LARGE MARGIN CLASSIFIERS

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CS 158 – Fall 2023

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Admin

- Assignment 5
  - Experiments
- Assignment 6: due *Friday* (10/13)
- **Next class: Meet in Edmunds 105**
- Midterm: out and due by the end of the day Friday
- Course feedback
  - Thanks!
  - We'll go over it at the beginning of next class

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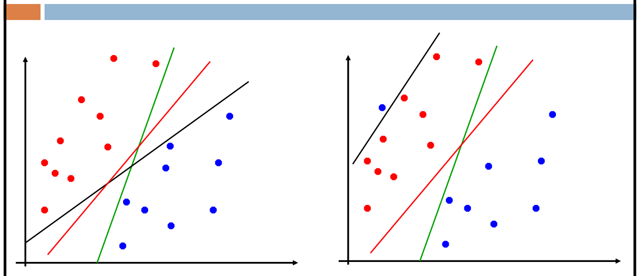
Which hyperplane?



Two main variations in linear classifiers:
- which hyperplane they choose when the data is linearly separable
- how they handle data that is not linearly separable

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Linear approaches so far

Perceptron:
- separable:
- non-separable:

Gradient descent:
- separable:
- non-separable:

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Linear approaches so far

Perceptron:
- separable:
  - finds ***some*** hyperplane that separates the data
- non-separable:
  - will continue to adjust as it iterates through the examples
  - final hyperplane will depend on which examples it saw recently

Gradient descent:
- separable and non-separable
  - finds the hyperplane that minimizes the objective function (loss + regularization)

Which hyperplane is this?

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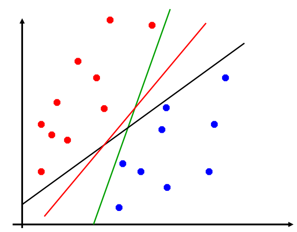
Which hyperplane would you choose?



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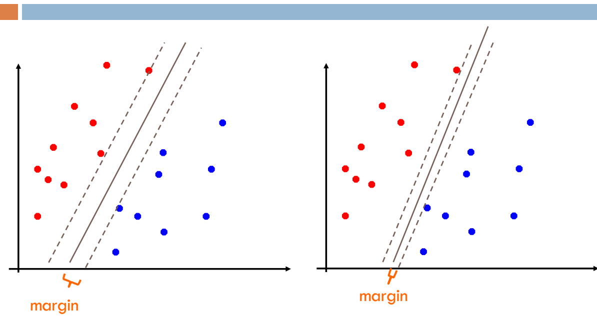
Large margin classifiers



Choose the line where the distance to the nearest point(s) is as large as possible

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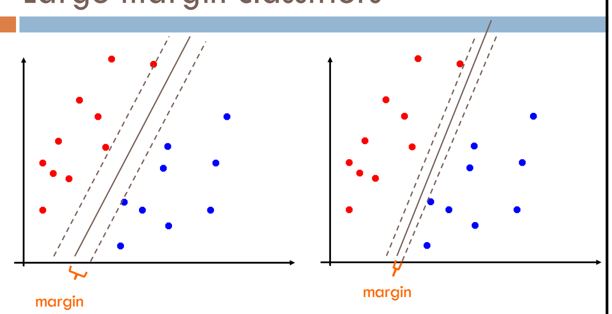
Large margin classifiers



The margin of a classifier is the distance to the closest points of either class

Large margin classifiers attempt to maximize this

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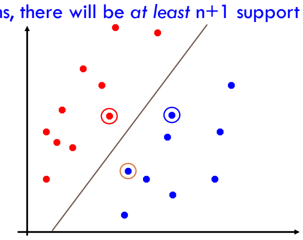
Support vectors

For any separating hyperplane, there exist some set of "closest points"

These are called the support vectors

For n dimensions, there will be *at least* n+1 support vectors



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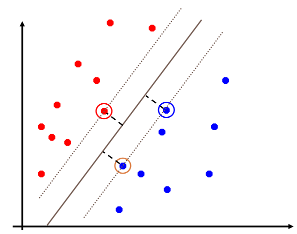
Measuring the margin

The margin is the distance to the support vectors, i.e. the "closest points", on either side of the hyperplane



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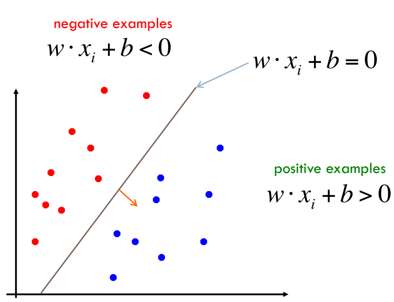
Measuring the margin



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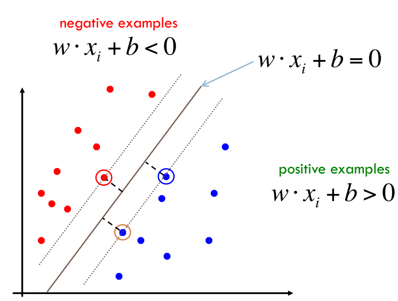
Measuring the margin

What are the equations for the margin lines?



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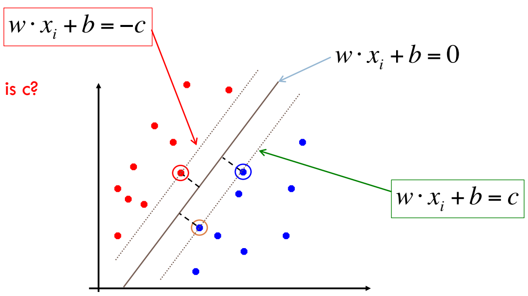
Measuring the margin

We know they're the same distance apart (otherwise, they wouldn't be support vectors!)

What is c?



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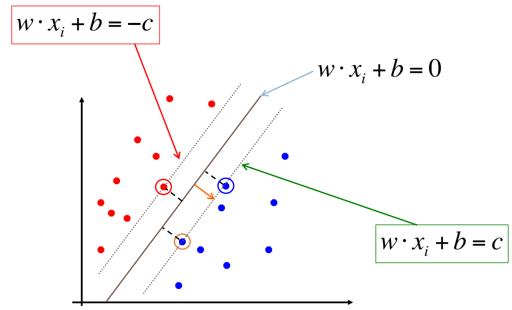
Measuring the margin

Depends! If we scale w, we vary the constant without changing the separating hyperplane



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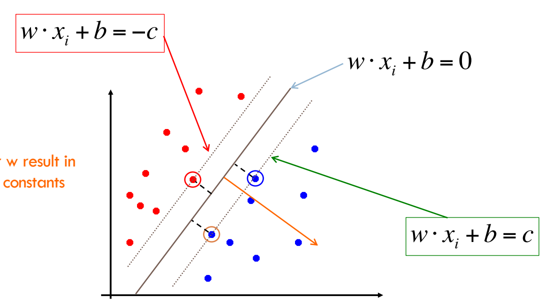
Measuring the margin

Depends! If we scale w, we vary the constant without changing the separating hyperplane

Larger w result in larger constants



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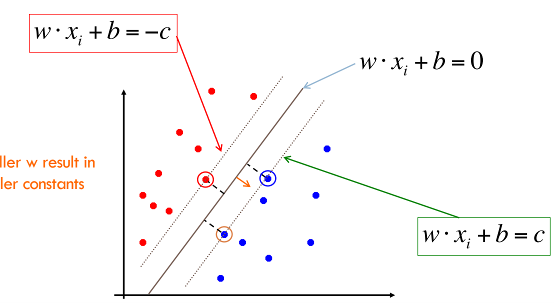
Measuring the margin

Depends! If we scale w, we vary the constant without changing the separating hyperplane

Smaller w result in smaller constants



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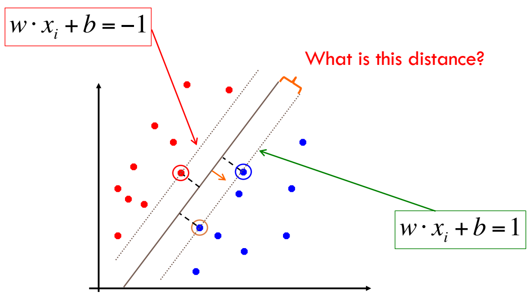
Measuring the margin

For now, let's assume c = 1.



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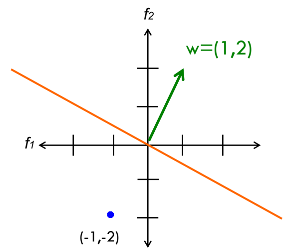
Distance from the hyperplane

How far away is this point from the hyperplane?



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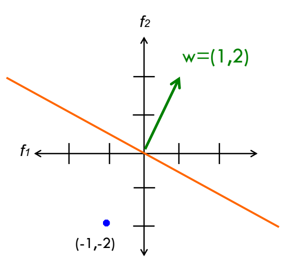
Distance from the hyperplane

How far away is this point from the hyperplane?



$$d = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$$

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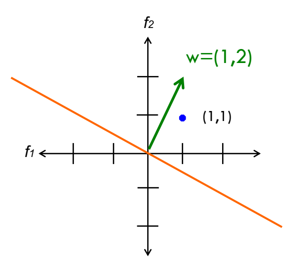
Distance from the hyperplane

How far away is this point from the hyperplane?



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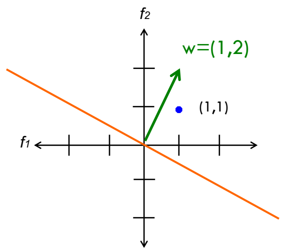
Distance from the hyperplane

How far away is this point from the hyperplane?



Is it?

$$d(x) = w \cdot x + b$$

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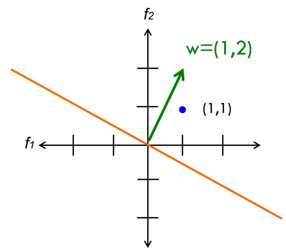
Distance from the hyperplane

Does that seem right? What's the problem?



$$d(x) = w \cdot x + b$$

$$= w_1 x_1 + w_2 x_2 + b$$

$$= 1 * 1 + 1 * 2 + 0$$

$$\boxed{= 3?}$$

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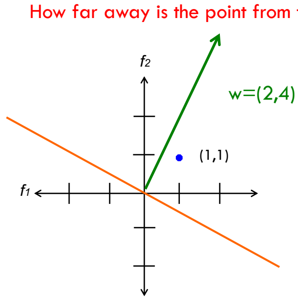
Distance from the hyperplane

How far away is the point from the hyperplane?



$$d(x) = w \cdot x + b$$

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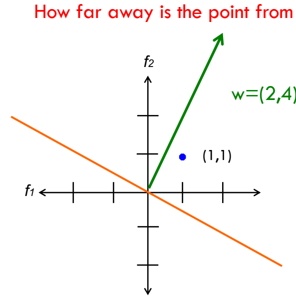
Distance from the hyperplane

How far away is the point from the hyperplane?



$$d(x) = w \cdot x + b$$

$$= w_1 x_1 + w_2 x_2 + b$$

$$= 2 * 1 + 4 * 1$$

$$= 6?$$

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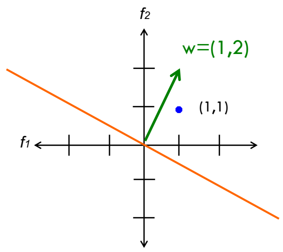
Distance from the hyperplane

How far away is this point from the hyperplane?



$$d(x) = \frac{w \cdot x + b}{\|w\|}$$

length normalized weight vectors

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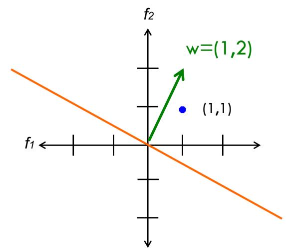
Distance from the hyperplane

How far away is this point from the hyperplane?



$$d(x) = \frac{w \cdot x + b}{\|w\|}$$

$$= \frac{(w_1 x_1 + w_2 x_2) + b}{\sqrt{5}}$$

$$= \frac{(1*1 + 1*2) + 0}{\sqrt{5}}$$

$$= 1.34$$

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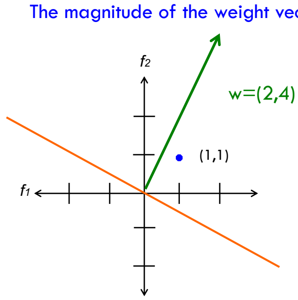
Distance from the hyperplane

The magnitude of the weight vector doesn't matter



$$d(x) = \frac{w \cdot x + b}{\|w\|}$$

length normalized weight vectors

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Distance from the hyperplane

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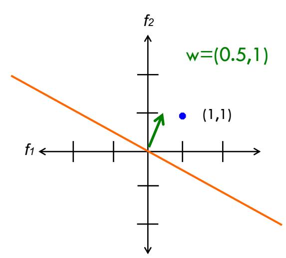

$$d(x) = \frac{w \cdot x + b}{\|w\|}$$

length normalized weight vectors

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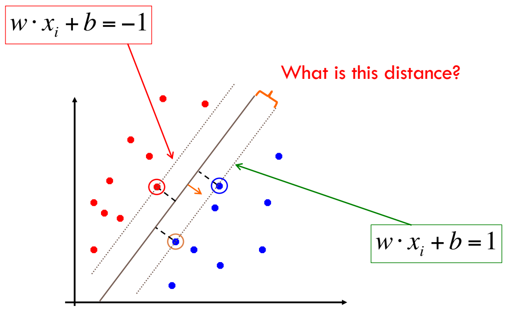
Measuring the margin

For now, let's just assume c = 1.



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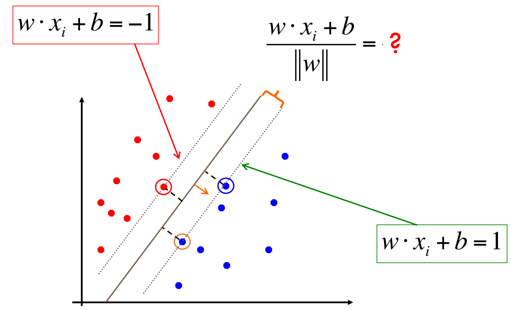

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Measuring the margin

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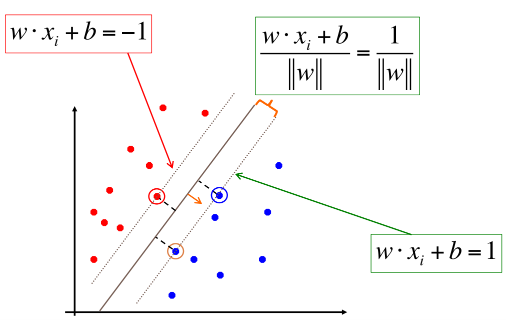

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Large margin classifier setup

Select the hyperplane with the largest margin where the points are classified correctly *and outside the margin!*

Setup as a constrained optimization problem:

$$\max_{w,b} \quad \text{margin}(w,b)$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 \quad \forall i$$

what does this say?

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Large margin classifier setup

Select the hyperplane with the largest margin where the points are classified correctly *and outside the margin!*

Setup as a constrained optimization problem:

$$\max_{w,b} \frac{1}{\|w\|}$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 \quad \forall i$$

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Maximizing the margin

$$\min_{w,b} \|w\|$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 \quad \forall i$$

Maximizing the margin is equivalent to minimizing $\|w\|$!
(subject to the separating constraints)

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Maximizing the margin

The minimization criterion wants w to be as small as possible

$$\min_{w,b} \|w\|$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 \quad \forall i$$

The constraints:

1. make sure the data is separable
2. encourages w to be larger (once the data is separable)

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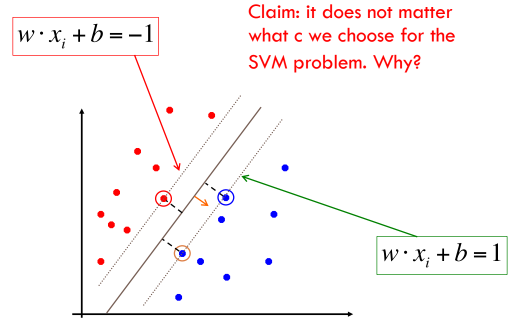
Measuring the margin

For now, let's just assume c = 1.

Claim: it does not matter what c we choose for the SVM problem. Why?



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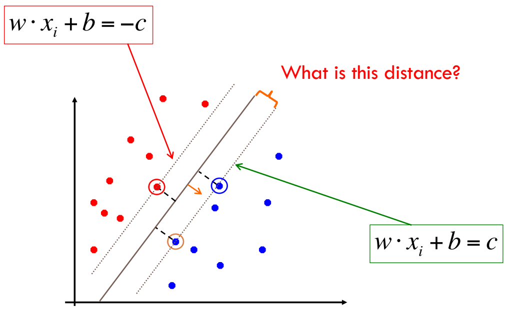
Measuring the margin



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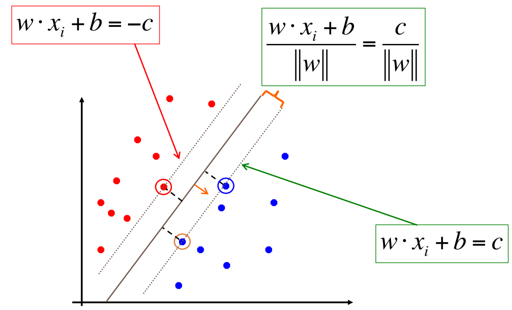
Measuring the margin



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Maximizing the margin

$$\min_{w,b} \frac{\|w\|}{c}$$

subject to:

$$y_i(w \cdot x_i + b) \geq c \quad \forall i$$

vs.

What's the difference?

$$\min_{w,b} \|w\|$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 \quad \forall i$$

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Maximizing the margin

$$\min_{w,b} \frac{\|w\|}{c}$$

subject to:

$$y_i(w \cdot x_i + b) \geq c \quad \forall i$$

vs.

$$\min_{w,b} \|w\|$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 \quad \forall i$$

Learn the exact same hyperplane just scaled by a constant amount

Because of this, often see it with c = 1

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For those that are curious…

$$\frac{\|w\|}{c} = \frac{\sqrt{w_1^2 + w_2^2 + \ldots + w_m^2 + b^2}}{c}$$
$$= \sqrt{\left(\frac{\sqrt{w_1^2 + w_2^2 + \ldots + w_m^2}}{c}\right)^2}$$
$$= \sqrt{\frac{w_1^2 + w_2^2 + \ldots + w_m^2}{c^2}}$$
$$= \sqrt{\frac{w_1^2}{c^2} + \frac{w_2^2}{c^2} + \ldots + \frac{w_m^2}{c^2}}$$
$$= \sqrt{\left(\frac{w_1}{c}\right)^2 + \left(\frac{w_2}{c}\right)^2 + \ldots + \left(\frac{w_m}{c}\right)^2}$$

scaled version of w

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Maximizing the margin: the real problem

$$\min_{w,b} \quad \|w\|^2$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 \quad \forall i$$

Why the squared?

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Maximizing the margin: the real problem

$$\min_{w,b} \quad \|w\| = \sqrt{\sum_i w_i^{\;2}}$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 \quad \forall i$$

$$\min_{w,b} \quad \|w\|^2 = \sum_i w_i^{\;2}$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 \quad \forall i$$

Minimizing $||\mathbf{w}||$ is equivalent to minimizing $||\mathbf{w}||^2$

The sum of the squared weights is a convex function!

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Support vector machine problem

$$\min_{w,b} \quad \|w\|^2$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 \quad \forall i$$

This is a version of a quadratic optimization problem

Maximize/minimize a quadratic function

Subject to a set of linear constraints

Many, many variants of solving this problem (we'll see one in a bit)

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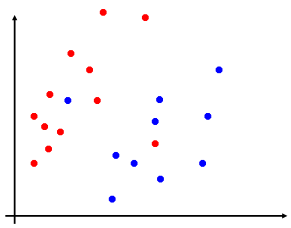
Soft Margin Classification



$$\min_{w,b} \; \|w\|^2$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 \quad \forall i$$

What about this problem?

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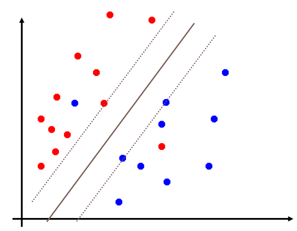
Soft Margin Classification



$$\min_{w,b} \; \|w\|^2$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 \quad \forall i$$

We'd like to learn something like this, but our constraints won't allow it ☹

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Slack variables

$$\min_{w,b} \; \|w\|^2$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 \quad \forall i$$

⬇

$$\min_{w,b} \; \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 - \varsigma_i \quad \forall i$$

$$\varsigma_i \geq 0$$

slack variables (one for each example)

What effect does this have?

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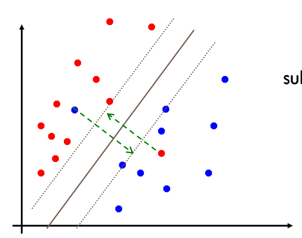
Slack variables



$$\min_{w,b} \; \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 - \varsigma_i \quad \forall i$$

$$\varsigma_i \geq 0$$

slack penalties

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Slack variables

margin

trade-off between margin maximization and penalization

$$\min_{w,b} \; \|w\|^2 + C\sum_i \varsigma_i$$

penalized by how far from "correct"

subject to:

$$y_i(w \cdot x_i + b) \geq 1 - \varsigma_i \quad \forall i$$

allowed to make a mistake

$$\varsigma_i \geq 0$$

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Soft margin SVM

$$\min_{w,b} \; \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 - \varsigma_i \quad \forall i$$

$$\varsigma_i \geq 0$$

Still a quadratic optimization problem!

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Demo

https://cs.stanford.edu/~karpathy/svmjs/demo/

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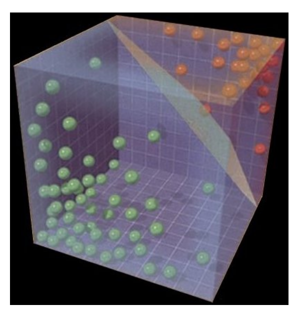
Solving the SVM problem



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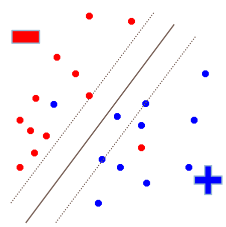
Understanding the Soft Margin SVM



$$\min_{w,b} \; \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \ge 1 - \varsigma_i \quad \forall i$$
$$\varsigma_i \ge 0$$

Given the optimal solution, w, b:

Can we figure out what the slack penalties are for each point?

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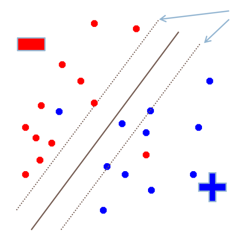
Understanding the Soft Margin SVM



What do the margin lines represent wrt w,b?

$$\min_{w,b} \; \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \ge 1 - \varsigma_i \quad \forall i$$
$$\varsigma_i \ge 0$$

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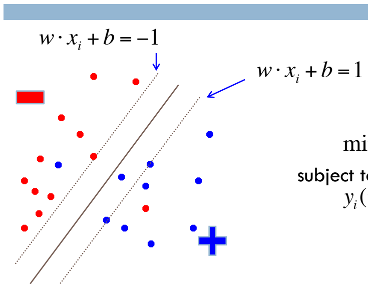
Understanding the Soft Margin SVM



$$\min_{w,b} \; \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \ge 1 - \varsigma_i \quad \forall i$$
$$\varsigma_i \ge 0$$

Or: $y_i(w \cdot x_i + b) = 1$

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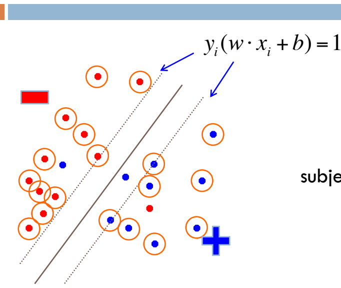
Understanding the Soft Margin SVM



$$\min_{w,b} \; \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \ge 1 - \varsigma_i \quad \forall i$$
$$\varsigma_i \ge 0$$

What are the slack values for points outside (or on) the margin AND correctly classified?

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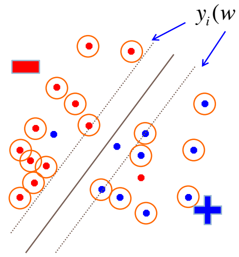
Understanding the Soft Margin SVM

$y_i(w \cdot x_i + b) = 1$



$$\min_{w,b} \ \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 - \varsigma_i \quad \forall i$$
$$\varsigma_i \geq 0$$

0! The slack variables have to be greater than or equal to zero and if they're on or beyond the margin then $y_i(wx_i+b) \geq 1$ already

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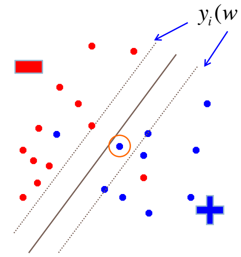
Understanding the Soft Margin SVM

$y_i(w \cdot x_i + b) = 1$



$$\min_{w,b} \ \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 - \varsigma_i \quad \forall i$$
$$\varsigma_i \geq 0$$

What are the slack values for points inside the margin AND classified correctly?

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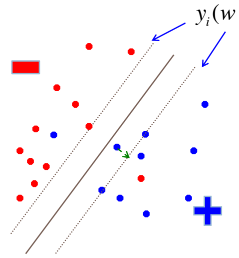
Understanding the Soft Margin SVM

$y_i(w \cdot x_i + b) = 1$



$$\min_{w,b} \ \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 - \varsigma_i \quad \forall i$$
$$\varsigma_i \geq 0$$

Difference from the point to the margin. Which is?

$$\varsigma_i = 1 - y_i(w \cdot x_i + b)$$

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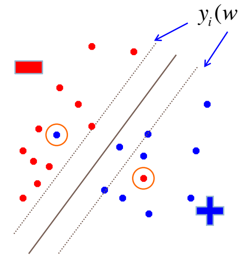
Understanding the Soft Margin SVM

$y_i(w \cdot x_i + b) = 1$



$$\min_{w,b} \ \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 - \varsigma_i \quad \forall i$$
$$\varsigma_i \geq 0$$

What are the slack values for points that are incorrectly classified?

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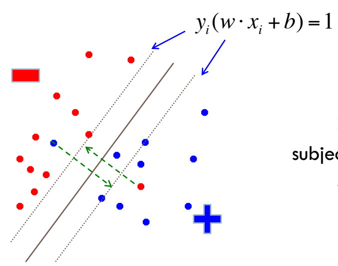
Understanding the Soft Margin SVM



$$\min_{w,b} \; \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 - \varsigma_i \quad \forall i$$
$$\varsigma_i \geq 0$$

Which is?

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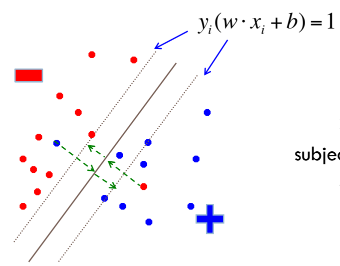
Understanding the Soft Margin SVM



$$\min_{w,b} \; \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 - \varsigma_i \quad \forall i$$
$$\varsigma_i \geq 0$$

"distance" to the hyperplane *plus* the "distance" to the margin

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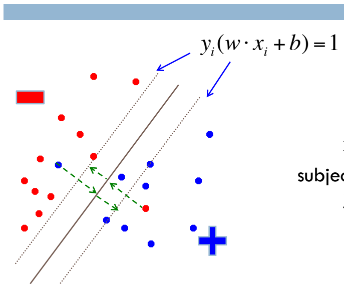
Understanding the Soft Margin SVM



$$\min_{w,b} \; \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 - \varsigma_i \quad \forall i$$
$$\varsigma_i \geq 0$$

"distance" to the hyperplane *plus* the "distance" to the margin

?

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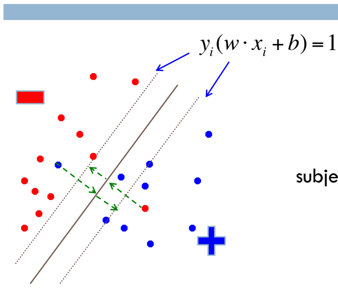
Understanding the Soft Margin SVM



$$\min_{w,b} \; \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 - \varsigma_i \quad \forall i$$
$$\varsigma_i \geq 0$$

"distance" to the hyperplane *plus* the "distance" to the margin

$-y_i(w \cdot x_i + b)$ Why -?

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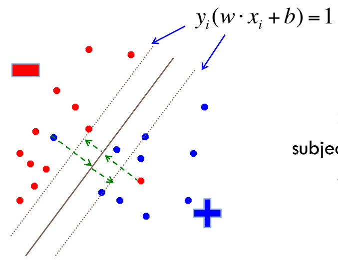
Understanding the Soft Margin SVM



$$\min_{w,b} \quad \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 - \varsigma_i \quad \forall i$$
$$\varsigma_i \geq 0$$

"distance" to the hyperplane *plus* the "distance" to the margin

$-y_i(w \cdot x_i + b)$?

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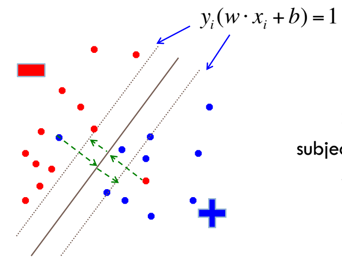
Understanding the Soft Margin SVM



$$\min_{w,b} \quad \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 - \varsigma_i \quad \forall i$$
$$\varsigma_i \geq 0$$

"distance" to the hyperplane *plus* the "distance" to the margin

$-y_i(w \cdot x_i + b)$ 1

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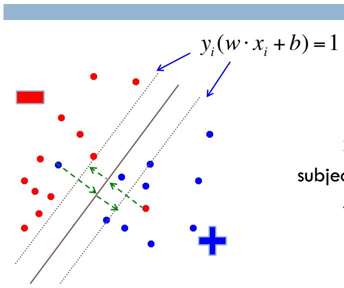
Understanding the Soft Margin SVM



$$\min_{w,b} \quad \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 - \varsigma_i \quad \forall i$$
$$\varsigma_i \geq 0$$

"distance" to the hyperplane *plus* the "distance" to the margin

$$\varsigma_i = 1 - y_i(w \cdot x_i + b)$$

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Understanding the Soft Margin SVM

$$\min_{w,b} \quad \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 - \varsigma_i \quad \forall i$$
$$\varsigma_i \geq 0$$

$$\varsigma_i = \begin{cases} 0 & \text{if } y_i(w \cdot x_i + b) \geq 1 \\ 1 - y_i(w \cdot x_i + b) & \text{otherwise} \end{cases}$$

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Understanding the Soft Margin SVM

$$\varsigma_i = \begin{cases} 0 & \text{if } y_i(w \cdot x_i + b) \geq 1 \\ 1 - y_i(w \cdot x_i + b) & \text{otherwise} \end{cases}$$

$$\varsigma_i = \max(0, 1 - y_i(w \cdot x_i + b))$$
$$= \max(0, 1 - yy')$$

Does this look familiar?

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Hinge loss!

0/1 loss: $l(y, y') = 1[yy' \leq 0]$

Hinge: $l(y, y') = \max(0, 1 - yy')$

Exponential: $l(y, y') = \exp(-yy')$

Squared loss: $l(y, y') = (y - y')^2$

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Understanding the Soft Margin SVM

$$\min_{w,b} \; \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 - \varsigma_i \quad \forall i$$
$$\varsigma_i \geq 0$$

$$\varsigma_i = \max(0, 1 - y_i(w \cdot x_i + b))$$

Do we need the constraints still?

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Understanding the Soft Margin SVM

$$\min_{w,b} \; \|w\|^2 + C\sum_i \varsigma_i$$

subject to:

$$y_i(w \cdot x_i + b) \geq 1 - \varsigma_i \quad \forall i$$
$$\varsigma_i \geq 0$$

$$\varsigma_i = \max(0, 1 - y_i(w \cdot x_i + b))$$

$$\min_{w,b} \; \|w\|^2 + C\sum_i \max(0, 1 - y_i(w \cdot x_i + b))$$

Unconstrained problem!

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Understanding the Soft Margin SVM

$$\min_{w,b} \quad \|w\|^2 + C\sum_i loss_{hinge}(y_i, y_i')$$

Does this look like something we've seen before?

$$\text{argmin}_{w,b} \sum_{i=1}^{n} loss(yy') + \lambda \; regularizer(w,b)$$

Gradient descent problem!

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Soft margin SVM as gradient descent

$$\min_{w,b} \quad \|w\|^2 + C\sum_i loss_{hinge}(y_i, y_i')$$

multiply through by 1/C and rearrange

$$\min_{w,b} \quad \sum_i loss_{hinge}(y_i, y_i') + \frac{1}{C}\|w\|^2$$

let λ=1/C

$$\min_{w,b} \quad \sum_i loss_{hinge}(y_i, y_i') + \lambda\|w\|^2$$

What type of gradient descent problem?

$$\text{argmin}_{w,b} \sum_{i=1}^{n} loss(yy') + \lambda \; regularizer(w,b)$$

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Soft margin SVM as gradient descent

One way to solve the soft margin SVM problem is using gradient descent

$$\min_{w,b} \quad \sum_i loss_{hinge}(y_i, y_i') + \lambda\|w\|^2$$

hinge loss

L2 regularization

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Gradient descent SVM solver

- pick a starting point (w)
- repeat until loss doesn't decrease in all dimensions:
  - pick a dimension
  - move a small amount in that dimension towards decreasing loss (using the derivative)

$$w_i = w_i - \eta \frac{d}{dw_i}(loss(w) + regularizer(w,b))$$

$$w_j = w_j + \eta \sum_{i=1}^{n} y_i x_i 1[y_i(w \cdot x + b) < 1] - \eta\lambda w_j$$

hinge loss L2 regularization

Finds the largest margin hyperplane while allowing for a soft margin

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Support vector machines: 2013

One of the most successful (if not the most successful) classification approach:

	2013	2016	2019	2022
decision tree	About 2,160,000	About 2,480,000	About 3,000,000	About 3,070,000
Support vector machine	About 1,960,000	About 2,430,000	About 3,020,000	About 3,250,000
k nearest neighbor	About 746,000	About 979,000	About 1,380,000	About 2,260,000
perceptron algorithm	About 84,300	About 104,000	About 153,000	About 230,000

Google scholar

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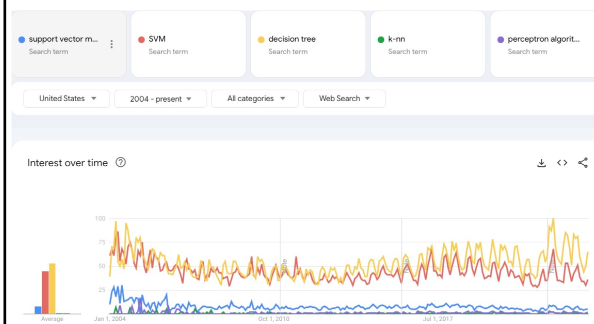
Trends over time



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