Lower Bound on Comparison-Based Sorting

https://cs.pomona.edu/classes/cs140/
Outline

**Topics and Learning Objectives**

• Discuss a lower bound for the running time of all comparison-based sorting algorithms

**Exercise**

• Lower bound
Extra Resources

• *Introduction to Algorithms, 3rd, Chapter 8*
Comparison-Based Sorting

**Claim:** The worst-case, lower bound on comparison-based sorting is $\Omega(n \lg n)$

Comparison-based sorting methods:
• Merge sort, quicksort, heapsort, insertion sort, bubble sort, ...
• General purpose routines

Non-comparison-based sorting methods:
• Bucket sort, counting sort, radix sort, ...
• These methods look at the values (not just at the relative ordering)
• They assume something about the distribution of the data
• They can operate in linear time
Proof

• Consider an array of the values 1..n
• The array has \( n! \) different orderings (permutations)
• We can only use the results of comparisons to reorder elements
• Suppose an algorithm makes \( k \) comparisons
• We don’t know what \( k \) is just yet
• How many possible distinct comparisons sequences do we have?

We need an equation based on \( k \)

• What is a reasonable upper bound on \( k \)?
• What is the lower bound on \( k \)?

\( 2^k \)

\( n \lg n \)

\( n^2 \)
Proof

Given each of the $n!$ inputs and the $k$ comparisons:

- We have $2^k$ distinct comparison sequences
- For each of the $k$ comparison we can return value $a$ or value $b$
- You can think of these comparisons as a decision tree

$A = [a_1, a_2, a_3]$

$k$ is the maximum depth of the tree
How many leaves as a function of $n$? $n!$

What is the height of the tree as a function of $k$? $k$

What is the maximum number of leaves in a depth $k$ binary tree? $2^k$

What is the minimum height of a binary tree with $n!$ leaves? $\lg(n!)$
Let’s find a bound on $k$

What is bigger?

- The number of leaves with $n!$ numbers OR
- The maximum number of leaves for a tree of height $k$?
Let’s find a bound on $k$

What is bigger?

- The number of leaves with $n!$ numbers OR
- The maximum number of leaves for a tree of height $k$?

$n! \leq 2^k$

Maximum number of leaves with depth $k$ (k Comparisons)

$$\ln(n!) \leq \ln(2^k)$$

$$\ln(n!) \leq k \cdot \ln(2)$$

$\Rightarrow \ln(n!) \leq k \cdot c_1$

Lower Bound!

Number of comparisons $k$ is at least...

$$\frac{1}{c_1} \ln(n!) \leq k$$
Let’s find a bound on $k$

Stirling's approximation:
\[ \ln(n!) = n \cdot \ln(n) - n + O(\ln(n)) \]

\[ n \cdot \ln(n) - n + O(\ln(n)) \leq k \cdot c_1 \]
\[ n \cdot \ln(n) + O(\ln(n)) \leq k \cdot c_1 \]
\[ c_3 n \ln(n) \leq k \cdot c_1 \]
\[ \frac{c_3}{c_1} n \ln(n) \leq k \]
\[ c_4 n \ln(n) \leq k \]
\[ k = \Omega(n \ln(n)) \]