Graph Representations

https://cs.pomona.edu/classes/cs140/
Topics and Learning Objectives

• Graphs with weights
• Directed graphs
• Edge lists
• Adjacency matrices
• Adjacency lists

Exercise
• Friend Circles
Comparisons

For each representation, we are going to ask the following questions:

1. How do we count the number of vertices, and how long does it take?
2. How do we count the number of edges, and how long does it take?
3. How do we add vertices, and how long does it take?
4. How do we add edges, and how long does it take?
5. How do you check for the existence of an edge, and how long does it take?
6. How do you find all neighbors of a vertex, and how long does it take?
7. How much memory is needed to store the graph?
(0, 1) OR (1, 0)

(0, 1, 5) OR (1, 0, 5)
Edge List Representation

\[
\text{edge}_\text{list} = [ \\
(0, 1), (0, 6), (0, 8), \\
(1, 4), (1, 6), (1, 9), \\
(2, 4), (2, 6), \\
(3, 4), (3, 5), (3, 8), \\
(4, 5), (4, 9), \\
(7, 8), (7, 9), \\
]\]
edge_list = [(0, 1), (0, 6), (0, 8),
(1, 4), (1, 6), (1, 9),
(2, 4), (2, 6),
(3, 4), (3, 5), (3, 8),
(4, 5), (4, 9),
(7, 8), (7, 9),
]

# 1. How do we count the number of vertices, and how long does it take?
def count_vertices_el(el):
    '''Retun the number of vertices in an edge list. Directedness does not matter.

This procedure loops over all m vertices. So:
T(n, m) = O(m)
'''
    V = set()
    for e in el:
        v1, v2 = e[0], e[1]

        # We have to add both v1 and v2 since v2 # might never show up as the first vertex in # an edge.
        V.add(v1)
        V.add(v2)

    return len(V)
# 2. How do we count the number of edges, and how long does it take?

def count_edges_el(el):
    '''Return the number of edges in an edge list.

    This function assumes that the edge list does not contain any self-connections (an edge connecting a vertex to itself).

    T(n, m) = O(1)
    '''
    return len(el)
### 3. How do we add vertices, and how long does it take?

```python
def add_vertex_el(el, v, is_weighted=False):
    '''Add a new vertex to an edge list.

    This function assumes that v is not already in the edge list.

    T(n, m) = O(1)
    '''
    new_edge = (v, v, 0) if is_weighted else (v, v)
    return add_edge_el(new_edge)
```

This function assumes that `v` is not already in the edge list.
# 4. How do we add edges, and how long does it take?

def add_edge_el(el, e):
    '''Add a new edge to an edge list.

    This function assumes that e is not already in the edge list.

    T(n, m) = O(1)
    '''
    el.append(e)
    return el
# 5. How do you check for the existence of an edge, and how long does it take?

def find_edge_el(el, e, is_ordered=False):
    '''Check for existence of edge in edge list.

    If is_ordered is True, then all edges are in sorted order.

    This function does not assume directed or undirected edges, so the order
    (v1, v2) vs (v2, v1) DOES matter.

    This function will not work well with floating-point weights.

    Must search through all edges in order, or using a binary search.
    T(n, m) = O(m), or
    T(n, m) = O(lg(m))
    '''

edge_list = [
    (0, 1), (0, 6), (0, 8),
    (1, 4), (1, 6), (1, 9),
    (2, 4), (2, 6),
    (3, 4), (3, 5), (3, 8),
    (4, 5), (4, 9),
    (7, 8), (7, 9),
]
edge_list = [
    (0, 1), (0, 6), (0, 8),
    (1, 4), (1, 6), (1, 9),
    (2, 4), (2, 6),
    (3, 4), (3, 5), (3, 8),
    (4, 5), (4, 9),
    (7, 8), (7, 9),
]

# Binary search
if is_ordered:
    left, right = 0, len(el) - 1
    while left <= right:
        mid = (left + right) // 2
        if e == el[mid]:
            return True
        elif e < el[mid]:
            right = mid - 1
        else:
            left = mid + 1

# Linear search
else:
    for edge in el:
        if e == edge:
            return True

return False
edge_list = [
    (0, 1), (0, 6), (0, 8),
    (1, 4), (1, 6), (1, 9),
    (2, 4), (2, 6),
    (3, 4), (3, 5), (3, 8),
    (4, 5), (4, 9),
    (7, 8), (7, 9),
]

# 6. How do you find all neighbors of a vertex, and how long does it take?
def find_neighbors_el(el, v, is_directed=False):
    '''Return all neighbors of a given vertex.

    This function does not assume edges are in sorted order.
    
    T(n, m) = O(m)
    
    neighbors = []
    for edge in el:
        v1, v2 = edge[0], edge[1]
        if v1 == v:
            neighbors.append(v2)
        elif not is_directed and v2 == v:
            neighbors.append(v1)
    return neighbors
edge_list = [(0, 1), (0, 6), (0, 8), (1, 4), (1, 6), (1, 9),
             (2, 4), (2, 6), (3, 4), (3, 5), (3, 8), (4, 5), (4, 9),
             (7, 8), (7, 9),]

# 7. How much memory is needed to store the graph?
def calc_memory_el(el):
    '''Calculate the approximate amount of memory used by el in bytes.
    Edge lists use 2 or 3 numbers for representing undirected, or directed graphs, respectively.
    \[ M(n, m) = O(m) \]
    \[
    m = \text{count_edges_el}(el)
    
    # Memory used for storing vertices
    \text{vertices\_per\_edge} = 2
    \text{bytes\_per\_int} = 4
    \text{memory\_usage} = m \times \text{vertices\_per\_edge} \times \text{bytes\_per\_int}
    
    # Check for edge weights
    \text{if} \ \text{len}(el[0]) == 3:
        \text{bytes\_per\_float} = 8
        \text{memory\_usage} += m \times \text{bytes\_per\_float}
    
    \text{return} \ \text{memory\_usage}
Adjacency Matrix Representation

adjacency_matrix = [
    [0, 1, 0, 0, 0, 0, 1, 0, 1, 0],
    [1, 0, 0, 0, 1, 0, 1, 0, 0, 1],
    [0, 0, 0, 0, 1, 0, 1, 0, 0, 0],
    [0, 0, 0, 0, 1, 1, 0, 0, 1, 0],
    [0, 1, 1, 1, 0, 0, 0, 1, 0, 0, 1],
    [0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0],
    [1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0],
    [0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1],
    [1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0],
    [0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0],
]

# Or T/F
# Symmetric matrix for undirected graphs
# 1. How do we count the number of vertices, and how long does it take?

def count_vertices_am(am):
    '''Retun the number of vertices in an adjacency matrix.
    
    T(n, m) = O(1)
    '''

    return len(am)
# 2. How do we count the number of edges, and how long does it take?

def count_edges_am(am, is_weighted=False, is_directed=False):
    '''Return the number of edges.
    We must loop over the entire matrix and look for non-zero entries.
    If is_weighted is True, then assume all entries are None or a floating-point number.
    If is_directed is False, then the matrix is symmetric.
    \[ T(n, m) = O(n^2) \]
    \[
    \begin{align*}
    \text{if ~is_weighted:} & \\
    \text{num_edges} &= \text{sum}(\text{sum}(1 \text{ for val in row} \\
    \text{if ~val ~is ~not ~None) ~for ~row ~in ~am}) \\
    \text{else:} & \\
    \text{num_edges} &= \text{sum}(\text{sum}(\text{row}) \text{ for row ~in ~am})
    \end{align*}
    \]
    return num_edges ~if ~is_directed ~else ~num_edges ~// ~2
adjacency_matrix = [
    [0, 1, 0, 0, 0, 1, 0, 1, 0],
    [1, 0, 0, 1, 0, 1, 0, 0, 1],
    [0, 0, 0, 1, 0, 1, 0, 0, 0],
    [0, 0, 0, 1, 1, 0, 0, 1, 0],
    [0, 1, 1, 0, 1, 0, 0, 0, 0],
    [1, 0, 0, 1, 0, 0, 0, 0, 0],
    [0, 1, 0, 0, 0, 0, 1, 1, 1],
    [1, 0, 1, 0, 0, 0, 0, 0, 0],
    [0, 1, 0, 0, 1, 0, 0, 1, 0],
    [0, 1, 0, 0, 0, 1, 0, 1, 0],
]

# 3. How do we add vertices, and how long does it take?
def add_vertex_am(am, is_weighted=False):
    '''Add a new vertex to an adjacency matrix.
    Use the next available index.
    We must create a new row and a new column
    T(n, m) = O(n)
    '''
    default_value = None if is_weighted else 0

    # Add the new column
    for row in am:
        row.append(default_value)

    # Add the new row
    new_n = count_edges_am(am)
    am.append([default_value] * new_n)
    return am
adjacency_matrix = [
[0, 1, 0, 0, 0, 0, 1, 0, 1, 0],
[1, 0, 0, 0, 1, 0, 0, 1, 0, 1],
[0, 0, 0, 0, 0, 1, 0, 0, 0, 0],
[0, 0, 0, 0, 1, 0, 0, 1, 0, 0],
[0, 1, 1, 1, 1, 0, 0, 0, 0, 0],
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
[0, 1, 1, 0, 1, 0, 0, 1, 0, 0],
[1, 0, 0, 0, 0, 0, 0, 0, 0, 0],
[0, 1, 0, 0, 0, 0, 0, 0, 0, 0],
[0, 1, 0, 0, 0, 0, 0, 0, 0, 0],
]

# 4. How do we add edges, and how long does it take?
def add_edge_am(am, e, is_directed=False):
    '''Add a new edge to an adjacency matrix.

    If is_directed is False, then the matrix is symmetric.

    T(n, m) = O(1)
    '''
    # Check for edge weight
    if len(e) == 2:
        e = (e[0], e[1], 1)
    v1, v2, w = e
    am[v1][v2] = w
    if not is_directed:
        am[v2][v1] = w
    return am
adjacency_matrix = [
    [0, 1, 0, 0, 0, 1, 0, 1, 0],
    [1, 0, 0, 1, 0, 1, 0, 0, 1],
    [0, 0, 1, 0, 0, 1, 0, 0, 1],
    [0, 0, 0, 1, 1, 0, 0, 1, 0],
    [0, 1, 1, 0, 1, 0, 0, 0, 1],
    [0, 0, 0, 1, 1, 0, 0, 0, 0],
    [1, 1, 0, 0, 0, 0, 0, 0, 0],
    [0, 0, 0, 0, 0, 0, 0, 1, 1],
    [1, 0, 0, 1, 0, 0, 1, 0, 0],
    [0, 1, 0, 1, 0, 0, 1, 0, 0]
]

# 5. How do you check for the existence of an edge, and how long does it take?
def find_edge_am(am, e, is_weighted=False):
    '''Check for existence of edge in adjacency matrix.

    This function assumes that edges contain only v1 and v2.

    T(n, m) = O(1)
    '''

    if is_weighted:
        v1, v2, _ = e
        return am[v1][v2] != None
    else:
        v1, v2 = e
        return am[v1][v2] == 1
# 6. How do you find all neighbors of a vertex, and how long does it take?

def find_neighbors_am(am, v, is_weighted=False):
    '''Return all neighbors of a given vertex.

    T(n, m) = O(n)
    '''

    default_value = None if is_weighted else 0
    neighbors = []

    for v2, w in enumerate(am[v]):
        if w != default_value:
            neighbors.append(v2)

    return neighbors

adjacency_matrix = [
    [0, 1, 0, 0, 0, 0, 1, 0, 1, 0],
    [1, 0, 0, 1, 0, 1, 0, 0, 1, 0],
    [0, 0, 0, 1, 0, 1, 0, 0, 0, 0],
    [0, 0, 0, 1, 1, 0, 0, 1, 0, 0],
    [0, 1, 1, 0, 1, 0, 0, 1, 0, 1],
    [0, 0, 0, 1, 1, 0, 0, 0, 0, 0],
    [1, 1, 0, 0, 0, 0, 0, 0, 0, 0],
    [0, 0, 0, 0, 0, 0, 0, 0, 1, 1],
    [1, 0, 0, 1, 0, 0, 1, 0, 0, 0],
    [0, 1, 0, 0, 1, 0, 0, 1, 0, 0],
]
adjacency_matrix = [
    [0, 1, 0, 0, 0, 1, 0, 1, 0],
    [1, 0, 0, 0, 1, 0, 1, 0, 0],
    [0, 0, 0, 0, 1, 0, 1, 0, 0],
    [0, 0, 0, 1, 1, 0, 0, 0, 0],
    [0, 1, 1, 0, 1, 0, 0, 0, 0],
    [1, 0, 0, 0, 0, 0, 0, 0, 0],
    [0, 0, 0, 1, 0, 0, 0, 0, 1],
    [1, 0, 0, 0, 0, 0, 0, 1, 0],
    [0, 1, 0, 0, 0, 0, 1, 0, 0],
]

# 7. How much memory is needed to store the graph?
def calc_memory_am(am, is_weighted=False):
    '''Calculate the approximate amount of memory used by am in bytes.
    Matrices store booleans or floats for representing un-weighted,
or weighted graphs, respectively.

    M(n, m) = O(n^2)
    '''

    n = count_vertices_am(am)
    bytes_per_bool = 1
    bytes_per_float = 8
    bytes_per_cell = bytes_per_float if is_weighted else bytes_per_bool
    return n * n * bytes_per_cell
Adjacency List Representation

```
adjacency_list = [
    [1, 6, 8],
    [0, 4, 6, 9],
    [4, 6],
    [4, 5, 8],
    [1, 2, 3, 5, 9],
    [3, 4],
    [0, 1, 2],
    [8, 9],
    [0, 3, 7],
    [1, 4, 7],
]
```
# 1. How do we count the number of vertices, and how long does it take?

def count_vertices_al(al):
    '''Retun the number of vertices in an adjacency list.

    T(n, m) = O(1)
    '''
    return len(al)
# 2. How do we count the number of edges, and how long does it take?

def count_edges_al(al, is_directed=False):
    '''Return the number of edges in an adjacency list.

    We must loop over each vertex and count edges.

    T(n, m) = O(n)
    ...
    
    if is_directed:
        return sum(len(l) for l in al)
    else:
        return sum(len(l) for l in al) // 2
adjacency_list = [
    [1, 6, 8],
    [0, 4, 6, 9],
    [4, 6],
    [4, 5, 8],
    [1, 2, 3, 5, 9],
    [3, 4],
    [0, 1, 2],
    [8, 9],
    [0, 3, 7],
    [1, 4, 7],
]

# 3. How do we add vertices, and how long does it take?
def add_vertex_al(al):
    '''Add a new vertex to an adjacency list.
    Use the next available index.
    This function assumes that v is not already in the adjacency list.
    T(n, m) = O(1)
    '''
    al.append([])
    return al
# 4. How do we add edges, and how long does it take?
def add_edge_al(al, e):
    '''Add a new edge to an adjacency list.'''

    v1, v2 = e[0], e[1]

    # Check for edge weight
    if len(e) == 3:
        v2 = (v2, e[2])

    al[v1].append(v2)
    return al
adjacency_list = [
    [1, 6, 8],
    [0, 4, 6, 9],
    [4, 6],
    [4, 5, 8],
    [1, 2, 3, 5, 9],
    [3, 4],
    [0, 1, 2],
    [8, 9],
    [0, 3, 7],
    [1, 4, 7],
]

# 5. How do you check for the existence of an edge, and how long does it take?
def find_edge_al(al, e, is_weighted=False):
    '''Check for existence of edge in adjacency list.

    This function assumes that edges contain only v1 and v2.

    T(n, m) = O(m)
    '''
    v1 = e[0]
    vw = (e[1], e[2]) if is_weighted else e[1]
    return vw in al[v1]
adjacency_list = [
    [1, 6, 8],
    [0, 4, 6, 9],
    [4, 6],
    [4, 5, 8],
    [1, 2, 3, 5, 9],
    [3, 4],
    [0, 1, 2],
    [8, 9],
    [0, 3, 7],
    [1, 4, 7],
]

# 6. How do you find all neighbors of a vertex, and how long does it take?
def find_neighbors_al(al, v, is_weighted=False):
    '''Return all neighbors of a given vertex.'''

    T(n, m) = O(1)
    '''
    return al[v]
adjacency_list = [
    [1, 6, 8],
    [0, 4, 6, 9],
    [4, 6],
    [4, 5, 8],
    [1, 2, 3, 5, 9],
    [3, 4],
    [0, 1, 2],
    [8, 9],
    [0, 3, 7],
    [1, 4, 7],
]

# 7. How much memory is needed to store the graph?
def calc_memory_al(al, is_weighted=False):
    '''Calculate the approximate amount of memory used by al in bytes.

    Adjacency lists use 1 or 2 numbers for representing each edge in weighted or unweighted graphs, respectively.

    M(n, m) = O(n + m)
    '''
    m = count_edges_al(al)

    # Memory used for storing vertices
    vertices_per_edge = 1
    bytes_per_int = 4
    memory_usage = m * vertices_per_edge * bytes_per_int

    # Check for edge weights
    if is_weighted:
        bytes_per_float = 8
        memory_usage += m * bytes_per_float
    return memory_usage
<table>
<thead>
<tr>
<th></th>
<th>m (\leq) n</th>
<th>Edge List</th>
<th>Adjacency Matrix</th>
<th>Adjacency List</th>
</tr>
</thead>
<tbody>
<tr>
<td>Count Vertices</td>
<td>O(m)</td>
<td>O(1)</td>
<td>O(1)</td>
<td>O(1)</td>
</tr>
<tr>
<td>Count Edges</td>
<td>O(1)</td>
<td>O(n^2)</td>
<td>O(n)</td>
<td>O(n)</td>
</tr>
<tr>
<td>Add Vertex</td>
<td>O(1)</td>
<td>O(n)</td>
<td>O(1)</td>
<td>O(1)</td>
</tr>
<tr>
<td>Add Edge</td>
<td>O(1)</td>
<td>(1)</td>
<td>O(1)</td>
<td>O(1)</td>
</tr>
<tr>
<td>Check Edge</td>
<td>O(m) or O(lg(m))</td>
<td>O(1)</td>
<td>O(m)</td>
<td>O(m)</td>
</tr>
<tr>
<td>Find Neighbors</td>
<td>O(m)</td>
<td>O(n)</td>
<td>O(1)</td>
<td>O(1)</td>
</tr>
<tr>
<td>Memory Usage</td>
<td>O(m)</td>
<td>O(n^2)</td>
<td>O(n + m)</td>
<td>O(n + m)</td>
</tr>
</tbody>
</table>
Exercise

Friend Circles Algorithm

call DFS on each node.