Kruskal’s MST Algorithm

https://cs.pomona.edu/classes/cs140/
Outline

**Topics and Learning Objectives**

- Introduce Kruskal’s algorithms for MSTs
- Discuss disjoint sets

**Exercise**

- MST exercise question 3
Minimum-Spanning-Tree Overview

Input: a graph where each edge has an associated cost

Output: a minimum-spanning-tree
1. Connects the entire graph as a tree, but
2. Has a minimal cost

Assumptions:
1. The input graph is connected
2. The edges costs are distinct (only necessary/useful for our proof)

Cut Property: if e is the cheapest edge crossing a cut, then it must be in the MST
Kruskal’s

A greedy algorithm for finding the minimum spanning tree

Why are we learning another one?
• Kruskal’s will motivate a new data structure: Union-Find (disjoint-set)
• It will also let us talk a bit about clustering

Can you think of another greedy algorithm for solving MST?

Prim’s : vertex
Kruskal’s Minimum Spanning Tree Algorithm

Sort $E$ by edge cost

$T = \text{empty}$

For $e$ in $E$:

if $T \cup \{e\}$ has no cycles

add $e$ to $T$
Proof of Kruskal’s Algorithm

Theorem: Kruskal’s algorithm is correct (computes the MST)

Let $T^*$ = the output of Kruskal’s algorithm
Graph/Cut/Tree Lemmas and Properties

• Empty Cut Lemma: a graph is not connected if there exists a cut \((A, B)\) with zero crossing edges

• Double Crossing Lemma: suppose the cycle \(C\) has an edge crossing the cut \((A, B)\), then there must be at least one more edge in \(C\) that crosses the cut

• No Cycle Corollary: if \(e\) is the only edge crossing some cut \((A, B)\), then it is not in any cycle

• Cut Property: if \(e\) is the cheapest edge that crosses the cut \((A, B)\) then it must be in the MST
Proof of Kruskal’s Algorithm

Theorem: Kruskal’s algorithm is correct (computes the MST)
Let $T^*$ = the output of Kruskal’s algorithm

Does Kruskal’s output a spanning tree (what are the properties)?
• No cycles
• Connected
Kruskal’s Minimum Spanning Tree Algorithm

Sort \( E \) by edge cost

\( T = \) empty

For \( e \) in \( E \):
    if \( T \cup \{e\} \) has no cycles
        add \( e \) to \( T \)
Proof of Kruskal’s Algorithm

Theorem: Kruskal’s algorithm is correct (computes the MST)
Let $T^*$ = the output of Kruskal’s algorithm

Does Kruskal’s output a spanning tree (what are the properties)?
• No cycles (this is given by the definition of the algorithm)
• Connected
Proof of Kruskal’s Algorithm

Proof of Connectivity

• Given the Empty Cut Lemma, we only need to show that Kruskal’s produces a tree $T^*$ that crosses every cut.

• Fix a cut $(A,B)$
• Since $G$ is connected, at least one of its edges crosses $(A,B)$
• Kruskal’s algorithm considers each edge once
• Let’s fast-forward to the first time that it encounters an edge crossing $(A,B)$
• Claim: this 1st edge is guaranteed to be in $T^*$
• Given the No Cycle Corollary the claim is true
Proof of Kruskal’s Algorithm

For the second part of the proof we need to prove that $T^*$ is minimal
• We just finished proving that Kruskal’s outputs some spanning tree $T^*$

Claim: every edge is justified by the Cut Property
• Remember that satisfying the Cut Property implies that we have an MST
• This was very explicit in Prim’s Algorithm
Prim’s Minimum Spanning Tree Algorithm

X = \{s\}
T = empty

while X is not V:
    let e = (u, v) be the cheapest edge of G with u in X and v not in X
    add e to T
    add v to X
Proof of Kruskal’s Algorithm

Proving that we can use the Cut Property

• Consider each iteration where edge \((u, v)\) is added to \(T^*\)
• Since \(T^* \cup \{(u, v)\}\) has no cycle, \(T^*\) currently has no \(u \rightarrow v\) path
• Thus, there must be a cut \((A, B)\) separating \(u\) and \(v\). For example:
  • All findable from \(u\) in \(A\)
  • All findable from \(v\) in \(B\)
  • All other vertices can be partitioned arbitrarily
• Hence, \((u, v)\) is the first crossing cut for \((A, B)\)
• Additionally, it must be the cheapest such cut since we sorted the edges
• Finally, the edge \((u, v)\) is justified by the Cut Property
Proof of Kruskal’s Algorithm

What have we done?

We proved that Kruskal’s outputs a spanning tree
• No cycles by definition
• Connectivity by the Empty Cut Lemma

We then proved that Kruskal’s outputs the minimum spanning tree
• The Cut Property implies that we are left with the MST
• We showed that Kruskal’s uses the Cut Property because the edges are sorted
Implementation of Kruskal’s
Kruskal’s Minimum Spanning Tree Algorithm

Sort $E$ by edge cost

$T = \text{empty}$

For $e$ in $E$:

if $T \cup \{e\}$ has no cycles

add $e$ to $T$

$O(m \lg n)$ or $O(m \lg m)$

Naïvely $O(n + m)$

$O(m \lg n) + O(m) \times O(n+m)$

$O(mn + m^2)$
Trick Question for the Day

Which is asymptotically bigger?

\[ O(m \log n) \text{ or } O(m \log m) \]
Kruskal’s Minimum Spanning Tree Algorithm

Sort $E$ by edge cost
$T = \text{empty}$

For $e$ in $E$:
    if $T \cup \{e\}$ has no cycles
        add $e$ to $T$

What can we change (should we change) to do better than $O(mn)$?
The Union-Find Data Structure

• Also known as the disjoint-set data structure
• Used to maintain a partition of objects
Union-Find

Operations:

- **Find(x):** return the name of the group to which x belongs
- **Union(Ci, Cj):** merge the two partitions into a single partition
How does this help us with Kruskal’s?

- What do we store in the data structure?
- What makes a group/partition?
Kruskal’s Minimum Spanning Tree Algorithm

Sort $E$ by edge cost
$T = \text{empty}$

For $e$ in $E$:
  if $T \cup \{e\}$ has no cycles
    add $e$ to $T$

$O(m)$

Naïvely $O(n + m)$

$O(1)$
Motivation

• Speed up the way in which we check for cycles.
• How would you implement the Union-Find data structure?
• Conceptually, we’re going to augment each vertex to include another piece of information: the name of its leader

• Invariant: each vertex points to its leader

• How long does it take to check for a cycle now?
Checking for cycles

• Given an edge \((u, v)\), we can check if \(u\) and \(v\) are in the same partition in constant time \(O(1)\).

\[
\text{Find}(u) = \text{Find}(v)\]
Maintaining the Invariant

• **Invariant: each vertex points to its leader**

What is the exact number at most?

How many possible vertex leaders do you need to fix after a union?

$O(1)$  $O(\log n)$  $O(n)$  $O(m)$
Union-Find Data Structure

• Put every element in its own partition
  • Every element has its own leader

• Join partitions by copying the leader of the larger partition elements to all elements of the smaller partition

• You can use an array or hash table to keep track of leaders

• No other information/memory is needed
Kruskal’s Minimum Spanning Tree Algorithm

Sort E by edge cost
T = empty

For e in E:
    if T U {e} has no cycles
        add e to T

What do we have as a running time now?
What happened?

Sort $E$ by edge cost
$T = \text{empty}$

For $e$ in $E$:

if $T \cup \{e\}$ has no cycles

add $e$ to $T$  \(\text{We don’t do this every iteration}\)

$O(n+m) \rightarrow O(1) \ (\text{checking leaders})$

$O(1) \rightarrow O(n) \ (\text{updating leaders})$
Maximum number of leader updates?

How many times can we update the leader of a single vertex?

• We only update the leader of a vertex if we merge it with a bigger partition.

• How many times can we merge a vertex into a bigger partition?
  • (Or: How many times can we double the size of a partition?)

This is our global view of something happening inside the loop.
Kruskal’s Minimum Spanning Tree Algorithm

Sort $E$ by edge cost

$T = \text{empty}$

For $e$ in $E$:

- If $T \cup \{e\}$ has no cycles
  - Add $e$ to $T$

$O(m \log m)$ * $O(1)$ just for the cycle check

$O(m) + O(n \log n) + O(m) \cdot O(1) = O(m \log m)$

Total
Cutting Edge

- Can we do better than $O(m \lg n)$?
  - Yes!
  - **Average** $O(m)$ using a randomized algorithm (1995)
  - We do not actually know if a deterministic $O(m)$ algorithm exists.
  - We do have a deterministic algorithm that is $O(m \alpha(n))$
  - $\alpha$ is the inverse Ackermann function
  - Which is slower than the Iterated logarithm: $\lg^*$
    - the number of times the logarithm function must be iteratively applied before the result is less than or equal to 1
  - An optimal deterministic algorithm was developed in 2002
  - But, we do not know the exact asymptotic complexity
    - Just that it is between $O(m)$ and $O(m \alpha(n))$