

In-Class Worksheet

CS 181 Advanced Algorithms — Spring 2026

Hall's theorem: Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. Then G has a perfect matching if and only if $\forall S \subseteq L$ we have $|N(S)| \geq |S|$.

Example 1: Consider a standard French deck of cards, with 4 suits and 13 values per suit, and shuffle it randomly. Deal 13 different piles, each pile containing 4 cards, the cards being face up. Show that you can always select exactly one card from each pile such that the 13 selected cards have the values A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K.

Example 2: Let $r, n \in \mathbb{N}$ with $r \leq n$. An $r \times n$ matrix M is a *Latin rectangle* of size (r, n) if all entries are integers from 1 to n and no number appears twice in any row or column.

So for example, if $r = 3$ and $n = 5$, here are some Latin rectangles:

$$\begin{pmatrix} 1 & 2 & 3 & 5 & 4 \\ 2 & 3 & 5 & 4 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 4 & 5 & 2 \\ 4 & 1 & 5 & 2 & 3 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}.$$

These are not valid Latin rectangles:

$$\begin{pmatrix} 1 & 2 & 5 & 3 & 4 \\ 2 & 3 & 5 & 4 & 1 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 4 & 5 & 1 \\ 4 & 1 & 5 & 2 & 3 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}.$$

- (a) A Latin square is a Latin rectangle where $r = n$. Give two different Latin squares of size 3×3
- (b) Let M be a Latin rectangle of size (r, n) with $r < n$. Show that one can add a row so the $(r + 1, n)$ rectangle is also Latin. Repeatedly doing this shows that any Latin rectangle of size (r, n) can be completed to an n by n Latin square.