Dijkstra’s Algorithm

https://cs.pomona.edu/classes/cs140/
Dijkstra's Single-Source Shortest Path Algorithm
Outline

Topics and Learning Objectives
• Discuss graphs with edge weights
• Discuss shortest paths
• Discuss Dijkstra’s algorithm including a proof

Exercise
• Dijkstra’s Algorithm
Extra Resources

• Introduction to Algorithms, 3rd, chapter 24
Dijkstra’s Algorithm

Find the shortest path between a start vertex s and every other vertex in the graph G

Can halt the algorithm if you only want to find shortest path to a specific vertex (for example, a destination city)

Uses:
• Network routing
• Path planning
• Etc.
Dijkstra’s Algorithm

Find the shortest path between a start vertex \( s \) and every other vertex in the graph \( G \)

Can halt the algorithm if you only want to find shortest path to a specific vertex (for example, a destination city)

Uses:
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Dijkstra’s Algorithm

Input
• A weighted graph $G = (V,E)$ and
• A source vertex $s$

Output
• for all $v$ in $V$ we output the length of the shortest path from $s \rightarrow v$
• you can also output the actual path, but we’ll just worry about length for now

Assumptions
• A path exists from $s$ to every other node (how can we check this property?)
• All edge weights are non-negative
What is the shortest path from S to all other vertices?
How did we do shortest path before?

• BFS

• How can we modify that process to work for graphs with weighted edges?

• Why would we not want to do that?
FUNCTION Dijkstra(G, start_vertex)
found = {}
lengths = {v: INFINITY FOR v IN G.vertices}
found.add(start_vertex)
lengths[start_vertex] = 0

WHILE found.length != G.vertices.length
    FOR v IN found
        FOR vOther, weight IN G.edges[v]
            IF vOther NOT IN found
                vOther_length = lengths[v] + weight
                IF vOther_length < min_length
                    min_length = vOther_length
                    vMin = vOther
        found.add(vMin)
        lengths[vMin] = min_length

RETURN lengths
FUNCTION Dijkstra(start_vertex)
found = {}
lenghts = FOR v IN G.vertices
lengths[v] = INFINITY

found.add(start_vertex)
lenghts[start_vertex] = 0

WHILE found.length != G.vertices.length
FOR v IN found
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IF vOther NOT IN found
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                IF vOther_length < min_length
                    min_length = vOther_length
                    vMin = vOther
                found.add(vMin)
                lengths[vMin] = min_length

RETURN lengths
Exercise
Dijkstra’s Algorithm with negative edges

• How might you deal with negative edges?
• How about adding some value to every edge?

What is the shortest path from s to t?
Dijkstra’s Algorithm with negative edges

• How might you deal with negative edges?
• How about adding some value to every edge?

What is the shortest path from s to t?
Dijkstra’s Algorithm with negative edges

• How might you deal with negative edges?
• How about adding some value to every edge?

We would add a different amount to each path!
Dijkstra’s Algorithm

• What have we done so far?
• We’ve only shown that it works for the given example.
• This is not enough to prove correctness.

• In general, examples are good for:
  • Demonstration
  • Contradictions

• They are not good for proving correctness.
Proof by Induction Cheat-sheet

Proof by induction that $P(n)$ holds for all $n$

1. $P(1)$ holds because <something about the code/problem>
2. Let’s assume that $P(k)$ (where $k < n$) holds.
3. $P(n)$ holds because of $P(k)$ and <something about the code>
4. Thus, by induction, $P(n)$ holds for all $n$
Correctness

**Theorem** for Dijkstra’s algorithm:

For every graph with non-negative edge lengths, Dijkstra’s algorithm computes all shortest path distances from `start_vertex` to every other vertex.

**Base Case:**
- `lengths[start_vertex] = 0`

Proof by induction that $P(n)$ holds for all $n$
- $P(1)$ holds because ...
- Let’s assume that $P(k)$ (where $k < n$) holds.
- $P(n)$ holds because of $P(k)$ and ...
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Correctness

**Theorem** for Dijkstra’s algorithm:

For every graph with non-negative edge lengths, Dijkstra’s algorithm computes all shortest path distances from `start_vertex` to every other vertex.

**Inductive Hypothesis:**

- Assume all previous iterations produce correct shortest paths.
- For all `v` in `found`, `lengths[v] = shortest path length from start_vertex to v`.

Proof by induction that `P(n)` holds for all `n`:

- `P(1)` holds because ...
- Let’s assume that `P(k)` (where `k < n`) holds.
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                    vOther_length = lengths[v] + weight
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                        min_length = vOther_length
                        vMin = vOther
                    found.add(vMin)
                    lengths[vMin] = min_length

    RETURN lengths

Proof by induction that P(n) holds for all n
• P(1) holds because ...
• Let’s assume that P(k) (where k < n) holds.
  • P(n) holds because of P(k) and ...
• Thus, by induction, P(n) holds for all n
Inductive Step

In the current iteration:

- We pick an edge \((v^*, v_{\text{Min}})\) based on Dijkstra’s greedy criterion
- add \(v_{\text{Min}}\) to found
- Set the path length of \(v_{\text{Min}}\) \(\rightarrow\) \(\text{lengths}[v_{\text{Min}}] = \text{lengths}[v^*] + \text{weight}_{v^*, v_{\text{Min}}}\)

What do we know about \(\text{lengths}[v^*]?)?

- Optimal path to \(v^*\), and we won’t find a better path to \(v_{\text{Min}}\)

Our inductive hypothesis states that it is the minimal path length

How do we prove this? Loop Invariant

Proof by induction that \(P(n)\) holds for all \(n\)

- \(P(1)\) holds because ...
- Let’s assume that \(P(k)\) (where \(k < n\)) holds.
- \(P(n)\) holds because of \(P(k)\) and ...
- Thus, by induction, \(P(n)\) holds for all \(n\)
Inductive Step

In the current iteration:
• We pick an edge \((v^*, v_{\text{Min}})\) based on Dijkstra’s greedy criterion
• add \(v_{\text{Min}}\) to found
• Set the path length of \(v_{\text{Min}} \rightarrow \text{lengths}[v_{\text{Min}}] = \text{lengths}[v^*] + \text{weight}_{v^*, v_{\text{Min}}}\)

What do we know about \(\text{lengths}[v^*]\)?
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How do we prove this?

Loop Invariant

By our inductive hypothesis, our theorem for Dijkstra’s is correct
Correctness

How many different types of paths do we consider each iteration?
Correctness

Dijkstra’s says that this is the best available path.
Correctness

How do we know that the path from \( v^* \) to \( v_{\text{Min}} \) is better than the path from \( v^* \) to \( y \)?

Both include the path from \( s \) to \( v^* \), and Dijkstra’s Algorithm always picks the minimal path length.
Correctness

How do we know that the path from \( v^* \) to \( y \) to \( v_{\text{Min}} \) is not even better than the path from \( v^* \) to \( v_{\text{Min}} \)?

Dijkstra's Algorithm only operates on graphs with positive edge weights. Thus, this new path must be greater than or equal to the \((v^*, v_{\text{Min}})\) edge.
How do we know that the path from $v^*$ to $v_{\text{Min}}$ is better than the path from $x$ to $v_{\text{Min}}$?

Dijkstra’s Algorithm compares these two options and picks the minimal path length.
Correctness

How do we know that the path from $x$ to $y$ to $v_{\text{Min}}$ is not even better than the path from $v^*$ to $v_{\text{Min}}$?

Dijkstra’s Algorithm only operates on graphs with positive edge weights. Thus, this new path must be greater than or equal to the $(v^*, v_{\text{Min}})$ edge.
Not taking the shortest edge. We are taking the shortest path!
Sometimes the shortest edge is on the shortest path.
Why doesn’t Dijkstra’s work on graphs with negative edges?
Correctness (summary)

• Given our assumption that we do not have negative edges
• And our inductive hypothesis that our path to \( v^* \) is the shortest
• And our analysis of Dijkstra’s greedy criterion

• We have shown that

\[
\text{lengths}[v_{\text{Min}}] = \text{lengths}[v^*] + \text{weight}_{v^*,v_{\text{Min}}} \text{ is the best available path length}
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    RETURN lengths

What is the running time?

How many times does the outer loop run?
O(n)

How many times do the inner two loops run?
O(m)

O(nm)