

Mathematical Foundations

Proof by induction

CS51 – Fall 2025

Today, we will see a new technique for proving that a propositional statement is true called *mathematical induction*. To prove a claim, we will rely on proving a smaller instance of the same claim.

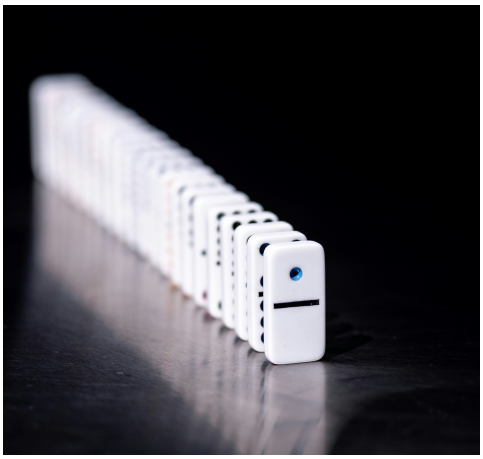
Proof by induction

- To prove that a property $P(n)$ holds for all non-negative integers n, we prove that:
  - $P(0)$ is true,
  - for every $n \geq 0$, if $P(n)$ is true, then $P(n+1)$ is true, too.
- The proof of $P(0)$ is called the **base case**.
- The proof that $P(n) \Rightarrow P(n+1)$ is called the **inductive case**.
- Practically, the principle of mathematical induction says the following: to prove that a statement $P(n)$ is true for all non-negative integers n, we can prove that P "starts being true" (the base case) and that P "never stops being true" (the inductive case).

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The structure of a proof by induction assumes that we have a property P(n) that holds true for all non-negative integers n. We start by proving that P(0) is true, what is known as the base case. And then we assume that for every n that is greater or equal to 0, if the property is true for n then we show that it is true for P(n+1). This s known as the inductive case. The intuition behind mathematical induction is that P start being true and never stops being true, which corresponds to the base and inductive case, respectively.

Proof by induction analogies



- We have an infinitely long line of dominoes, numbered $0, 1, 2, \ldots, n, \ldots$
- To convince someone that the n -th domino falls over, you can convince them that:
  - the 0-th domino falls over, and
  - whenever one domino falls over, the next domino falls over too. (One domino falls, and they keep on falling. Thus, for every $n \geq 0$, the $n+1$ -th domino falls, if we assume that the n-th domino falls.)
- Similar analogies with climbing a ladder or whispering a secret down an alley.

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Here are some analogies to make induction more intuitive:

- Dominoes falling: We have an infinitely long line of dominoes, numbered 0, 1, 2, . . . , n, ... To convince someone that the nth domino falls over, you can convince them that • the 0th domino falls over, and • whenever one domino falls over, the next domino falls over too. (One domino falls, and they keep on falling. Thus, for every n ≥ 0, the n+1-th domino falls, if the n-th domino falls.)
- Climbing a ladder: We have a ladder with rungs numbered 0, 1, 2, . . . , n, ... To convince someone that a climber climbing the ladder reaches the nth rung, you can convince them that • the climber steps onto rung #0. • if the climber steps onto one rung, then she also steps onto the next rung. (The climber starts to climb, and the climber never stops climbing. Thus, for every n ≥ 0, the climber reaches the nth rung.)
- Whispering down the alley: We have an infinitely long line of people, with the people numbered 0, 1, 2, . . . , n, ... To argue that everyone in the line learns a secret, we can argue that • person #0 learns the secret. • if person #n learns the secret, then she tells person #(n + 1) the secret. (The person at the front of the line learns the secret, and everyone who learns it tells the secret to the next person in line. Thus, for every n ≥ 0, the nth

person learns the secret.)

An aside: summation notation

- Let $x_1, x_2, \ldots, x_n$ be a sequence of n numbers. We write $\sum_{i=1}^{n} x_i$ to denote the sum of these n numbers.
- That is $\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$
- We read this as the sum for i equals 1 to n of x_i.
- i is called the index of summation or the index variable.
- There is a correspondence with for-loops like:
  - ```
    result=0
    for i in range(1,n+1):
       result += xn
    return result
    ```
- Note that $\sum_{i=1}^{0} x_i = 0$: when you add nothing together, you end up with zero.

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Before trying together our first proof, I want to talk about the summation notation. Let's assume that we have n numbers, x1, x2, …, xn. We write \sum_{i=1}^{n} x_i to denote the sum of these n numbers, that is x1+x2+…+xn. We read this as the sum for i equals 1 to n of xi. I is called the index of summation or the index variable and you can easily think that the summation notation corresponds to a simple for loop. Note that the sum for I equals 1 to 0 of xi is equal to 0. Since we added nothing together, we ended up with zero.
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Examples: summation notation

- Let $a_1 = 2, a_2 = 4, a_3 = 8, a_4 = 16$. Then:
  - $\sum_{i=2}^{4} a_i$
    - $\sum_{i=2}^{4} a_i = 4 + 8 + 16 = 28$
  - $\sum_{i=1}^{3} (a_i + 1)$
    - $\sum_{i=1}^{3} (a_i + 1) = (2 + 1) + (4 + 1) + (8 + 1) = 17$
  - $\sum_{i=1}^{4} i$
    - $\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$

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Here are a few examples that we will work through together

Practice time: summation notation

- Let $a_1 = 2, a_2 = 4, a_3 = 8, a_4 = 16$. Evaluate the following expressions:
  - $\sum_{i=1}^{4} i^2$
  - $\sum_{i=2}^{4} i^2$
  - $\sum_{i=1}^{4} (a_i + i^2)$
  - $\sum_{i=1}^{4} 5$

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Your turn now to evaluate these sums.

Answer: summation notation

- Let $a_1 = 2, a_2 = 4, a_3 = 8, a_4 = 16$. Evaluate the following expressions:
  - $\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$
  - $\sum_{i=2}^{4} i^2 = 2^2 + 3^2 + 4^2 = 29$
  - $\sum_{i=1}^{4} (a_i + i^2) = (2 + 1^2) + (4 + 2^2) + (8 + 3^2) + (16 + 4^2) = 60$
  - $\sum_{i=1}^{4} 5 = 5 + 5 + 5 + 5 = 20$

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Did you get these?

Example: sum of powers of two

- Using mathematical induction, prove that for any non-negative integer n, we have that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.
- Let's start with a plausibility check to test the formula for small values of n.
  - When $n = 1$: $\sum_{i=0}^{1} 2^i = 2^0 + 2^1 = 3$. $2^{n+1} - 1 = 2^2 - 1 = 3$.
  - When $n = 2$: $\sum_{i=0}^{2} 2^i = 2^0 + 2^1 + 2^2 = 7$. $2^{n+1} - 1 = 2^3 - 1 = 7$.
  - When $n = 3$: $\sum_{i=0}^{3} 2^i = 2^0 + 2^1 + 2^2 + 2^3 = 15$. $2^{n+1} - 1 = 2^4 - 1 = 15$.
- These small claims all check out, so it's reasonable to try to prove the claim.

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Now that we established the notation for summation, let's go about proving using mathematical induction that for any non-negative integer n, the sum of i equals 0 to n of 2^I is equal to 2^{n+1}-1. Before you jump into the proof by induction, it's always a good idea to test the statement by plugging in a few small values for n. For example, we see that it holds for n=1, 2, 3. Since they all check out, it's reasonable to try to prove the claim for any non-negative integer n.

Example: Induction to prove sum of powers of two

- Let $P(n)$ denote the property $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$. We will prove that for every non-negative integer, $P(n)$ is true by induction on n.
- *Base case* $(n = 0)$: We must prove $P(0)$, that is, $\sum_{i=0}^{0} 2^i = 2^{0+1} - 1$.
  This fact is easy to prove, because both sides are equal to 1:
  - $\sum_{i=0}^{0} 2^i = 2^0 = 1$
  - $2^{0+1} - 1 = 2 - 1 = 1$

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We will start by denoting the property by P(n). We have to say that we will do the proof using mathematical induction on n.
For the base case, we must prove P(0) which is easy to prove since both sides are equal to 0. So we proved P(0).

Example: Induction to prove sum of powers of two

- *Inductive case* ($n \geq 0$): We must prove that if $P(n) \Longrightarrow P(n+1)$, for an arbitrary $n \geq 0$.
- We prove this by assuming the antecedent/hypothesis (we assume $P(n)$ and prove $P(n+1)$).
- The assumption $P(n)$ is $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$. (*)
- We can now prove $P(n+1)$ (under the assumption (*)) by showing that the left- and right-hand sides of $P(n+1)$ are equal, that is, we want to prove that $\sum_{i=0}^{n+1} 2^i = 2^{(n+1)+1} - 1$:
- $\sum_{i=0}^{n+1} 2^i = \left[\sum_{i=0}^{n} 2^i\right] + 2^{n+1}$ *by the definition of summation*
  $= [2^{n+1} - 1] + 2^{n+1}$ *by (*)*
  $= 2^{n+1} - 1 + 2^{n+1} = 2{\times}2^{n+1} - 1 = 2^{(n+1)+1} - 1.$ *by algebraic manipulation*
- We have thus shown that $\sum_{i=0}^{n+1} 2^i = 2^{(n+1)+1} - 1$, in other words we have proven $P(n+1)$.
- By the principle of mathematical induction, we have shown $P(n)$ holds for all integers $n \geq 0$.

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Next, we will prove the inductive case for every n>=0. We will stary by assuming that the P(n) holds and prove that it holds for P(n+1). Once we do so, by the principle of mathematical induction, we have shown that P(n) holds for all non-negative integers.

Template for proofs by induction

1. Clearly state the claim to be proven. Clearly state that the proof will be by induction, and clearly state the variable upon which induction will be performed.
2. State and prove the base case.
3. State and prove the inductive case. Within the statement and proof of inductive case:
   a) state the inductive hypothesis
   b) state what we need to prove
   c) Prove it, making use of the inductive hypothesis and stating where it was used.

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In general, whenever you want to prove a claim using mathematical induction, there is a template that you should follow:

1. Clearly state the claim to be proven. Clearly state that the proof will be by induction, and clearly state the variable upon which induction will be performed.
2. State and prove the base case.
3. State and prove the inductive case. Within the statement and proof of inductive case:
   a) state the inductive hypothesis
   b) state what we need to prove
   c) Prove it, making use of the inductive hypothesis and stating where it was used.

Example: Induction to prove sum of powers of -1

- 1. Claim: For any integer $n \geq 0$, we have that $\sum_{i=0}^{n}(-1)^{i} = \begin{cases} 1, & if\ n\ is\ even \\ 0, & if\ n\ is\ odd. \end{cases}$
- Let $P(n)$ denote the property $\sum_{i=0}^{n}(-1)^{i} = \begin{cases} 1, & if\ n\ is\ even \\ 0, & if\ n\ is\ odd. \end{cases}$. We will prove that for every non-negative integer, $P(n)$ is true by induction on n.
- *2. Base case* $(n = 0)$: We must prove $P(0)$. Indeed, $\sum_{i=0}^{0}(-1)^{i} = (-1)^{0} = 1$ and 0 is even.

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Let's go over one more example together where we will prove the claim that the sum of I equals 0 to n of -1^I is either equal to 1, if n is even, or 0 if n is odd. This will be the property P(n) we want to prove by induction. Our base case is P(0). The sum of I equals 0 to 0 of -1^0 is equal to -1^0 equals to 1. Since 0 counts as an even number, indeed the sum is equal to 1 and we proved P(0).

Example: Induction to prove sum of powers of -1

- *3.a. Inductive case* $(n \geq 0)$: We assume the inductive hypothesis $P(n)$ for an arbitrary $n \geq 0$, namely:
- $\sum_{i=0}^{k}(-1)^{i} = \begin{cases} 1, & if\ n\ is\ even \\ 0, & if\ n\ is\ odd \end{cases}$ (*). b. We must prove $P(n+1)$, that is $\sum_{i=0}^{n+1}(-1)^{i} = \begin{cases} 1, & if\ n+1\ is\ even \\ 0, & if\ n+1\ is\ odd \end{cases}$.
- c. $\sum_{i=0}^{n+1}(-1)^{i} = \left[\sum_{i=0}^{n}(-1)^{i}\right] + (-1)^{(n+1)}$ *by the definition of summation*

  $= \begin{cases} 1 + (-1)^{(n+1)}, & if\ n\ is\ even \\ 0 + (-1)^{(n+1)}, & if\ n\ is\ odd \end{cases}$ *by using inductive hypothesis(*)*

  $= \begin{cases} 1 + (-1)^{(n+1)}, & if\ n+1\ is\ odd \\ 0 + (-1)^{(n+1)}, & if\ n+1\ is\ even \end{cases}$ n *is odd* $\Leftrightarrow n+1$ *is even*

  $= \begin{cases} 1 + (-1), & if\ n+1\ is\ odd \\ 0 + (1), & if\ n+1\ is\ even \end{cases}$ $(-1)^{(n+1)} = -1$ *when* $(n+1)$ *is odd,* $(-1)^{(n+1)} = 1$ when $(n+1)$ is even

  $= \begin{cases} 0, & if\ n+1\ is\ odd \\ 1, & if\ n+1\ is\ even \end{cases}$ Thus, we have shown $P(n)$ holds for all integers $n \geq 0$.

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We proceed with proving the inductive case. We start by assuming the inductive hypothesis P(n) for every n>=0 and prove that P(n+1) is also true. Thus, we prove that P(n) holds for all non-negative integers.

Practice time

- You are given the following theorem about the sum of the first n non-negative integers:
- For any integer $n \geq 0$, we have that $\sum_{i=0}^{n} i = 0 + 1 + 2 + \cdots + \mathrm{n} = \frac{n(n+1)}{2}$.
  - For example, for $n = 4$, we have $0 + 1 + 2 + 3 + 4 = 10 = \frac{4(4+1)}{2}$
- Using the principle of mathematic induction and the 3 steps of the template we just saw, prove it.

- This summation is not an arbitrary arithmetic example. You will encounter it again and again when analyzing the time complexity of fundamental algorithms. More to come...

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Your turn to prove a remarkably useful summation that you will face a lot when analyzing the time complexity of fundamental algorithms.

Answer

- 1. Let $P(n)$ denote the property $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$. We will prove that for every non-negative integer, $P(n)$ is true by induction on n.
- *2. Base case* $(n = 0)$: We must prove $P(0)$. Indeed, $\sum_{i=0}^{0} i = 0$ and $\frac{0(0+1)}{2} = 0$, too. Thus $P(0)$ holds.

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Answer

- *3.a. Inductive case* $(n \geq 0)$: We assume the inductive hypothesis $P(n)$ for an arbitrary $n \geq 0$, namely:
- $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ (*). b. We must prove $P(n+1)$, that is $\sum_{i=0}^{n+1} i = \frac{(n+1)((n+1)+1)}{2}$.
- c. $\sum_{i=0}^{n+1} i = [\sum_{i=0}^{n} i] + (n+1)$ *by the definition of summation*
  $= \left[\frac{n(n+1)}{2}\right] + (n+1)$. *by using inductive hypothesis(*)*
  $= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}$ *using algebraic manipulations*

  Thus, we have shown $P(n)$ holds for all integers $n \geq 0$.

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Loop invariants and induction

- The parallels between loop invariants and induction are pretty direct. A loop invariant is a property that remains true at the beginning and end of each iteration of a loop. The first corresponds to the base case and the second to the inductive case.

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If you think about it, the parallels between loop invariants and induction are pretty direct. A loop invariant is a property that remains true at the beginning and end of each iteration of a loop. The first corresponds to the base case and the second to the inductive case.