PROBABILITY

David Kauchak
CS159 – Spring 2023

1

Admin

Assignment 1 advice

- test individual components of your regex first, then put them all together
- write test cases

Office hours posted

Mentor hours posted

2

Why probability?

Prostitutes Appeal to Pope

Language is ambiguous

Probability theory gives us a tool to model this ambiguity in reasonable ways.

3

Basic Probability Theory: terminology

An **experiment** has a set of potential outcomes, e.g., throw a dice, "look at" another sentence

The **sample space** of an experiment is the set of all possible outcomes, e.g., {1, 2, 3, 4, 5, 6}

In NLP our sample spaces tend to be ***very*** large

- All words, bigrams, 5-grams
- All sentences of length 20 (given a finite vocabulary)
- All sentences
- All parse trees over a given sentence

4

Basic Probability Theory: terminology

An **event** is a subset of the sample space

Dice rolls
- {2}
- {3, 6}
- even = {2, 4, 6}
- odd = {1, 3, 5}

NLP
- a particular word/part of speech occurring in a sentence
- a particular topic discussed (politics, sports)
- sentence with a parasitic gap
- pick your favorite phenomena…

5

Events

We're interested in probabilities of events
- p({2})
- p(even)
- p(odd)
- p(parasitic gap)
- p(first word in a sentence is "banana")

6

Random variables

A random variable is a mapping from the sample space to a number (think events)

It represents all the possible values of something we want to measure in an experiment

For example, random variable, *X*, could be the number of heads for a coin tossed three times

space	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
X	3	2	2	1	2	1	1	0

Really for notational convenience, since the event space can sometimes be irregular

7

Random variables

We can then talk about the probability of the different values of a random variable

The definition of probabilities over *all* of the possible values of a random variable defines a **probability distribution**

space	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
X	3	2	2	1	2	1	1	0

X	P(X)
3	P(X=3) = **?**
2	P(X=2) = **?**
1	P(X=1) = **?**
0	P(X=0) = **?**

8

Random variables

We can then talk about the probability of the different values of a random variable

The definition of probabilities over *all* of the possible values of a random variable defines a **probability distribution**

space	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
X	3	2	2	1	2	1	1	0

X	P(X)
3	P(X=3) = 1/8
2	P(X=2) = 3/8
1	P(X=1) = 3/8
0	P(X=0) = 1/8

9

Probability distribution

To be explicit

- A probability distribution assigns probability values to all possible values of a random variable
- These values must be >= 0 and <= 1
- These values must sum to 1 for all possible values of the random variable

X	P(X)
3	P(X=3) = 1/2
2	P(X=2) = 1/2
1	P(X=1) = 1/2
0	P(X=0) = 1/2

X	P(X)
3	P(X=3) = -1
2	P(X=2) = 2
1	P(X=1) = 0
0	P(X=0) = 0

10

Unconditional/prior probability

Simplest form of probability distribution is

- P(X)

Prior probability: without any additional information:

- What is the probability of heads on a coin toss?
- What is the probability of a sentence containing a pronoun?
- What is the probability of a sentence containing the word "banana"?
- What is the probability of a document discussing politics?
- …

11

Prior probability

What is the probability of getting HHH for three coin tosses, assuming a fair coin?

1/8

What is the probability of getting THT for three coin tosses, assuming a fair coin?

1/8

12

Joint distribution

We can also talk about probability distributions over multiple variables

P(X,Y)

- probability of X *and* Y
- a distribution over the cross product of possible values

NLPPass	P(NLPPass)
true	0.89
false	0.11

EngPass	P(EngPass)
true	0.92
false	0.08

NLPPass, EngPass	P(NLPPass, EngPass)
true, true	.88
true, false	.01
false, true	.04
false, false	.07

13

Joint distribution

Still a probability distribution

- all values between 0 and 1, inclusive
- all values sum to 1

All questions/probabilities of the two variables can be calculated from the joint distribution

NLPPass, EngPass	P(NLPPass, EngPass)
true, true	.88
true, false	.01
false, true	.04
false, false	.07

What is P(ENGPass)?

14

Joint distribution

Still a probability distribution

- all values between 0 and 1, inclusive
- all values sum to 1

All questions/probabilities of the two variables can be calculated from the joint distribution

NLPPass, EngPass	P(NLPPass, EngPass)
true, true	.88
true, false	.01
false, true	.04
false, false	.07

0.92

How did you figure that out?

15

Joint distribution

$$P(x) = \sum_{y \in Y} p(x,y)$$

Called “marginalization”, aka summing over a variable

NLPPass, EngPass	P(NLPPass, EngPass)
true, true	.88
true, false	.01
false, true	.04
false, false	.07

16

Conditional probability

As we learn more information, we can update our probability distribution

P(X|Y) models this (read "probability of X *given* Y")

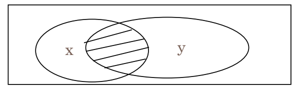
- What is the probability of heads *given* that both sides of the coin are heads?
- What is the probability the document is about politics, given that it contains the word "Clinton"?
- What is the probability of the word "banana" given that the sentence also contains the word "split"?

Notice that it is still a distribution over the values of X

17

Conditional probability

$$p(X \mid Y) = \ ?$$

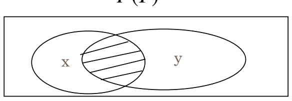


In terms of pior and joint distributions, what is the conditional probability distribution?

18

Conditional probability

$$p(X \mid Y) = \frac{P(X,Y)}{P(Y)}$$

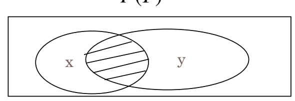


Given that y has happened, in what proportion of those events does x also happen

19

Conditional probability

$$p(X \mid Y) = \frac{P(X,Y)}{P(Y)}$$



Given that y has happened, what proportion of those events does x also happen

NLPPass, EngPass	P(NLPPass, EngPass)
true, true	.88
true, false	.01
false, true	.04
false, false	.07

What is: p(NLPPass=true | EngPass=false)?

20

Conditional probability

NLPPass, EngPass	P(NLPPass, EngPass)
true, true	.88
true, false	.01
false, true	.04
false, false	.07

$$p(X \mid Y) = \frac{P(X,Y)}{P(Y)}$$

What is:
p(NLPPass=true | EngPass=false)?

$$\frac{P(true, false) = 0.01}{P(EngPass = false) = 0.01 + 0.07 = 0.08} = 0.125$$

Notice this is very different than p(NLPPass=true) = 0.89

21

A note about notation

When talking about a particular assignment, you should technically write p(X=x), etc.

However, when it's clear, we'll often shorten it

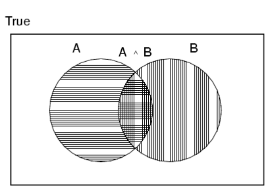
Also, we may also say P(X) or p(x) to generically mean any particular value, i.e. P(X=x)

$$\frac{P(true, false) = 0.01}{P(EngPass = false) = 0.01 + 0.07 = 0.08} = 0.125$$

22

Properties of probabilities

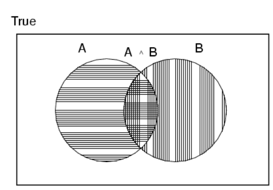
P(*A or B*) = ?



23

Properties of probabilities

P(*A or B*) = P(*A*) + P(*B*) - P(*A*,*B*)



24

Properties of probabilities

$P(\neg E) = 1 - P(E)$

More generally:

- Given events $E = e_1, e_2, \ldots, e_n$

$$p(e_i) = 1 - \sum_{j=1:n, j \neq i} p(e_j)$$

$P(E1, E2) \leq P(E1)$

25

Chain rule (aka product rule)

$$p(X \mid Y) = \frac{P(X,Y)}{P(Y)} \Rightarrow p(X,Y) = P(X \mid Y)P(Y)$$

We can view calculating the probability of X *AND* Y occurring as two steps:

1. Y occurs with some probability P(Y)
2. Then, X occurs, given that Y has occurred

or you can just trust the math… ☺

26

Chain rule

$$p(X,Y,Z) = P(X \mid Y,Z)P(Y,Z)$$

$$p(X,Y,Z) = P(X,Y \mid Z)P(Z)$$

$$p(X,Y,Z) = P(X \mid Y,Z)P(Y \mid Z)P(Z)$$

$$p(X,Y,Z) = P(Y,Z \mid X)P(X)$$

$$p(X_1, X_2, \ldots, X_n) = \ ?$$

27

Applications of the chain rule

We saw that we could calculate the individual prior probabilities using the joint distribution

$$p(x) = \sum_{y \in Y} p(x,y)$$

What if we don't have the joint distribution, but do have conditional probability information:

- P(Y)
- P(X|Y)

$$p(x) = \sum_{y \in Y} p(y) p(x \mid y)$$

28

Bayes' rule (theorem)

$$p(X \mid Y) = \frac{P(X,Y)}{P(Y)} \Rightarrow p(X,Y) = P(X \mid Y)P(Y)$$

$$p(Y \mid X) = \frac{P(X,Y)}{P(X)} \Rightarrow p(X,Y) = P(Y \mid X)P(X)$$

$$p(X \mid Y) = \frac{P(Y \mid X)P(X)}{P(Y)}$$

29

Obtaining probabilities

We've talked a lot about probabilities, but not where they come from

How do we calculate:
- the probability of heads?
- the probability that a sentence contains a pronoun?
- the probability that a sentence contains "banana", given that it also contains the word split?

30

Estimating probabilities

What is the probability of a sentence contains a pronoun?

We don't know!

We can ***estimate*** it based on data, though:

$$\frac{\text{number of sentences with a pronoun}}{\text{total number of sentences}}$$

This is called the maximum likelihood estimation. Why?

31

Maximum likelihood estimates

$$p(y) = \frac{count(y)}{n} \qquad \frac{\text{number of examples with thing y}}{\text{total number of examples}}$$

$$p(x|y) = \frac{count(x,y)}{count(y)} \qquad \frac{\text{number of examples with thing y and thing x}}{\text{number of examples with thing y}}$$

32

Bayes rule

Allows us to talk about P(Y|X) rather than P(X|Y)

Sometimes this can be more intuitive

Why?

$$p(X \mid Y) = \frac{P(Y \mid X)P(X)}{P(Y)}$$

33

Bayes rule

p(disease | symptoms)

How would you estimate this?

$$\frac{\text{count(disease, symptoms)}}{\text{count(symptoms)}}$$

Find a bunch of people with those symptoms and see how many have the disease

Is this feasible?

34

Bayes rule

p(disease | symptoms) $\propto$ p(symptoms | disease)

How would you estimate this?

Find a bunch of people with the disease and see how many have this set of symptoms. Much easier!

$$\frac{\text{count(disease, symptoms)}}{\text{count(disease)}}$$

35

Bayes rule

p(linguistic phenomena | features)

- For all examples that had those features, how many had that phenomena?
- p(features | linguistic phenomena)
  - For all the examples with that phenomena, how many had this feature

p(cause | effect) vs. p(effect | cause)

36

Gaps

V

I just won't put these away.

direct object

These, I just won't put away.

filler

I just won't put ___ away.

gap

37

Gaps

What did you put ___ away?

gap

The socks that I put ___ away.

gap

38

Gaps

Whose socks did you fold ___ and put ___ away?

gap gap

Whose socks did you fold ___ ?

gap

Whose socks did you put ___ away?

gap

39

Parasitic gaps

These I'll put ___ away without folding ___ .

gap gap

These I'll put ___ away.

gap

These without folding ___ .

gap

40

Parasitic gaps

These I'll put ____ away without folding ____ .
gap gap

1. Cannot exist by themselves (parasitic)

These I'll put my pants away without folding ____ .
gap

2. They're optional

These I'll put ____ away without folding them.
gap

41

Parasitic gaps

http://literalminded.wordpress.com/2009/02/10/dougs-parasitic-gap/

42

Frequency of parasitic gaps

Parasitic gaps occur on average in 1/100,000 sentences

Problem:

You have developed a complicated set of regular expressions to try and identify parasitic gaps. If a sentence has a parasitic gap, it correctly identifies it 95% of the time. If it doesn't, it will incorrectly say it does with probability 0.005. Suppose we run it on a sentence and the algorithm says it has a parasitic gap, what is the probability it actually is?

43

Prob of parasitic gaps

You have developed a complicated set of regular expressions to try and identify parasitic gaps. If a sentence has a parasitic gap, it correctly identifies it 95% of the time. If it doesn't, it will incorrectly say it does with probability 0.005. Suppose we run it on a sentence and the algorithm says it has a parasitic gap, what is the probability it actually does?

G = gap
T = test positive

What question do we want to ask?

44

Prob of parasitic gaps

You have developed a complicated set of regular expressions to try and identify parasitic gaps. If a sentence has a parasitic gap, it correctly identifies it 95% of the time. If it doesn't, it will incorrectly say it does with probability 0.005. Suppose we run it on a sentence and the algorithm says it has a parasitic gap, what is the probability it actually does?

G = gap
T = test positive

$$p(g \mid t) = ?$$

45

Prob of parasitic gaps

You have developed a complicated set of regular expressions to try and identify parasitic gaps. If a sentence has a parasitic gap, it correctly identifies it 95% of the time. If it doesn't, it will incorrectly say it does with probability 0.005. Suppose we run it on a sentence and the algorithm says it has a parasitic gap, what is the probability it actually does?

G = gap
T = test positive

$$p(g \mid t) = \frac{p(t \mid g)p(g)}{p(t)}$$

$$= \frac{p(t \mid g)p(g)}{\sum_{g \in G} p(g)p(t \mid g)} = \frac{p(t \mid g)p(g)}{p(g)p(t \mid g) + p(\overline{g})p(t \mid \overline{g})}$$

46

Prob of parasitic gaps

You have developed a complicated set of regular expressions to try and identify parasitic gaps. If a sentence has a parasitic gap, it correctly identifies it 95% of the time. If it doesn't, it will incorrectly say it does with probability 0.005. Suppose we run it on a sentence and the algorithm says it has a parasitic gap, what is the probability it actually does?

G = gap
T = test positive

$$p(g \mid t) = \frac{p(t \mid g)p(g)}{p(g)p(t \mid g) + p(\overline{g})p(t \mid \overline{g})}$$

$$= \frac{0.95 * 0.00001}{0.00001 * 0.95 + 0.99999 * 0.005} \approx 0.002$$

47