

Example: divisibility by a prime

- $n + 1$ is either prime or composite.
- If $n + 1$ is prime, then $n + 1$ is evenly divisible by a prime number, namely itself.
- If $n + 1$ is not prime, then there exists an integer a , $2 \leq a \leq n$ so that $a|(n + 1)$.
 - By the inductive hypothesis, a is evenly divisible by a prime number p , therefore $n + 1$ is also divisible by p .
- Hence, regardless of whether $n + 1$ is prime or not, $n + 1$ is divisible by a prime number and we have proven $P(n + 1)$. By the principle of mathematical induction, we have proven our claim $P(n)$ holds for any integer $n \geq 2$.

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Thus, $P(n+1)$ holds since $n+1$ is divisible by a prime number regardless of whether itself is a prime number or a composite. By the principle of mathematical induction, we have proven our claim.

Practice time

- Define a sequence of numbers a_1, a_2, a_3, \dots as follows:
- $a_1 = 0$
- $a_2 = 2$
- $a_n = 3 \times a_{\lfloor \frac{n}{2} \rfloor} + 2$, for all integers $n \geq 3$.
- 1) Calculate a_3, a_4, a_5, a_6, a_7 .
- 2) Let $P(n)$ be the property " a_n is even." Use strong mathematical induction to show that this property holds for all integers $n \geq 1$.

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Your turn now.

Answer – Part 1

- $a_1 = 0$
- $a_2 = 2$
- $a_3 = 3 \times a_{\lfloor \frac{3}{2} \rfloor} + 2 = 3 \times a_1 + 2 = 3 \times 0 + 2 = 2$
- $a_4 = 3 \times a_{\lfloor \frac{4}{2} \rfloor} + 2 = 3 \times a_2 + 2 = 3 \times 2 + 2 = 8$
- $a_5 = 3 \times a_{\lfloor \frac{5}{2} \rfloor} + 2 = 3 \times a_2 + 2 = 3 \times 2 + 2 = 8$
- $a_6 = 3 \times a_{\lfloor \frac{6}{2} \rfloor} + 2 = 3 \times a_3 + 2 = 3 \times 2 + 2 = 8$
- $a_7 = 3 \times a_{\lfloor \frac{7}{2} \rfloor} + 2 = 3 \times a_3 + 2 = 3 \times 2 + 2 = 8$

Answer – Part 2

- 1. Using strong induction, we will prove the property $P(n)$ that a_n is even for all integers $n \geq 1$.
- 2. Base cases ($n = 1, n = 2$): The property holds because $a_1 = 0$ and $a_2 = 2$.

Answer – Part 2

- 3. Inductive case ($n \geq 1$): Assume the inductive hypothesis $P(1) \wedge P(2) \wedge \dots \wedge P(n)$, that is assume for every integer $1, 2, 3, \dots, n, n > 2$, we have that a_1, a_2, \dots, a_n are even
- We must prove $P(n + 1)$, that is we must prove that a_{n+1} is even.
- By definition $a_{n+1} = 3 \times a_{\lfloor \frac{n+1}{2} \rfloor} + 2$ for all integers greater or equal to 3.
- $a_{\lfloor \frac{n+1}{2} \rfloor}$ is even by induction hypothesis, because $n + 1 > 2$ and so $0 \leq \lfloor \frac{n+1}{2} \rfloor < n + 1$.
- Thus, $3 \times a_{\lfloor \frac{n+1}{2} \rfloor}$ is even (because odd \times even = even).
- Hence, $3 \times a_{\lfloor \frac{n+1}{2} \rfloor} + 2$ (because even + even = even). Consequently, a_{n+1} is even!

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Weak vs strong induction

- Anything that can be proven using weak induction can also be shown using strong induction.
- However, if you can prove something using weak induction, you should.