LOGISTIC REGRESSION

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CS158 – Fall 2019

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Admin

Assignment 7

Grading update

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Priors

Coin1 data: 3 Heads and 1 Tail
Coin2 data: 30 Heads and 10 tails
Coin3 data: 2 Tails
Coin4 data: 497 Heads and 503 tails

If someone asked you what the probability of heads was for each of these coins, what would you say?

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Basic steps for probabilistic modeling

	Probabilistic models
Step 1: pick a model	Which model do we use, i.e. how do we calculate p(*feature, label*)?
Step 2: figure out how to estimate the probabilities for the model	How do train the model, i.e. how to we we estimate the probabilities for the model?
Step 3 (optional): deal with overfitting	How do we deal with overfitting?

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Training revisited

What we're really doing during training is selecting the Θ that maximizes:

$$p(\theta \mid data)$$

i.e.

$$\theta = \text{argmax}_{\theta}\, p(\theta \mid data)$$

That is, we pick the most likely model parameters given the data

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Estimating revisited

We want to incorporate a prior belief of what the probabilities might be

To do this, we need to break down our probability

$$p(\theta \mid data) = ?$$

(Hint: Bayes rule)

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Estimating revisited

What are each of these probabilities?

$$p(\theta \mid data) = \frac{p(data \mid \theta)\, p(\theta)}{p(data)}$$

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Priors

likelihood of the data under the model

probability of different parameters, call the prior

$$p(\theta \mid data) = \frac{p(data \mid \theta)\, p(\theta)}{p(data)}$$

probability of seeing the data (regardless of model)

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Priors

$$\theta = \text{argmax}_\theta \frac{p(data \mid \theta)p(\theta)}{p(data)}$$

Does p(data) matter for the argmax?

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Priors

likelihood of the data under the model

probability of different parameters, call the prior

$$\theta = \text{argmax}_\theta \, p(data \mid \theta)p(\theta)$$

What does MLE assume for a prior on the model parameters?

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Priors

likelihood of the data under the model

probability of different parameters, call the prior

$$\theta = \text{argmax}_\theta \, p(data \mid \theta)p(\theta)$$

- Assumes a uniform prior, i.e. all Θ are equally likely!
- Relies solely on the likelihood

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A better approach

$$\theta = \text{argmax}_\theta \, p(data \mid \theta)p(\theta)$$

$$likelihood(data) = \prod_{i=1}^{n} p_\theta(x_i)$$

We can use any distribution we'd like.

This allows us to impart addition bias into the model

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Another view on the prior

Remember, the max is the same if we take the log:

$$\theta = \text{argmax}_\theta \log(p(data \mid \theta)) + \log(p(\theta))$$

$$\text{log-likelihood} = \sum_{i=1}^{n} \log(p(x_i))$$

We can use any distribution we'd like.

This allows us to impart addition bias into the model

Does this look like something we've seen before?

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Regularization vs prior

$$\theta = \text{argmax}_\theta \log(p(data \mid \theta)) + \log(p(\theta))$$

fit: likelihood based on the data / loss function based on the data

model bias: prior / regularizer

$$\text{argmin}_{w,b} \sum_{i=1}^{n} loss(yy') + \lambda regularizer(w)$$

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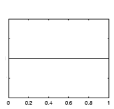
Prior for NB

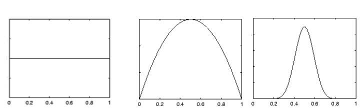
$$\theta = \text{argmax}_\theta \log(p(data \mid \theta)) + \log(p(\theta))$$

Uniform prior



$$p(x_i \mid y) = \frac{count(x_i, y)}{count(y)}$$

Dirichlet prior



λ= 0 increasing →

$$p(x_i \mid y) = \frac{count(x_i, y) + \lambda}{count(y) + possible\_values\_of\_x_i * \lambda}$$

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Prior: another view

$$p(x_1, x_2, ..., x_m, y) = p(y) \prod_{j=1}^{m} p(x_i \mid y)$$

MLE: $$p(x_i \mid y) = \frac{count(x_i, y)}{count(y)}$$

What happens to our likelihood if, for one of the labels, we never saw a particular feature?

Goes to 0!

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Prior: another view

$$p(x_i \mid y) = \frac{count(x_i, y)}{count(y)}$$

$$p(x_i \mid y) = \frac{count(x_i, y) + \lambda}{count(y) + possible\_values\_of\_x_i * \lambda}$$

Adding a prior avoids this!

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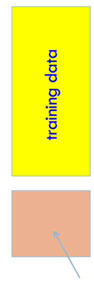
Smoothing



for each label, pretend like we've seen each feature value occur in λ additional examples

$$p(x_i \mid y) = \frac{count(x_i, y)}{count(y)}$$

$$p(x_i \mid y) = \frac{count(x_i, y) + \lambda}{count(y) + possible\_values\_of\_x_i * \lambda}$$

Sometimes this is also called smoothing because it is seen as smoothing or interpolating between the MLE and some other distribution

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Priors

Coin1 data: 3 Heads and 1 Tail

Coin2 data: 30 Heads and 10 tails

Coin3 data: 2 Tails

Coin4 data: 497 Heads and 503 tails

$$p(x_i \mid y) = \frac{count(x_i, y) + \lambda}{count(y) + possible\_values\_of\_x_i * \lambda}$$

Does this do the right thing in these cases?

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Basic steps for probabilistic modeling

Probabilistic models

Step 1: pick a model

Which model do we use, i.e. how do we calculate p(*feature, label*)?

Step 2: figure out how to estimate the probabilities for the model

How do train the model, i.e. how to we we estimate the probabilities for the model?

Step 3 (optional): deal with overfitting

How do we deal with overfitting?

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Joint models vs conditional models

We've been trying to model the joint distribution (i.e. the data generating distribution):

$$p(x_1, x_2, \ldots, x_m, y)$$

However, if all we're interested in is classification, why not directly model the conditional distribution:

$$p(y \mid x_1, x_2, \ldots, x_m)$$

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A first try: linear

$$p(y \mid x_1, x_2, \ldots, x_m) = x_1 w_1 + w_2 x_2 + \ldots + w_m x_m + b$$

Any problems with this?

- Nothing constrains it to be a probability
- Could still have combination of features and weight that exceeds 1 or is below 0

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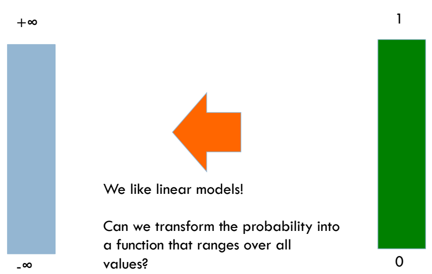
The challenge

$x_1 w_1 + w_2 x_2 + \ldots + w_m x_m + b$ $\quad\quad$ $p(y \mid x_1, x_2, \ldots, x_m)$

Linear model $\quad\quad$ probability



We like linear models!

Can we transform the probability into a function that ranges over all values?

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Odds ratio

Rather than predict the probability, we can predict the ratio of 1/0 (positive/negative)

Predict the **odds** that it is 1 (true): How much more likely is 1 than 0.

Does this help us?

$$\frac{P(1 \mid x_1, x_2, \ldots, x_m)}{P(0 \mid x_1, x_2, \ldots, x_m)} = \frac{P(1 \mid x_1, x_2, \ldots, x_m)}{1 - P(1 \mid x_1, x_2, \ldots, x_m)} = x_1 w_1 + w_2 x_2 + \ldots + w_m x_m + b$$

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Odds ratio

$x_1 w_1 + w_2 x_2 + \ldots + w_m x_m + b$

$$\frac{P(1 \mid x_1, x_2, \ldots, x_m)}{1 - P(1 \mid x_1, x_2, \ldots, x_m)}$$

Linear model

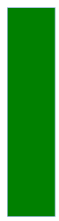
odds ratio

+∞

-∞

Where is the dividing line between class 1 and class 0 being selected?

+∞



0

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Odds ratio

$P(1 \mid x_1, x_2, \ldots, x_m) > P(0 \mid x_1, x_2, \ldots, x_m)$

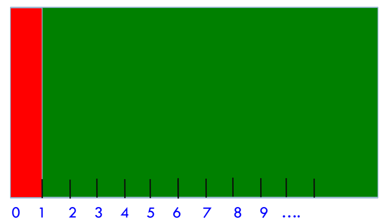
$P(1 \mid x_1, x_2, \ldots, x_m) > 1 - P(1 \mid x_1, x_2, \ldots, x_m)$

$$\frac{P(1 \mid x_1, x_2, \ldots, x_m)}{1 - P(1 \mid x_1, x_2, \ldots, x_m)}$$

We're trying to find some transformation that transforms the odds ratio to a number that is -∞ to +∞

Does this suggest another transformation?

odds ratio



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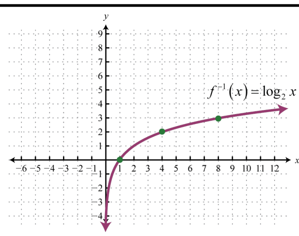

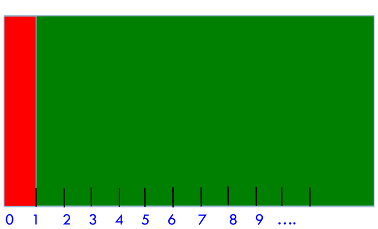

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Log odds (logit function)

$$x_1 w_1 + w_2 x_2 + \ldots + w_m x_m + b = \log \frac{P(1 \mid x_1, x_2, \ldots, x_m)}{1 - P(1 \mid x_1, x_2, \ldots, x_m)}$$

Linear regression

odds ratio

+∞

-∞

How do we get the probability of an example?

+∞

-∞

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Log odds (logit function)

$$\log \frac{P(1 \mid x_1, x_2, \ldots, x_m)}{1 - P(1 \mid x_1, x_2, \ldots, x_m)} = w_1x_2 + w_2x_2 + \ldots + w_mx_m + b$$

$$\frac{P(1 \mid x_1, x_2, \ldots, x_m)}{1 - P(1 \mid x_1, x_2, \ldots, x_m)} = e^{w_1x_2 + w_2x_2 + \ldots + w_mx_m + b}$$

$$P(1 \mid x_1, x_2, \ldots, x_m) = (1 - P(1 \mid x_1, x_2, \ldots, x_m))e^{w_1x_2 + w_2x_2 + \ldots + w_mx_m + b}$$

…

$$P(1 \mid x_1, x_2, \ldots, x_m) = \frac{1}{1 + e^{-(w_1x_2 + w_2x_2 + \ldots + w_mx_m + b)}}$$

anyone recognize this?

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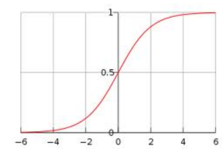
Logistic function

$$\text{logistic} = \frac{1}{1 + e^{-x}}$$



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Logistic regression

How would we classify examples once we had a trained model?

$$\log \frac{P(1 \mid x_1, x_2, \ldots, x_m)}{1 - P(1 \mid x_1, x_2, \ldots, x_m)} = w_1x_2 + w_2x_2 + \ldots + w_mx_m + b$$

If the sum > 0 then p(1)/p(0) > 1, so positive

if the sum < 0 then p(1)/p(0) < 1, so negative

Still a *linear* classifier (decision boundary is a line)

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Training logistic regression models

How should we learn the parameters for logistic regression (i.e. the w's and b)?

$$\log \frac{P(1 \mid x_1, x_2, \ldots, x_m)}{1 - P(1 \mid x_1, x_2, \ldots, x_m)} = w_1x_2 + w_2x_2 + \ldots + w_mx_m + b$$

parameters

$$P(1 \mid x_1, x_2, \ldots, x_m) = \frac{1}{1 + e^{-(w_1x_2 + w_2x_2 + \ldots + w_mx_m + b)}}$$

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MLE logistic regression

Find the parameters that maximize the likelihood (or log-likelihood) of the data:

$$\text{log-likelihood} = \sum_{i=1}^{n} \log(p(x_i))$$

$$= \sum_{i=1}^{n} \log\left(\frac{1}{1+e^{-y_i(w_1x_2+w_2x_2+\ldots+w_mx_m+b)}}\right)$$

assume labels 1, -1

$$= \sum_{i=1}^{n} -\log(1+e^{-y_i(w_1x_2+w_2x_2+\ldots+w_mx_m+b)})$$

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MLE logistic regression

$$\text{log-likelihood} = \sum_{i=1}^{n} -\log(1+e^{-y_i(w_1x_2+w_2x_2+\ldots+w_mx_m+b)})$$

We want to maximize, i.e.

$$MLE(data) = \text{argmax}_{w,b}\ \text{log-likelihood}(data)$$

$$= \text{argmax}_{w,b} \sum_{i=1}^{n} -\log(1+e^{-y_i(w_1x_2+w_2x_2+\ldots+w_mx_m+b)})$$

$$= \arg\min_{w,b} \sum_{i=1}^{n} \log(1+e^{-y_i(w_1x_2+w_2x_2+\ldots+w_mx_m+b)})$$

Look familiar? Hint: anybody read the book?

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MLE logistic regression

$$\text{argmin}_{w,b} \sum_{i=1}^{n} \log(1+e^{-y_i(w_1x_2+w_2x_2+\ldots+w_mx_m+b)})$$

Surrogate loss functions:

Zero/one:	$\ell^{(0/1)}(y,\hat{y}) = \mathbf{1}[y\hat{y} \le 0]$
Hinge:	$\ell^{(\text{hin})}(y,\hat{y}) = \max\{0, 1 - y\hat{y}\}$
Logistic:	$\ell^{(\log)}(y,\hat{y}) = \frac{1}{\log 2}\log(1+\exp[-y\hat{y}])$
Exponential:	$\ell^{(\exp)}(y,\hat{y}) = \exp[-y\hat{y}]$
Squared:	$\ell^{(\text{sqr})}(y,\hat{y}) = (y-\hat{y})^2$

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logistic regression: three views

$$\log \frac{P(1 \mid x_1,x_2,\ldots,x_m)}{1-P(1 \mid x_1,x_2,\ldots,x_m)} = w_0 + w_1x_2 + w_2x_2 + \ldots + w_mx_m$$

linear classifier

$$P(1 \mid x_1,x_2,\ldots,x_m) = \frac{1}{1+e^{-(w_0+w_1x_2+w_2x_2+\ldots+w_mx_m)}}$$

conditional model
logistic

$$\text{argmin}_{w,b} \sum_{i=1}^{n} \log(1+e^{-y_i(w_1x_2+w_2x_2+\ldots+w_mx_m+b)})$$

linear model
minimizing logistic loss

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Overfitting

$$\text{argmin}_{w,b} \sum_{i=1}^{n} \log(1 + e^{-y_i(w_1x_2 + w_2x_2 + \ldots + w_mx_m + b)})$$

If we minimize this loss function, in practice, the results aren't great and we tend to overfit

Solution?

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Regularization/prior

$$\text{argmin}_{w,b} \sum_{i=1}^{n} \log(1 + e^{-y_i(w_1x_2 + w_2x_2 + \ldots + w_mx_m + b)}) + \lambda\, regularizer(w,b)$$

or

$$\text{argmin}_{w,b} \sum_{i=1}^{n} \log(1 + e^{-y_i(w_1x_2 + w_2x_2 + \ldots + w_mx_m + b)}) - \log(p(w,b))$$

What are some of the regularizers we know?

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Regularization/prior

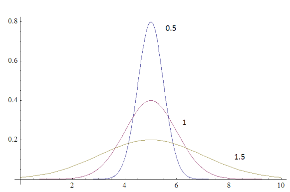
L2 regularization:

$$\text{argmin}_{w,b} \sum_{i=1}^{n} \log(1 + e^{-y_i(w_1x_2 + w_2x_2 + \ldots + w_mx_m + b)}) + \lambda \|w\|^2$$

Gaussian prior:

Gaussians are defined by a mean (μ) and a variance (σ^2)

p(w,b) ~



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Regularization/prior

L2 regularization:

$$\text{argmin}_{w,b} \sum_{i=1}^{n} \log(1 + e^{-y_i(w_1x_2 + w_2x_2 + \ldots + w_mx_m + b)}) + \lambda \|w\|^2$$

Gaussian prior:

$$\text{argmin}_{w,b} \sum_{i=1}^{n} \log(1 + e^{-y_i(w_1x_2 + w_2x_2 + \ldots + w_mx_m + b)}) + \frac{1}{2\sigma^2} \|w\|^2$$

Does the λ make sense? $\lambda = \frac{1}{2\sigma^2}$

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Regularization/prior

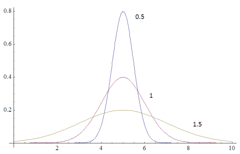
L2 regularization:

$$\text{argmin}_{w,b} \sum_{i=1}^{n} \log(1+e^{-y_i(w_1x_2+w_2x_2+...+w_mx_m+b)}) + \lambda \|w\|^2$$

Gaussian prior:

$$\text{argmin}_{w,b} \sum_{i=1}^{n} \log(1+e^{-y_i(w_1x_2+w_2x_2+...+w_mx_m+b)}) + \frac{1}{2\sigma^2}\|w\|^2$$

$$\lambda = \frac{1}{2\sigma^2}$$



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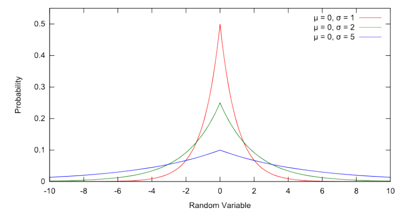
Regularization/prior

L1 regularization:

$$\text{argmin}_{w,b} \sum_{i=1}^{n} \log(1+e^{-y_i(w_1x_2+w_2x_2+...+w_mx_m+b)}) + \lambda \|w\|$$

Laplacian prior:

p(w,b) ~



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Regularization/prior

L1 regularization:

$$\text{argmin}_{w,b} \sum_{i=1}^{n} \log(1+e^{-y_i(w_1x_2+w_2x_2+...+w_mx_m+b)}) + \lambda \|w\|$$

Laplacian prior:

$$\text{argmin}_{w,b} \sum_{i=1}^{n} \log(1+e^{-y_i(w_1x_2+w_2x_2+...+w_mx_m+b)}) + \frac{1}{\sigma}\|w\|$$

$$\lambda = \frac{1}{\sigma}$$

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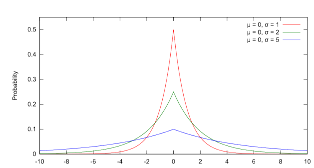
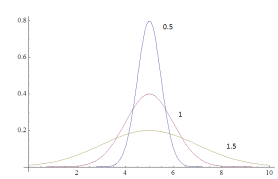
L1 vs. L2

L1 = Laplacian prior

L2 = Gaussian prior





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Logistic regression

Why is it called logistic regression?

It is a classifier??

$$\log \frac{P(1 \mid x_1, x_2, \ldots, x_m)}{1 - P(1 \mid x_1, x_2, \ldots, x_m)} = w_1 x_2 + w_2 x_2 + \ldots + w_m x_m + b$$

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A digression: regression vs. classification

Raw data	Label
	0
	0
	1
	1
	0

extract features

features	Label
$f_1, f_2, f_3, \ldots, f_n$	
$f_1, f_2, f_3, \ldots, f_n$	
$f_1, f_2, f_3, \ldots, f_n$	
$f_1, f_2, f_3, \ldots, f_n$	
$f_1, f_2, f_3, \ldots, f_n$	

classification: discrete (some finite set of labels)

regression: real value

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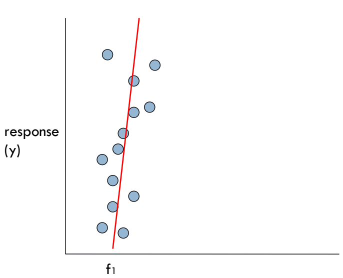
linear regression



Given some points, find the ***line*** that best fits/explains the data

Our model is a line, i.e. we're assuming a linear relationship between the feature and the label value

$$h(y) = w_1 x_1 + b$$

How can we find this line?

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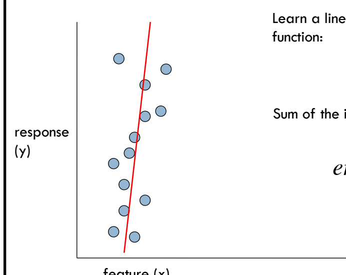
Linear regression



Learn a line *h* that minimizes some loss/error function:

$$error(h) = ?$$

Sum of the individual errors:

$$error(h) = \sum_{i=1}^{n} \left| y_i - h(f_i) \right|$$

0/1 loss!

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Error minimization

How do we find the minimum of an equation?

$$error(h) = \sum_{i=1}^{n} |y_i - h(f_i)|$$

Take the derivative, set to 0 and solve (going to be a min or a max)

Any problems here?

Ideas?

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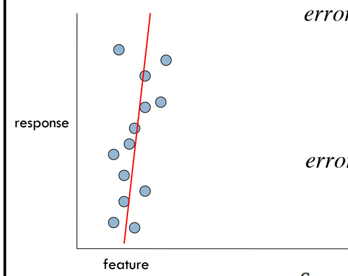
Linear regression



$$error(h) = \sum_{i=1}^{n} |y_i - h(f_i)|$$

$$error(h) = \sum_{i=1}^{n} (y_i - h(f_i))^2$$

squared error is convex!

Squared: $\ell^{(\text{sqr})}(y, \hat{y}) = (y - \hat{y})^2$

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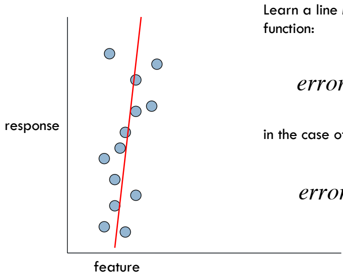
Linear regression



Learn a line *h* that minimizes an error function:

$$error(h) = \sum_{i=1}^{n} (y_i - h(f_i))^2$$

in the case of a 2d line:

$$error(h) = \sum_{i=1}^{n} (y_i - \underbrace{(w_1 x_1 + w_0)}_{\text{function for a line}})^2$$

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Linear regression

We'd like to *minimize* the error

Find w_1 and w_0 such that the error is minimized

$$error(h) = \sum_{i=1}^{n} (y_i - (w_1 f_i + w_0))^2$$

We can solve this in closed form

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Multiple linear regression

If we have *m* features, then we have a line in *m* dimensions

$$h(\bar{f}) = w_0 + w_1 f_1 + w_2 f_2 + \ldots + w_m f_m$$

weights

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Multiple linear regression

We can still calculate the squared error like before

$$h(\bar{f}) = w_0 + w_1 f_1 + w_2 f_2 + \ldots + w_m f_m$$

$$error(h) = \sum_{i=1}^{n} (y_i - (w_0 + w_1 f_1 + w_2 f_2 + \ldots + w_m f_m))^2$$

Still can solve this exactly!

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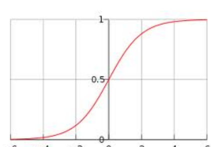
Logistic function

$$\text{logistic} = \frac{1}{1 + e^{-x}}$$



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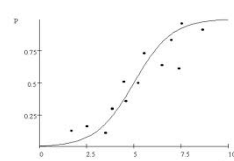
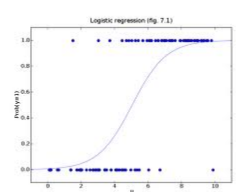
Logistic regression

Find the best fit of the data based on a logistic





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Basic steps for probabilistic modeling

	Probabilistic models
Step 1: pick a model	Which model do we use, i.e. how do we calculate p(*feature, label*)?
Step 2: figure out how to estimate the probabilities for the model	How do train the model, i.e. how to we we estimate the probabilities for the model?
Step 3 (optional): deal with overfitting	How do we deal with overfitting?

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Probabilistic models summarized

Two classification models:
- Naïve Bayes (models joint distribution)
- Logistic Regression (models conditional distribution)
  - In practice this tends to work better if all you want to do is classify

Priors/smoothing/regularization
- Important for both models
- In theory: allow us to impart some prior knowledge
- In practice: avoids overfitting and often tune on development data

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