

Mathematical Foundations

# Strong induction

CS51 – Spring 2026

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# Proof by induction

- From last time:
  - To prove that a property  $P(n)$  holds for all non-negative integers  $n$ , we prove that:
    - $P(0)$  is true,
    - for every  $n \geq 0$ , if  $P(n)$  is true, then  $P(n + 1)$  is true, too.
  - The proof of  $P(0)$  is called the **base case**.
  - The proof that  $P(n) \Rightarrow P(n + 1)$  is called the **inductive case**.
  - Practically, the principle of mathematical induction says the following: to prove that a statement  $P(n)$  is true for all non-negative integers  $n$ , we can prove that  $P$  “starts being true” (the base case) and that  $P$  “never stops being true” (the inductive case).

# Proof by strong induction

- To prove that a property  $P(n)$  holds for all non-negative integers  $n$ , we prove that:
  - $P(0)$  is true,
  - for every  $n \geq 0$ , if  $P(0) \wedge P(1) \wedge \dots \wedge P(n)$  is true, then  $P(n + 1)$  is true, too.
- The proof of  $P(0)$  is called the **base case**.
- The proof that  $P(0) \wedge P(1) \wedge \dots \wedge P(n) \Rightarrow P(n + 1)$  is called the **inductive case**.
- Practically, what we saw last time is known as **weak induction**. In weak induction: we show that if  $P$  is true this time, then it will be true next time. We only assume  $P(n)$ .
- In **strong induction**, we show that if  $P$  has been true up until now, then it will be true next time. We assume all of previous steps  $P(0) \wedge P(1) \wedge \dots \wedge P(n)$  to show  $P(n + 1)$ .
- Stronger doesn't mean better, just that we have a stronger inductive hypothesis.

# An aside: floor and ceiling

- The **floor** of a number  $x$ , written as  $\lfloor x \rfloor$  is the largest integer that is less than or equal to  $x$ .
  - For example,  $\lfloor 4.7 \rfloor = 4$ .
- The **ceiling** of a number  $x$ , written as  $\lceil x \rceil$  is the smallest integer that is greater than or equal to  $x$ .
  - For example,  $\lceil 4.7 \rceil = 5$ .

# A reminder: moduli and division

- For any integers  $k > 0$  and  $n$ , the integer  $n \bmod k$  (also denoted as  $n \% k$ ) is the remainder when we divide  $n$  by  $k$ . The value of  $n \bmod k$  is  $n - k \left\lfloor \frac{n}{k} \right\rfloor$ .
  - For example,  $8 \bmod 3 = 2$  because  $8 = 3 \times 2 + 2$  or by definition  $8 - 3 \left\lfloor \frac{8}{3} \right\rfloor = 8 - 3 \times 2 = 2$ .
- For any integer  $k > 0$  and  $n$ , we say that  $k$  evenly divides  $n$  (or  $n$  is evenly divisible by  $k$ ), written  $k|n$ , if  $\frac{n}{k}$  is an integer. Note that  $k|n$  is equivalent to  $n \bmod k = 0$ .
  - For example,  $5|10$  but  $5 \nmid 9$ .
- A positive integer  $n \geq 1$  is **prime** if the only positive integers that evenly divide  $n$  are 1 and  $n$  itself. If a number  $n > 1$  is not prime it is called **composite**.

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- We must prove  $P(n + 1)$ , that is we must prove  $n + 1$  is also evenly divisible by a prime number.

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  - By the inductive hypothesis,  $a$  is evenly divisible by a prime number  $p$ , therefore  $n + 1$  is also divisible by  $p$ .
- Hence, regardless of whether  $n + 1$  is prime or not,  $n + 1$  is divisible by a prime number and we have proven  $P(n + 1)$ . By the principle of mathematical induction, we have proven our claim  $P(n)$  holds for any integer  $n \geq 2$ .

# Practice time

- Define a sequence of numbers  $a_1, a_2, a_3, \dots$  as follows:
- $a_1 = 0$
- $a_2 = 2$
- $a_n = 3 \times a_{\lfloor \frac{n}{2} \rfloor} + 2$ , for all integers  $n \geq 3$ .
- 1) Calculate  $a_3, a_4, a_5, a_6, a_7$ .
- 2) Let  $P(n)$  be the property “ $a_n$  is even.” Use strong mathematical induction to show that this property holds for all integers  $n \geq 1$ .

# Answer – Part 1

- $a_1 = 0$
- $a_2 = 2$
- $a_3 = 3 \times a_{\lfloor \frac{3}{2} \rfloor} + 2 = 3 \times a_1 + 2 = 3 \times 0 + 2 = 2$
- $a_4 = 3 \times a_{\lfloor \frac{4}{2} \rfloor} + 2 = 3 \times a_2 + 2 = 3 \times 2 + 2 = 8$
- $a_5 = 3 \times a_{\lfloor \frac{5}{2} \rfloor} + 2 = 3 \times a_2 + 2 = 3 \times 2 + 2 = 8$
- $a_6 = 3 \times a_{\lfloor \frac{6}{2} \rfloor} + 2 = 3 \times a_3 + 2 = 3 \times 2 + 2 = 8$
- $a_7 = 3 \times a_{\lfloor \frac{7}{2} \rfloor} + 2 = 3 \times a_3 + 2 = 3 \times 2 + 2 = 8$

# Answer – Part 2

- 1. Using strong induction, we will prove the property  $P(n)$  that  $a_n$  is even for all integers  $n \geq 1$ .
- 2. Base cases ( $n = 1, n = 2$ ): The property holds because  $a_1 = 0$  and  $a_2 = 2$ .

# Answer – Part 2

- 3. Inductive case ( $n \geq 1$ ): Assume the inductive hypothesis  $P(1) \wedge P(2) \wedge \dots \wedge P(n)$ , that is assume for every integer  $1, 2, 3, \dots, n, n > 2$ , we have that  $a_1, a_2, \dots, a_n$  are even
- We must prove  $P(n + 1)$ , that is we must prove that  $a_{n+1}$  is even.
- By definition  $a_{n+1} = 3 \times a_{\lfloor \frac{n+1}{2} \rfloor} + 2$  for all integers greater or equal to 3.
- $a_{\lfloor \frac{n+1}{2} \rfloor}$  is even by induction hypothesis, because  $n + 1 > 2$  and so  $0 \leq \lfloor \frac{n+1}{2} \rfloor < n + 1$ .
- Thus,  $3 \times a_{\lfloor \frac{n+1}{2} \rfloor}$  is even (because odd  $\times$  even = even).
- Hence,  $3 \times a_{\lfloor \frac{n+1}{2} \rfloor} + 2$  (because even + even = even). Consequently,  $a_{n+1}$  is even!

# Weak vs strong induction

- Anything that can be proven using weak induction can also be shown using strong induction.
- However, if you can prove something using weak induction, you should.