Red-Black Trees
(A Balanced BST)

https://cs.pomona.edu/classes/cs140/

Some notes taken from
http://www.geeksforgeeks.org/
Outline

Topics and Learning Objectives

• Discuss tree balancing (rotations, insertions, deletions)
• Prove the balancing characteristic of red-black trees
• Discuss the running time of red-black tree operations

Assessments

• Red-black tree activity
Extra Resources

• Introduction to Algorithms, 3rd, chapter 13
Implementations

Although Red-Black trees are not the most modern choice, they do appear in

• Java: TreeMap<K,V>

• C++: std::map
Balanced Binary Search Trees

- Why is balancing important?
- What is the worst case for a binary tree?

Balanced tree: the height of a balanced tree stays $O(\lg n)$ after insertions and deletions

- Many different types of balanced search trees:
  - AVL Tree, Splay Tree, B Tree, Red-Black Tree
Red-Black Trees Invariants

1. Each node must be labeled either red or black

2. The root must be labeled black

3. The tree cannot have two red nodes in a row (for any red node its parent, left, and right must be black)

4. Every root-NULL path must include the same number of black nodes

Can a Red-Black tree of any height have only black nodes? Yes
Red-Black Trees

Can a “chain” be a red-black tree?

1. Each node must be labeled either red or black
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We could also move the black color down one level
Red-Black Trees

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How did Red-Black Trees get their name?
Red-Black Tree Height

• Claim: every Red-Black tree has a height $t_{\text{height}} \leq 2 \log(n + 1)$

• Observation: if every root-NULL path has $\geq k$ nodes, then the tree includes a perfectly balanced top portion with $k$ levels

$k = 2$

$k = 3$
Red-Black Tree Height

• Claim: every Red-Black tree has a \( t_{\text{height}} \leq 2 \log(n + 1) \)

• Observation: if every root-NULL path has \( \geq k \) nodes, then the tree includes a perfectly balanced top portion with \( k \) levels

What is \( k \)?

\( k = 2 \)

\( k = 3 \)
Red-Black Tree Height

• Claim: every Red-Black tree has a $t_{\text{height}} \leq 2 \lg(n + 1)$

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What is the minimum number of nodes ($n$) in the tree based on $k$?
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<table>
<thead>
<tr>
<th>$k$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<tr>
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<td>31</td>
</tr>
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<td>6</td>
<td>63</td>
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Red-Black Tree Height

• Claim: every Red-Black tree has a \( t_{\text{height}} \leq 2 \lg(n + 1) \)

• Observation: if every root-NULL path has \( \geq k \) nodes, then the tree includes a perfectly balanced top portion with \( k \) levels

What is the minimum number of nodes \( (n) \) in the tree based on \( k \)?
Red-Black Tree Height

• So we have:

\[ n \geq 2^k - 1 \]

\[ \lg(n + 1) \geq k \]

• So, we now have an upper bound on \( k \).

• But how does \( k \) help us bound the actual height of the tree?

• What does \( k \) tell us about the number of black nodes you can have?

• What is the maximum number of black nodes on any root-Null path?

**Observation:** if every root-NULL path has \( \geq k \) nodes, then the tree includes a perfectly balanced top portion with \( k \) levels
Red-Black Tree Height

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At most k black nodes
At most \( \lg(n + 1) \) black nodes
Red-Black Tree Height

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$$n \geq 2^k - 1$$

$$\lg(n + 1) \geq k$$

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What does k tell us about the number of black nodes you can have?

What is the maximum number of black nodes on any root-Null path?

Observation: if every root-NULL path has $\geq k$ nodes, then the tree includes a perfectly balanced top portion with k levels

At most k black nodes

At most $\lg(n + 1)$ black nodes
Red-Black Tree Height

• Thus: in a Red-Black tree with $n$ nodes, there is a root-NULL path with at most $\log (n + 1)$ black nodes

• By invariant (4): every root-NULL path has $\leq \log(n + 1)$ black nodes

• By invariant (3): every root-NULL path has $\leq \log(n + 1)$ red nodes

• Thus, a total of $\leq 2\log(n + 1)$ nodes on every root-NULL path

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4. Every root-NULL path must include the same number of black nodes
Red-Black Trees

• If our tree can be colored as a Red-Black tree, then every root-NULL path has \( \leq 2\log(n + 1) \) nodes total.

• The longest path will dictate the height of the tree.

• So, height of the tree is at most \( 2\log(n + 1) \)\[
\log(n+1) = \log n + \log(1 + 1/n) = \log n + C
\]

• A tree cannot contain a *chain* of three nodes.

• Thus, the height of the tree is \( O(\log n) \).

• Why is this important?
Exercise question 2

Draw a **Worst-Case** (most lopsided) Red-Black Tree with a minimum of 3 black nodes on every root-NUL path

\[ h \leq 2 \log_2 (n + 1) \]
Red-Black Trees, Inserting a Node

1. Insert the new node

2. Color it red

3. Fix colors to enforce Red-Black Tree invariants
   1. This is a recursive process
Red-Black Trees, Inserting a Node

1. Insert the new node (always insert as a leaf)  

2. If the inserted node is the root, then color it black, otherwise color it red  

3. If the new node is not root and its parent is black, then we are done  

4. Otherwise, look at the node's aunt  
   a) If aunt is red  
      i. Change color of parent and aunt to black  
      ii. Change color of the new node and the grandparent to red  
      iii. Go to step (2) and treat grandparent as new node

“Aunt” is usually called “Uncle”
**Red-Black Trees, Inserting a Node**

1. Insert the new node (always insert as a leaf)  
   - **Why?**
2. If the inserted node is the root, then color it black, otherwise color it red  
   - **Why?**
3. If the new node is not root and its parent is black, then we are done
4. Otherwise, look at the node's aunt
   a) If aunt is red
      i. Change color of parent and aunt to black
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Red-Black Trees, Inserting a Node

1. Insert the new node (always insert as a leaf)
2. If the inserted node is the root, then color it black, otherwise color it red
3. If the new node is not root and its parent is black, then we are done
4. Otherwise, look at the node's aunt
   a) If aunt is red
   b) If aunt is black
      I. Put the new node, its parent, and the grandparent “in order” with the middle node as the root
      II. We have four possibilities for the current positions of N, P, and G
Red-Black Trees, Inserting a Node: Left-Left

1. Right rotate around the grandparent
Tree Rotations: Right
Tree Rotations: Right

Original
Tree Rotations: Right

Original

Right-Rotated
Tree Rotations: Left
Tree Rotations: Left

Original

Diagram 1:
- z
- x
  - a
  - y
    - b
    - c

Diagram 2:
- z
- x
  - a
  - b
  - y
    - c
Tree Rotations: Left
Red-Black Trees, Inserting a Node: Left, Left

1. Right rotate around the grandparent
Red-Black Trees, Inserting a Node: Left, Left

1. Right rotate around the grandparent

2. Swap the colors of the grandparent and the parent
Red-Black Trees, Inserting a Node: Left, Left

1. Right rotate around the grandparent

2. Swap the colors of the grandparent and the parent
Red-Black Trees, Inserting a Node: Left, Right

1. Left rotate around the parent

```
                G
               / \
              P   A
             /   / \
            N   PL AL AR
           /   / \
          NL NR  
```
Red-Black Trees, Inserting a Node: Left, Right

1. Left rotate around the parent

2. Right rotate around the grandparent
Red-Black Trees, Inserting a Node: Left, Right

1. Left rotate around the parent

2. Right rotate around the grandparent

3. Swap the colors of the grandparent and the new node
Red-Black Trees, Inserting a Node: Left, Right

1. Left rotate around the parent

2. Right rotate around the grandparent

3. Swap the colors of the grandparent and the new node
Red-Black Trees, Inserting a Node

• What about the Right-Right and Right-Left options?
• They are the inverse of the cases we’ve just covered.
• What are the running times of these procedures?
  • Inserting the new node? $O(1)$
  • Recoloring? $O(1) \rightarrow O(\log(n))$
  • Restructuring?
• We’re not going to cover deletion, but what are your thoughts?
  • Operation? (http://www.geeksforgeeks.org/red-black-tree-set-3-delete-2/)
  • Running time? $O(\log(n))$
FUNCTION RBTreeInsert(tree, new_node) 34

# Search for position of new_node
parent = NONE
current_node = tree.root
WHILE current_node != NONE
    parent = current_node
    IF new_node.key < current_node.key
        current_node = current_node.left
    ELSE
        current_node = current_node.right
new_node.parent = parent
FUNCTION RBTreeInsert(tree, new_node)

    # Search for position of new_node

    ...

    # Insert new_node as root or left/right child

    IF parent == NONE
        tree.root = new_node
    ELSE IF new_node.key < parent.key
        parent.left = new_node
    ELSE
        parent.right = new_node
FUNCTION RBTreeInsert(tree, new_node)
   # Search for position of new_node
   ...

   # Insert new_node as root or left/right child
   ...

   # Initialize the new_node
   new_node.left = NONE
   new_node.right = NONE
   new_node.color = RED

   RBTreeFixColors(tree, new_node)
FUNCTION RBTreeFixColors(tree, node)

WHILE node.parent.color == RED
    # Look for aunt/uncle node
    IF node.parent == node.parent.parent.parent.left
        aunt = node.parent.parent.parent.right
    IF aunt.color == RED
        node.parent.color = BLACK
        aunt.color = BLACK
        node.parent.parent.color = RED
        node = node.parent.parent.parent
FUNCTION RBTreeFixColors(tree, node)

WHILE node.parent.color == RED

# Look for aunt/uncle node

IF node.parent == node.parent.parent.left
  aunt = node.parent.parent.right
IF aunt.color == RED
  ...
ELSE
  IF node == node.parent.right
    node = node.parent
    LeftRotate(tree, node)
    node.parent.color = BLACK
  node.parent.parent.color = RED
  RightRotate(tree, node.parent.parent.parent)
FUNCTION RBTreeFixColors(tree, node)

WHILE node.parent.color == RED
    # Look for aunt/uncle node
    IF node.parent == node.parent.parent.parent.left
        aunt = node.parent.parent.parent.right
    ELSE
        aunt = node.parent.parent.parent.left
        IF aunt.color == RED
            node.parent.color = BLACK
        aunt.color = BLACK
            node.parent.parent.parent.color = RED
        node = node.parent.parent.parent

10  25
15  22
 9  30
 8  35
 20
 35

FUNCTION RBTreeFixColors(tree, node)

WHILE node.parent.color == RED
    # Look for aunt/uncle node
    ...
    ELSE
        aunt = node.parent.parent.left
        ...
    ELSE
        IF node == node.parent.left
            node = node.parent
            RightRotate(tree, node)
            node.parent.color = BLACK
            node.parent.parent.color = RED
            LeftRotate(tree, node.parent.parent)
            ...
        ELSE
            ...
        ENDIF
FUNCTION RBTreeFixColors(tree, node)

WHILE node.parent.color == RED
    # Look for aunt/uncle node
    IF node.parent == node.parent.parent.left
        aunt = node.parent.parent.right
    ... 
    ELSE
        aunt = node.parent.parent.left
    ...

    tree.root.color = BLACK
Is this a valid BST?
Is this a valid BST?
Is this a valid R-B Tree?
Is this a valid R-B Tree?
Is this the only valid coloring?
Is this the only valid coloring?
Is this the only valid coloring?
Exercise question 3

Insert: 9
Insert: 9

Recolor: 8, 10, 15

Valid R-B Tree?

Right rotate G

Check the Aunt
Recolor
Valid R-B Tree?
BST Summary

• Most BST operations take $O(\text{height})$ time.
• With an unbalanced tree this could be as bad as $O(n)$
• We want to ensure that the height of the tree is $O(\lg n)$
• Red-Black trees provide one mechanism for creating balanced trees, meaning that they guarantee $O(\lg n)$ for applicable BST operation
• This requires extra work while inserting and deleting in the form of tree rotations

• Bottom line: as long as our tree satisfies the Red-Black tree invariants (which it does with appropriate insert/delete procedures), then we can assume optimal running time for BSTs