Master Method

https://cs.pomona.edu/classes/cs140/
Topics and Learning Objectives

• Learn about the master method for solving recurrences
• Understand how to draw general recursion trees

Exercise

• Applying the Master Method
Extra Resources

• Chapter 4 (sections 4-6) in CLRS

• Master Method
Master Method

• For “solving” recurrences

\[ T(n) = \text{the # of operations required to complete algorithm} \]
\[ T(n) = 2T(n/2) + 7n \]

Base Case: \[ T(1) \leq \text{base-case-work} \]
Recurrence: \[ T(n) \leq \text{recursive-work + combine-work} \]
Recurrence Equation

• When an algorithm contains a recursive call to itself

• We usually specify its running time by a recurrence equation

• We also sometimes just call this a “recurrence”

• A recurrence equation describes the overall running time on a problem of size $n$ in terms of the running time on smaller inputs (some fraction of $n$)

\[ T(n) = T\left(\frac{n}{3}\right) \]
FUNCTION MergeSort(array)

n = array.length

IF n == 1

RETURN array

left_sorted = MergeSort(array[0..<n//2])

right_sorted = MergeSort(array[n//2..<n])

array_sorted = Merge(left_sorted, right_sorted)

RETURN array_sorted

Recurrence Equation

T(n) = 2 \( T(n/2) \) + \( O(n) \)
Master Method

“Black Box” for solving recurrences

Assumes all subproblems are of equal size (most algorithms do this)

• The same amount of data is given to each recursive call

An algorithm that splits the subproblems into 1/3 and 2/3 (or an algorithm that splits data randomly) must be solved in a different manner. We’ll look at other methods later
Master Method Recurrence Equation

\[ T(n) \leq a \ T\left(\frac{n}{b}\right) + O(n^d) \]

T(n) : total amount of operations
a : recursive calls (# of subproblems), always \( \geq 1 \)
b : fraction of input size (shrinkage), always \( > 1 \)
d : extra work needed to combine, always \( \geq 0 \)

What does zero mean for \( d \)?
Master Method Cases

\[ T(n) \leq_{\alpha} T\left(\frac{n}{b}\right) + O(n^{\alpha}) \]

\[ T(n) = \begin{cases} 
O(n^d \log n), & a = b^d \\
O(n^d), & a < b^d \\
O(n^{\log_b a}), & a > b^d 
\end{cases} \]
Master Method Cases

\[ T(n) \leq a \ T\left(\frac{n}{b}\right) + O(n^d) \]

\[ T(n) = \begin{cases} 
    O(n^d \lg n), & a = b^d \text{ Case 1} \\
    O(n^d), & a < b^d \text{ Case 2} \\
    O(n^{\log_b a}), & a > b^d \text{ Case 3} 
\end{cases} \]
Exercise Question 1

- Merge sort

\[ T(n) = 2T(\frac{n}{2}) + O(n) \]

\[ a = 2 \]
\[ b = 2 \]
\[ d = 1 \]

a ? b  
\[ T(n) = \begin{cases} 
O(n^d \log n), & a = b^d \\
O(n^d), & a < b^d \\
O(n^{\log_b a}), & a > b^d 
\end{cases} \]
Exercise Question 2

• Binary search

\[ T(n) = T(n/2) + O(1) \]

- \( a = 1 \)
- \( b = 2 \)
- \( d = 0 \)

- \( \frac{a}{b} > 1 \) \( \Leftrightarrow \) case 1
- \( O(\log n) \)

- \( \frac{a}{b} = 1 \) \( \Leftrightarrow \) case 2
- \( O(n^d \log n) \)

- \( \frac{a}{b} < 1 \) \( \Leftrightarrow \) case 3
- \( O(n^d) \)

- \( \frac{a}{b} > 1 \) \( \Leftrightarrow \) case 4
- \( O(n^{\log_b a}) \)

\[ T(n) \leq a \cdot T\left(\frac{n}{b}\right) + O(n^d) \]

- \( T(n) : \) total amount of operations
- \( a : \) recursive calls (number of subproblems), always \( \geq 1 \)
- \( b : \) fraction of input size (shrinkage), always \( > 1 \)
- \( d : \) extra work needed to combine, always \( \geq 0 \)
Exercise Question 3

• Closest pair

\[ T(n) \leq a T\left(\frac{n}{b}\right) + O(n^d) \]

T(n) : total amount of operations
a : recursive calls (# of subproblems), always \( \geq 1 \)
b : fraction of input size (shrinkage), always > 1
d : extra work needed to combine, always \( \geq 0 \)

\[
T(n) = \begin{cases} 
O(n^d \log n), & a = b^d \\
O(n^d), & a < b^d \\
O(n^{\log_b a}), & a > b^d 
\end{cases}
\]
Exercise Question 4

- $T(n) \leq 2 \ T(n/2) + O(n^2)$

$T(n)$ : total amount of operations

- $a$ : recursive calls (number of subproblems), always $\geq 1$
- $b$ : fraction of input size (shrinkage), always $> 1$
- $d$ : extra work needed to combine, always $\geq 0$

$T(n) = \begin{cases} 
O(n^d \log n), & a = b^d \\
O(n^d), & a < b^d \\
O(n^{\log_b a}), & a > b^d 
\end{cases}$
Iterative Matrix Multiplication

1. FUNCTION IMM(X, Y)
2. Z = create_new_matrix(X.size, X.size)
3.
4. FOR i IN [0..<X.size]
5.    FOR j IN [0..<X.size]
6.      FOR k IN [0..<X.size]
7.        Z[i][j] += X[i][k] * Y[k][j]
8.
9. RETURN Z

\[ z_{ij} = \sum_{k=1}^{n} x_{ik} y_{kj} \]

What are “a”, “b”, and “d”?

\[ T(n) = O(n^3) \]
Recursive Matrix Multiplication

1. **FUNCTION** \( \text{RMM}(X, Y) \)
2. \[ \text{IF} \ X.\text{size} == 1 \]
3. \[ \text{return} \ X \ast Y \]
4. \[ Z = \text{create\_new\_matrix}(X.\text{size}, X.\text{size}) \]
5. \[ Z(1,1) = \text{RMM}(X(1,1), Y(1,1)) + \text{RMM}(X(1,2), Y(2,1)) \] # Upper left
6. \[ Z(1,2) = \text{RMM}(X(1,1), Y(1,2)) + \text{RMM}(X(1,2), Y(2,2)) \] # Upper right
7. \[ Z(2,1) = \text{RMM}(X(2,1), Y(1,1)) + \text{RMM}(X(2,2), Y(2,1)) \] # Lower left
8. \[ Z(2,2) = \text{RMM}(X(2,1), Y(1,2)) + \text{RMM}(X(2,2), Y(2,2)) \] # Lower right
9. \[ \text{return} \ Z \]

What are “a”, “b”, and “d”?
Exercise Question 5

• Recursive matrix multiplication

\[ T(n) = 8 T\left(\frac{n}{2}\right) + O(n^2) \]
\[ a = 8, \quad b = 2, \quad d = 2 \]
\[ 8 > 2^2 \]
\[ O(n \log_2 8) = O(n^3) \]

\[ T(n) \leq a T\left(\frac{n}{b}\right) + O(n^d) \]

T(n) : total amount of operations
a : recursive calls (# of subproblems), always >= 1
b : fraction of input size (shrinkage), always > 1
d : extra work needed to combine, always >= 0

\[ T(n) = \begin{cases} 
O(n^d \lg n), & a = b^d \\
O(n^d), & a < b^d \\
O(n^{\log_b a}), & a > b^d 
\end{cases} \]
Strassen’s Matrix Multiplication

1. FUNCTION SMM(X, Y)
2. IF X.size == 1
3. RETURN X * Y
4. PA = SMM(X(1,1), Y(1,2) - Y(2,2))
5. PB = SMM(X(1,1) + X(1,2), Y(2,2))
6. PC = SMM(X(2,1) + X(2,2), Y(1,1))
7. PD = SMM(X(2,2), Y(2,1) - Y(1,1))
8. PE = SMM(X(1,1) + X(2,2), Y(1,1) + Y(2,2))
9. PF = SMM(X(1,2) - X(2,2), Y(2,1) + Y(2,2))
10. PG = SMM(X(1,1) - X(2,1), Y(1,1) + Y(1,2))
11. Z(1,1) = PE + PD - PB + PF
12. Z(1,2) = PA + PB
13. Z(2,1) = PC + PD
14. Z(2,2) = PA + PE - PC - PG
15. RETURN Z
16. RETURN Z
17. RETURN Z

What are “a”, “b”, and “d”? 

\[ a = 7 \leq 9 \]
\[ b = 2 \]
\[ d = 2 \]
Exercise

• Strassen’s matrix multiplication

\[ T(n) = T \left( \frac{n}{2} \right) + O(n^2) \]

\[ a = 7, \quad b = 2, \quad d = 2 \]

\[ T(n) = \begin{cases} 
O(n^d \log n), & a = b^d \\
O(n^d), & a < b^d \\
O(n^{\log_b a}), & a > b^d 
\end{cases} \]
Master Method Proof

Assume
• \( T(1) = O(1) \) (this is our base-case)
• \( T(n) \leq a \frac{T(n/b)}{b} + cn^d \)
• \( n \) is a power of \( b \) (not necessary, but makes the math easier)

• How did we analyze our first recurrence/divide-and-conquer algorithm?
Generalizing the Recursion Tree Analysis

For merge sort

• What was the # number of subproblems for a given level L?
• What was the size of each of the subproblems at level L?
• How many total levels were there?
Merge Sort Exercise

1. How many sub-problems are there at level ‘L’? (Note: the top level is ‘Level 0’, the second level is ‘Level 1’, and the bottom level is ‘Level log₂(n)’)
   Answer: 2^L

2. How many elements are there for a given sub-problem found in level ‘L’?
   Answer: n/2^L

3. How many computations are performed at a given level? (Note the cost of a ‘merge’ operation was 21m)
   Answer: 2^L * 21(n/2^L) = 21n

4. What is the total computational cost of merge sort?
   Answer: 21n (log₂(n) + 1)
Generalizing the Recursion Tree Analysis

For merge sort
• What was the # number of subproblems for a given level L?
• What was the size of each of the subproblems at level L?
• How many total levels were there?

In the general case
• What is the # number of subproblems for a given level L?
• What is the size of each of the subproblems at level L?
• How many total levels are there?
$T(n) = \Theta(n^{\log_b n}) + O(n^a)$
How many sub-problems at level L?
How many elements for each problem at level L?
How much work is done outside of recursion?

Level 0

Level 1

Level \log_2 n
What is the total work done at level L?

\[ a^L c \left( \frac{n}{b^L} \right)^d \]
What is the total work done at level L?

Work at Level L

$$a^L c \left(\frac{n}{b_L}\right)^d$$
What is the total work done at level L?

Work at Level L

Rewrite to group together terms dependent on level

\[ T(n) = \begin{cases} 
  O(n^d \lg n), & \text{if } a = b^d \\
  O(n^d), & \text{if } a < b^d \\
  O(n^{\log_b a}), & \text{if } a > b^d 
\end{cases} \]
What is the total work done for the tree?

Work at Level L

$$a^L c (n/b_L)^d$$

Rewrite to group together terms dependent on level

$$cn^d (a/b_d)^L$$

Work done for the entire tree

$$T(n) \leq cn^d \sum_{L=0}^{\log_b n} (a/b_d)^L$$
Work done by a recursive algorithm

\[ T(n) \leq cn^d \sum_{L=0}^{\log_b n} \left( \frac{a}{bd} \right)^L \]
Let’s look at the cases again

<table>
<thead>
<tr>
<th>Condition</th>
<th>Description</th>
<th>Time Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a = b^d)</td>
<td>work stays roughly the same at each level</td>
<td>(O(n^d \lg n))</td>
</tr>
<tr>
<td>(a &lt; b^d)</td>
<td>work goes down at each level</td>
<td>(O(n^d))</td>
</tr>
<tr>
<td>(a &gt; b^d)</td>
<td>work goes up at each level</td>
<td>(O(n^{\log_b a}))</td>
</tr>
</tbody>
</table>
Review

• We have three difference cases of trees
  1. Work is similar at each level
  2. Work decreases at each level
  3. Work increases at each level

• These trees lead to our three cases for the Master Method
• What really matters is the ratio between $a$ and $b^d$

$$T(n) = \begin{cases} 
O(n^d \lg n), & a = b^d \quad \text{Case 1} \\
O(n^d), & a < b^d \quad \text{Case 2} \\
O(n^{\log_b a}), & a > b^d \quad \text{Case 3} 
\end{cases}$$

From where do we get the cases?
Couple of helpers

$$\sum_{i=0}^{k} 1 = ?$$
Couple of helpers

\[ \sum_{i=0}^{k} 1 = k + 1 \]

\( \log_{a}(n) = O(\log_{2}(n)) \) for all values of \( a \geq 1 \)
Couple of helpers

\[ \sum_{i=0}^{k} 1 = k + 1 \]

\[ \log_a(n) = \frac{\log_2(n)}{\log_2(a)} = c \log_2(n) = O(\log_2(n)) \text{ for all values of } a \geq 1 \]

\[ \sum_{i=0}^{k} r^i = ? \]

\[ O(\log n) \]

\[ \log_2 n \]
Couple of helpers

\[ \sum_{i=0}^{k} 1 = k + 1 \]

\[ \log_a(n) = \frac{\log_2(n)}{\log_2(a)} = c \log_2(n) = O(\log_2(n)) \text{ for all values of } a \geq 1 \]

\[ \sum_{i=0}^{k} r^i = \frac{r^{k+1} - 1}{r - 1} \]
Proving the Master Method: **Case 1**

Claim: \( T(n) = O(n^d \log n) \)
\[ T(n) \leq cn^d \sum_{L=0}^{\log_b n} \left( \frac{a}{bd} \right)^L \leq cn^d (\log_b n + 1) = O(n^d \log n) \]

\begin{align*}
Cn^d (\log_b n + 1) &= O(n^d \log n) \quad \forall \ n \geq n_0 \\
Cn^d \log_b n + Cn^d &\leq Cn^d \log_b n + Cn^d (\log_b n) \leq C_1 n^d \log n \\
2Cn^d \log_b n &= 2Cn^d \log n \cdot \frac{1}{\log b} \leq C_1 n^d \log n \quad \forall \ n \geq b \\
C_1 &= \frac{2C}{\log b^L} \quad n_0 = b
\end{align*}
Master Method

\[ T(n) \leq a \cdot T\left(\frac{n}{b}\right) + O(n^d) \]

\[ T(n) = \begin{cases} 
O(n^d \lg n), & a = b^d \\
O(n^d), & a < b^d \\
O(n^{\log_b a}), & a > b^d 
\end{cases} \]
Proving the Master Method: Case 2

\[ T(n) \leq cn^d \sum_{L=0}^{\log_b n} \left( \frac{a}{b^d} \right)^L \]

\[ cn^d \left( \frac{a}{b^d} \right)^{\log_b n + 1 - 1} \]

\[ cn^d \frac{1 - \left( \frac{a}{b^d} \right)^{\log_b n + 1}}{1 - \left( \frac{a}{b^d} \right)} \]

\[ a < b^d \]

\[ \sum_{r=0}^{k} r^i = \frac{r^{k+1} - 1}{r - 1} \]

Multiply top and bottom by -1

The ratio is ≤ 1, so we can remove the term and keep the original inequality
Proving the Master Method: Case 2

\[ T(n) \leq cn^d \left( \frac{1}{1-(a/bd)} \right) \]

Claim: \( T(n) = O(n^d) \)

\( a/bd \) is constant with respect to \( n \)

\[ cn^dc_2 \leq c_3 n^d \quad \forall \ n \geq n_0 \]

\[ c_3 = cc_2 \]

\[ n_0 = 1 \]
Master Method

\[ T(n) \leq a \cdot T\left(\frac{n}{b}\right) + O(n^d) \]

\[ T(n) = \begin{cases} 
O(n^d \lg n), & a = b^d \\
O(n^d), & a < b^d \\
O(n^{\log_b a}), & a > b^d 
\end{cases} \]
Proving the Master Method: Case 3

$$T(n) \leq cn^d \sum_{L=0}^{\log_b n} \left( \frac{a}{bd} \right)^L$$

$$cn^d \sum_{L=0}^{\log_b n} \left( \frac{a}{bd} \right)^L$$

$$cn^d \left( \frac{a}{bd} \right)^{\log_b n}$$

Claim: $$T(n) = O(n^{\log_b a})$$

$$ca^{\log_b n} = O(n^{\log_b a})$$

$$a > b^d$$

The last term dominates:

$$\left( \frac{a}{bd} \right)^{\log_b n}$$

Distribute the exponent and simplify:

$$\log_b (\frac{1}{bd}) \log_b n$$

$$\log_b n - d \log_b n$$

$$\log_b n - a$$

$$\log_b n - d$$
Master Method

\[ T(n) \leq a T\left(\frac{n}{b}\right) + O(n^d) \]

\[ T(n) = \begin{cases} 
O(n^d \lg n), & a = b^d \\
O(n^d), & a < b^d \\
O(n^{\log_b a}), & a > b^d
\end{cases} \]
Master Method Summary

1. We analyzed a generalized recursion tree
2. Counted the amount of work done at each level
3. Counted the amount of work done by the tree
4. Found that we have three different types of trees
   1. Same rate throughout (case 1: \( a = b^d \))
   2. Root dominates (case 2: \( a < b^d \))
   3. Leaves dominate (case 3: \( a > b^d \))
5. Saw that these tree relate to the difference master method cases

\[
T(n) \leq a \, T\left(\frac{n}{b}\right) + O(n^d)
\]

- \( T(n) \): total amount of operations
- \( a \): recursive calls (# of subproblems), always \( \geq 1 \)
- \( b \): fraction of input size (shrinkage), always \( > 1 \)
- \( d \): extra work needed to combine, always \( \geq 0 \)

\[
T(n) = \begin{cases} 
O(n^d \lg n), & a = b^d \\
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O(n^{\log_b a}), & a > b^d 
\end{cases}
\]