Huffman Codes

https://cs.pomona.edu/classes/cs140/

Outline

Topics and Learning Objectives

- Introduce Huffman Codes for compression

Exercise

- None

Extra Resources

- Algorithms Illuminated Part 3, Chapter 14

Huffman Codes

- This will be our final greedy algorithm / application
- Huffman Codes are used for compression

- In general they can be thought of as:
  - A mapping of some set of characters/symbols to binary strings

- For example: let’s encode the letters [a-z] and {., ?, !, ;, :}.
- How many bits would you use?
- Does this type of encoding sound familiar at all?

Huffman Codes

- In general we use Σ to represent the set of characters
- Let Σ = {A, B, C, D}
- What is one possible binary encoding?

A	B	C	D
00	01	10	11

- How many bits does it take to store 100 characters?

Huffman Codes

Can we do better than this fixed-length encoding (use fewer bits)?

Σ =	A	B	C	D
Fixed Encoding	00	01	10	11
(Bad) Variable Encoding	0	01	10	1

What does the string 001 encode?

AB	CD	AAD
001	101	001

Huffman Codes

The problem with this encoding is called prefixing.

Σ =	A	B	C	D
Fixed Encoding	00	01	10	11
(Bad) Variable Encoding	0	01	10	1

- This is **not** a prefix-free encoding.
- Problem: we don’t know where one character ends and the next begins.
- Solution: ensure that the encoding is prefix-free.

Example Prefix-Free Encoding

Σ =	A	B	C	D
Fixed Encoding	00	01	10	11
Prefix-free Encoding				

Example Prefix-Free Encoding

Σ =	A	B	C	D
Fixed Encoding	00	01	10	11
Prefix-free Encoding	0	10	110	111

Now, we know exactly when one character ends and another starts.

Why would this be a good idea?

- What if we needed to store a bunch of A's but only a few C's?

Example Prefix-Free Encoding

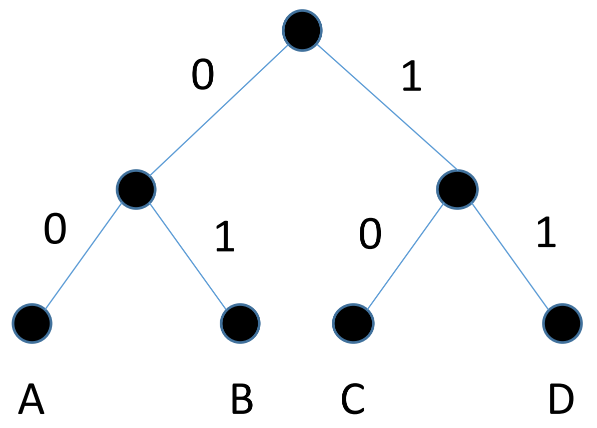
Σ =	A	B	C	D
Fixed Encoding	00	01	10	11
Prefix-free Encoding	0	10	110	111
Frequency	**60%**	**25%**	**10%**	**5%**

What are the average bit lengths for these two encodings?

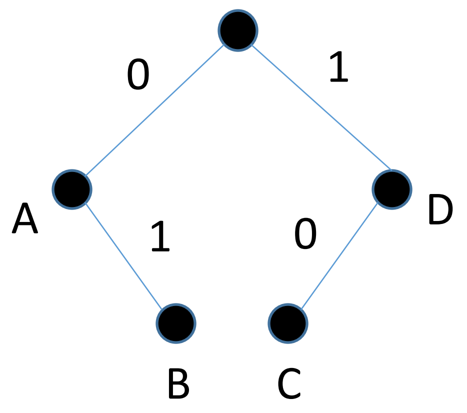
Discovering the Best Encoding

Let's think of Huffman Codes as trees. Σ = A, B, C, D

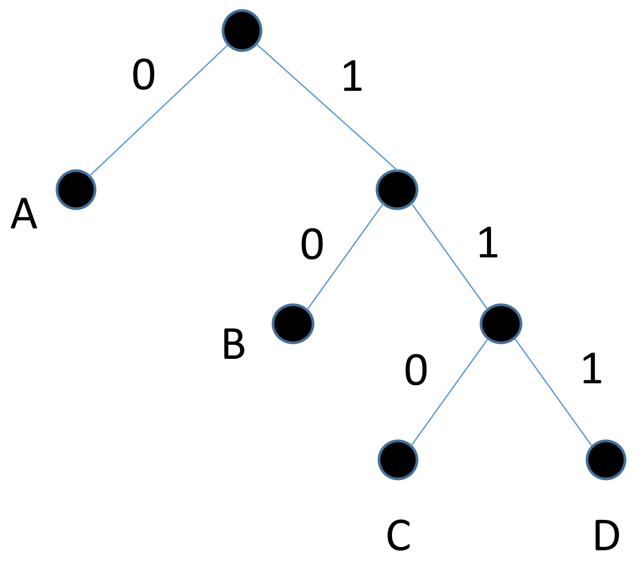
Fixed Encoding
{00,01,10,11}



First Variable Encoding
{0,01,10,1}



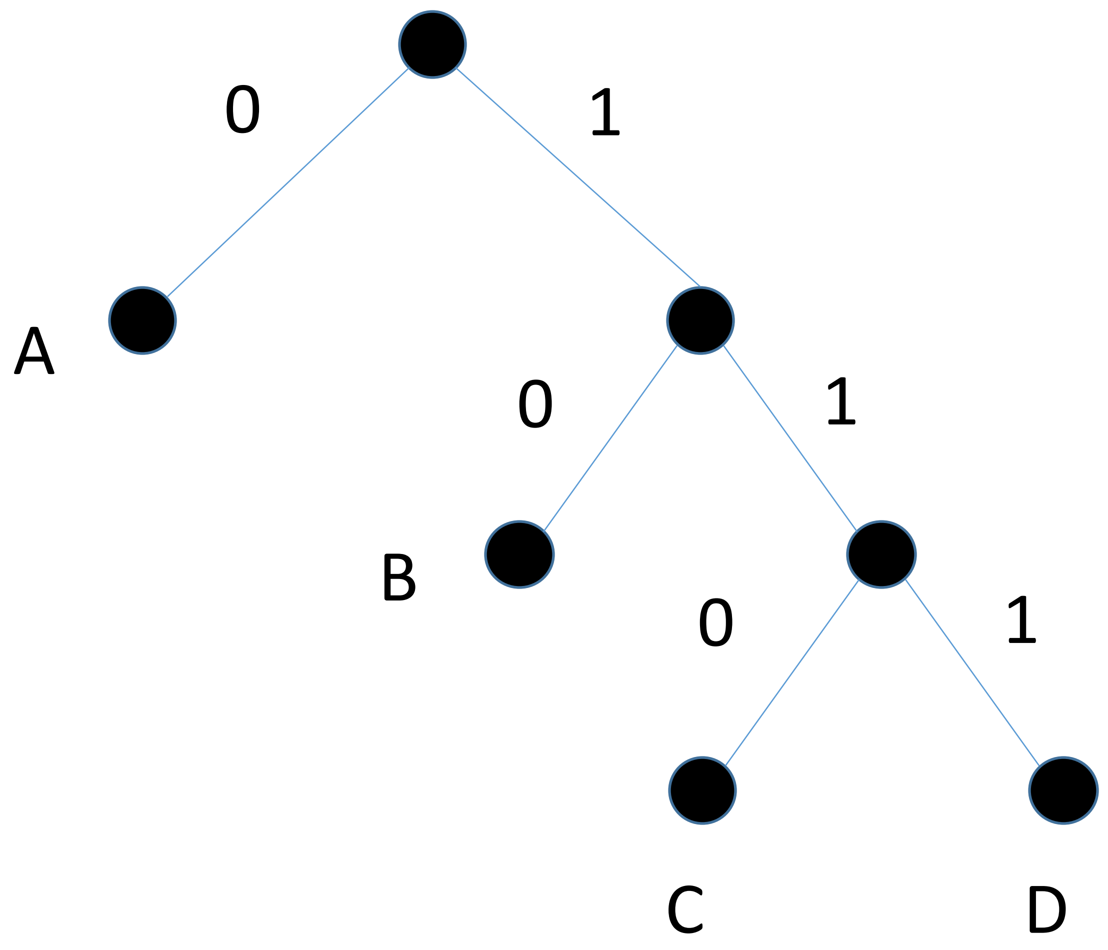
Prefix-Free Encoding
{0,10,110,111}



Huffman Codes as Trees

- Go to left child on a ‘0’
- Go to right child on a ‘1’
- For each symbol in Σ, exactly one node should be labeled x
- Prefix-free encoding require all labeled nodes to be leaves
- Trees are just a tool for helping us construct optimal encodings
- Decode: follow the input string until you reach a leaf
- Encode(x): the path followed from the root to x
- The encoding length of x is the same as its depth

Decode the string: 0110111



Huffman Codes

Problem: *how do we choose/design our encodings?*

- Input: a set of symbols Σ and their probabilities/frequencies p_i
- Notation: if T is a tree with leaves as symbols of Σ, then let

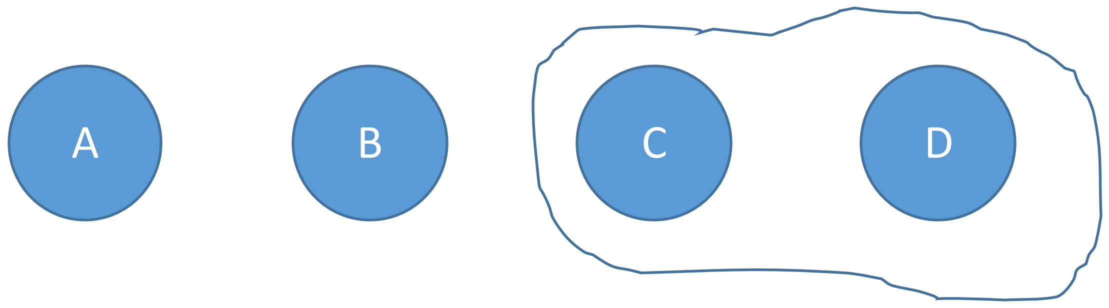
$$L(T) = \sum_{i=1}^{|\Sigma|} p_i * depth_i$$

- L(T) is the average encoding length
- The output of our algorithm will be a binary tree T that minimizes L(T)

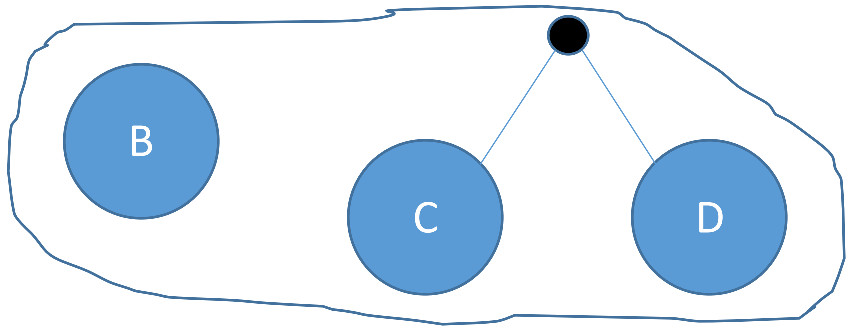
Huffman’s Algorithm (compression)

Huffman’s approach is the start at the leaves and build the the tree bottom-up

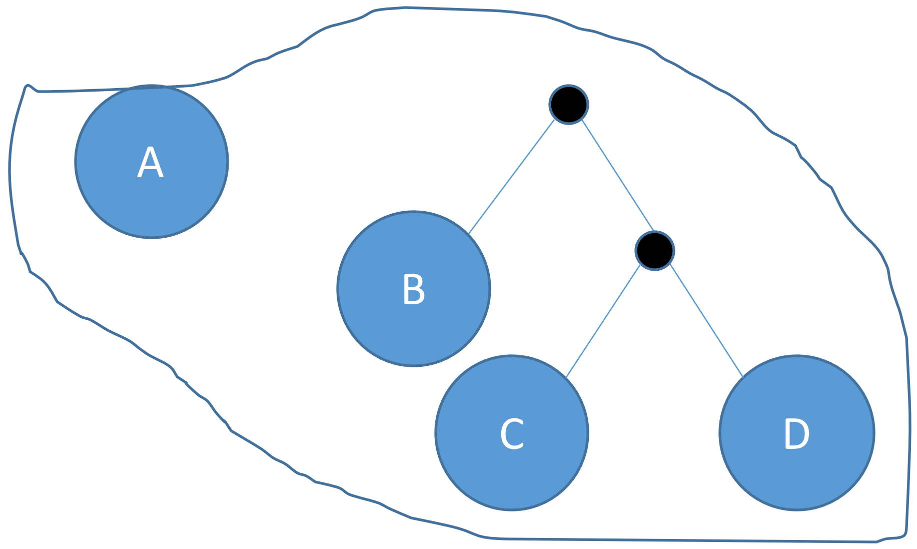
Iteration 1



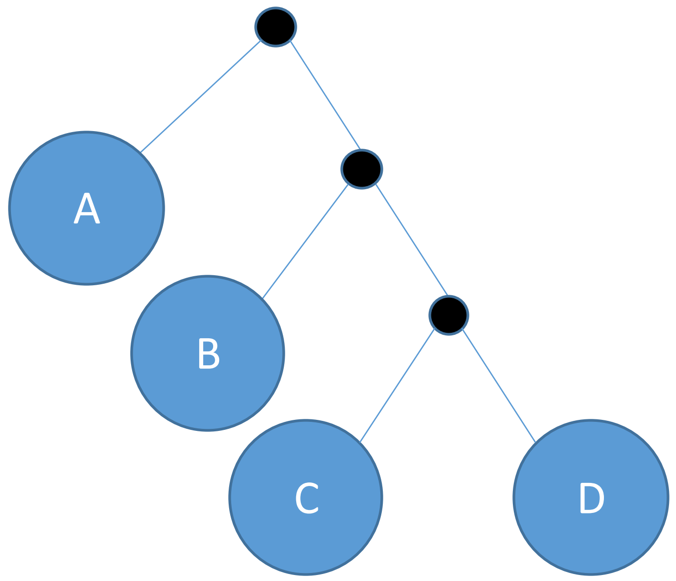
Iteration 2



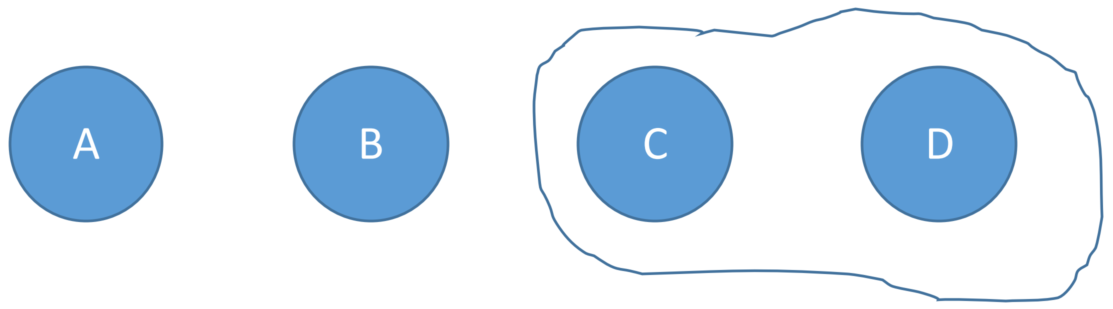
Iteration 3



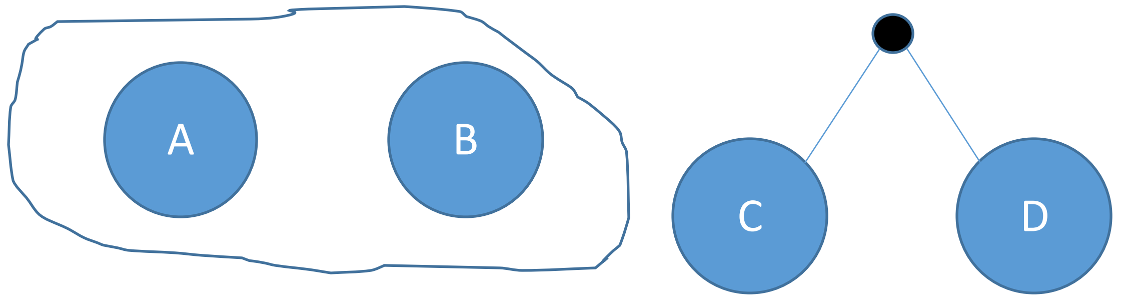
Iteration 4



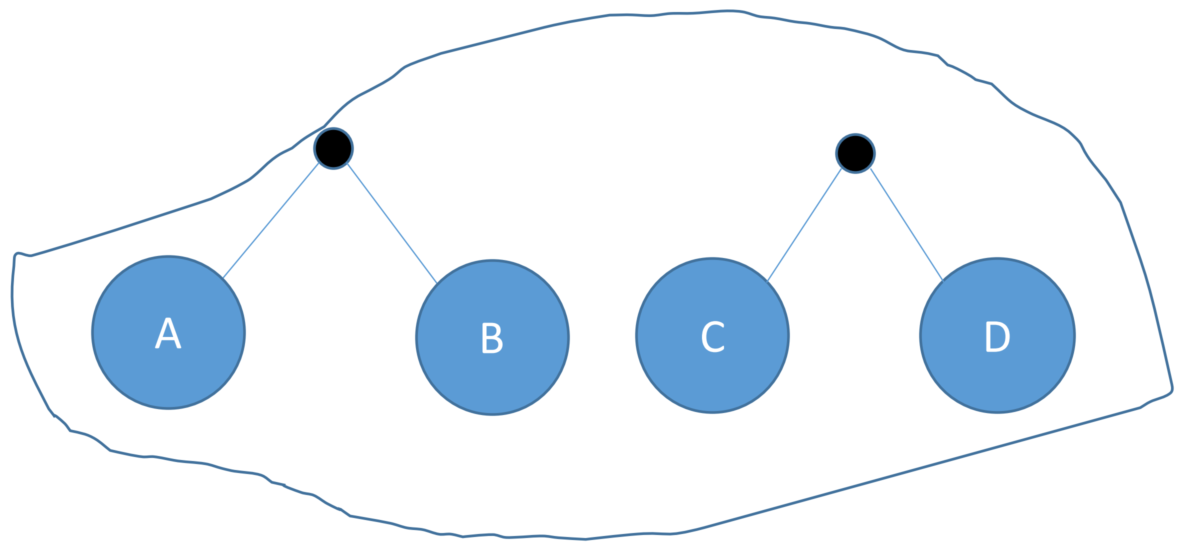
Iteration 1



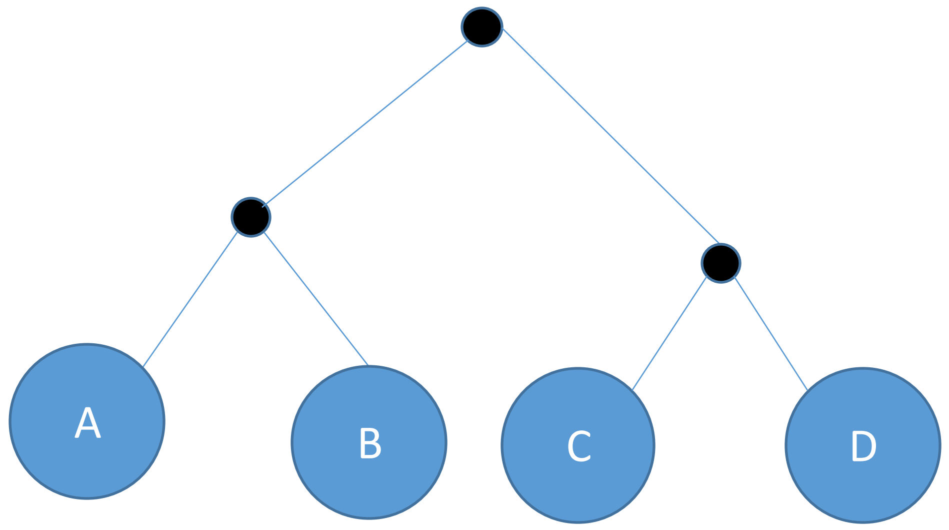
Iteration 2



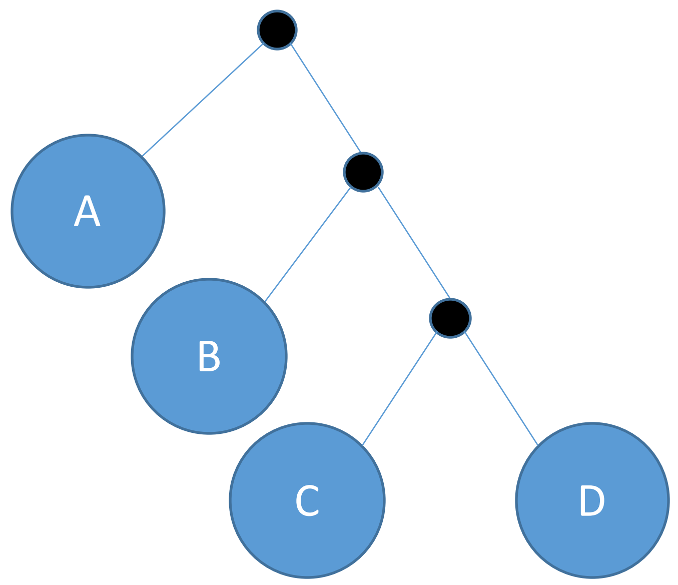
Iteration 3

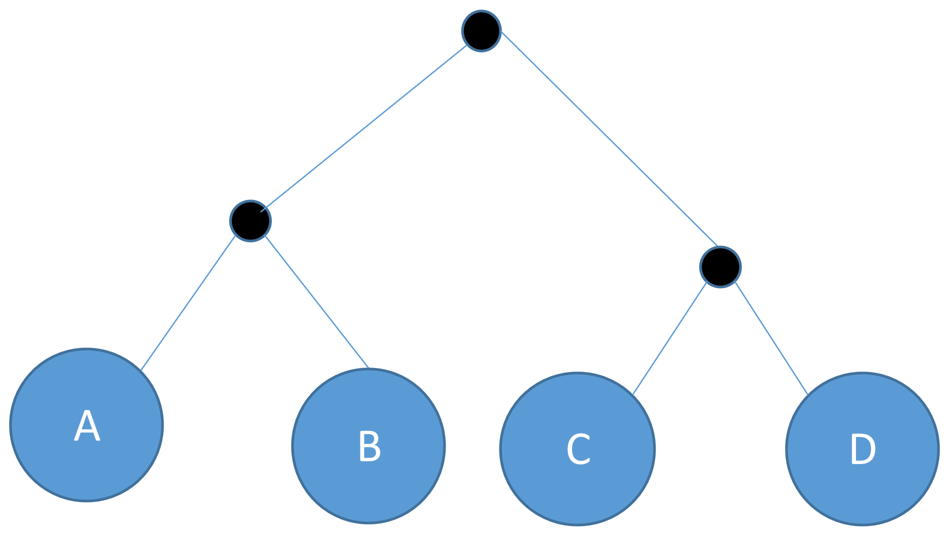


Iteration 4



Which Tree is Better?





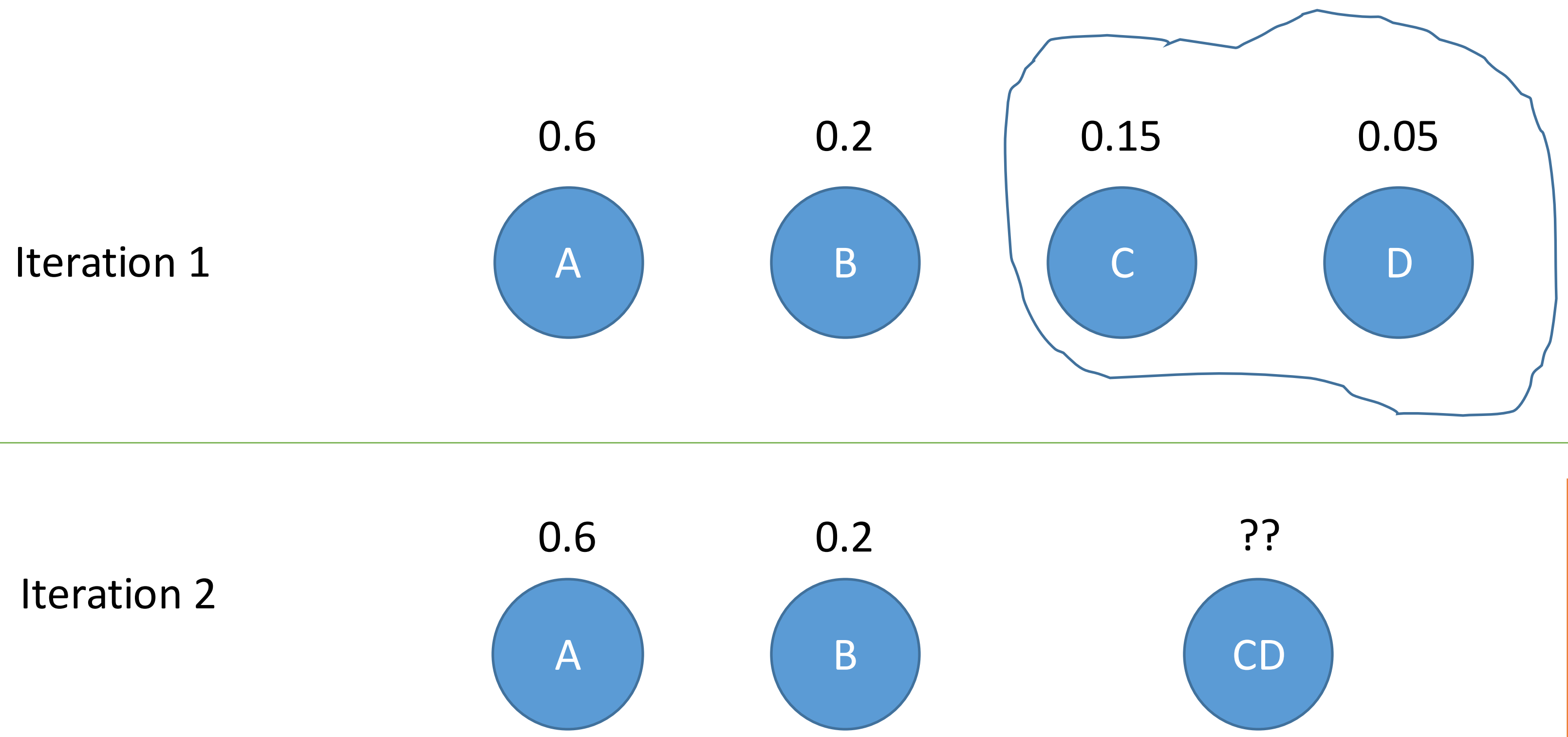
It depends on the frequencies!

Huffman's Algorithm

- We're building from the leaves up.
- How do we know which two symbols we should merge?
- How does the final encoding length of a given symbol in Σ relate to the number of merges it experiences?

- Each merge adds one node to the path from the root to x!
- So, how do we minimize the weighted average encoding length?
- Huffman's Greedy Criteria: Merge the least frequent characters first.

How do we compare nodes after a merge?



a) $p_c + p_d$
b) $\mathrm{Min}[p_c , p_d]$
c) $\mathrm{Max}[p_c , p_d]$
d) $p_c * p_d$

```
FUNCTION Huffman(symbols, frequencies)

    forest = [(f, s) FOR f, s IN Zip(symbols, frequencies)]
    heapifyMin(forest)

    WHILE forest.length ≥ 2
        treeA = extract_min(forest)
        treeB = extract_min(forest)
        treeMerged = merge(treeA, treeB)
        heap_add(forest, treeMerged)

    # Only one tree remaining in forest
    RETURN forest[0]
```

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$\Sigma =$	A	B	C	D	E	F
$P =$	3	2	6	8	2	6

```
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```

What is the running time?

Note: faster algorithms do exist for this problem

Correctness Proof

Theorem: Huffman's algorithm computes a binary tree that minimizes the average encoding length of all symbols

$$L(T) = \sum_{i=1}^{|\Sigma|} p_i * depth_i$$

Strategy:

- Induction
- Exchange argument

Proof by induction that P(n) holds for all n

- Base Case: P(1) holds because …
- Inductive Hypothesis: Let's assume that P(k) holds, where k < n
- Inductive Step: P(n) holds because of P(k) and …
- Thus, by induction, P(n) holds for all n

Inductive Proof

Proof by induction that P(n) holds for all n

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- Inductive Step: P(n) holds because of P(k) and …
- Thus, by induction, P(n) holds for all n

Base Case:

- If n = 1 or n = 2 there is only one option for average encoding length
- Thus the base cases are trivially true

Inductive Hypothesis:

- Huffman’s algorithm produces the optimal coding with ≤ k symbols where k < n

Inductive Step…

Main Ideas for Inductive Step

Let symbols ø and π be the symbols with the smallest and second smallest frequencies, respectively

1. Huffman's Algorithm outputs the optimal tree in which ø and π are siblings
   - Out of all possible trees where ø and π are siblings

2. The optimal tree is the one in which ø and π are siblings
   - Out of all possible trees in general

Part 1

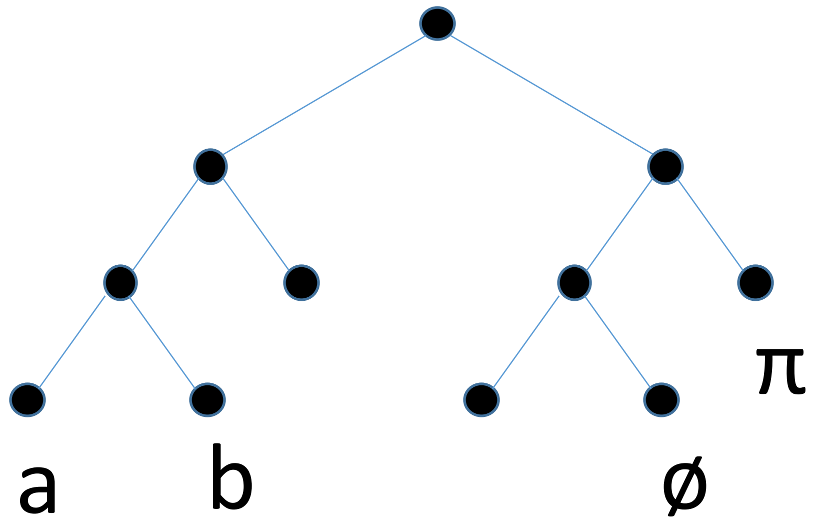
Huffman’s outputs the optimal tree in which ø and π are siblings

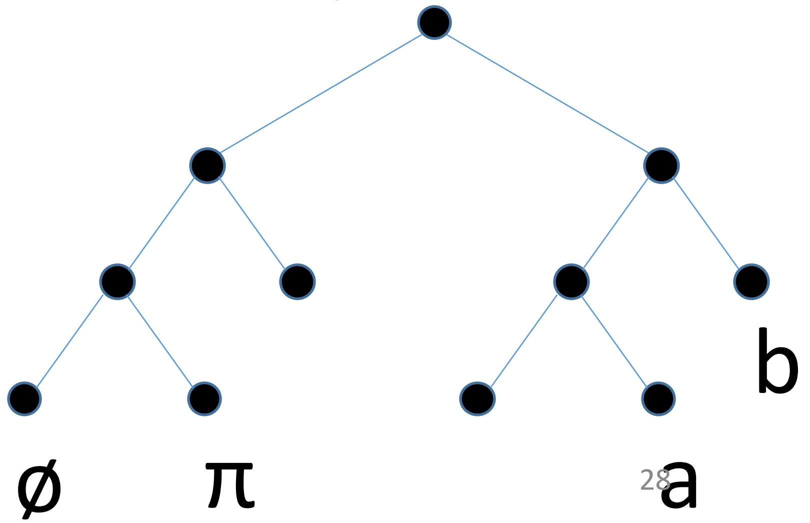
- After combining symbols ø and π into a single “øπ” symbol we have reduced the total number of symbols by 1
- Given our inductive hypothesis, we know that Huffman’s algorithm outputs the optimal tree for k symbols where k < n
- Thus, Huffman’s outputs the optimal tree after combining ø and π

Part 2

The optimal tree is the one in which ø and π are siblings

- Consider the case where ø and π are not siblings
- And we then exchange ø and π with two nodes that are siblings
- The average encoding length goes down (or stays the same)!





Summary

- Prefix-free, variable-length binary codes have smaller average encoding lengths (per symbol) than fixed-length codes

- These Huffman Codes can be visualized as a binary tree

- Huffman's Algorithm works by greedily combining trees in the forest until you are left with a single tree in O(n lg n) time

- We proved correctness with induction and an exchange argument