Dijkstra’s Algorithm

https://cs.pomona.edu/classes/cs140/

Dijkstra’s Shortest Path Algorithm

Dijkstra’s Single-Source Shortest Path Algorithm

Outline

Topics and Learning Objectives

- Discuss graphs with edge weights
- Discuss shortest paths
- Discuss Dijkstra’s algorithm including a proof

Exercise

- Dijkstra’s Algorithm

Extra Resources

- Introduction to Algorithms, 3rd, chapter 24
- Algorithms Illuminated Part 2: Chapter 9

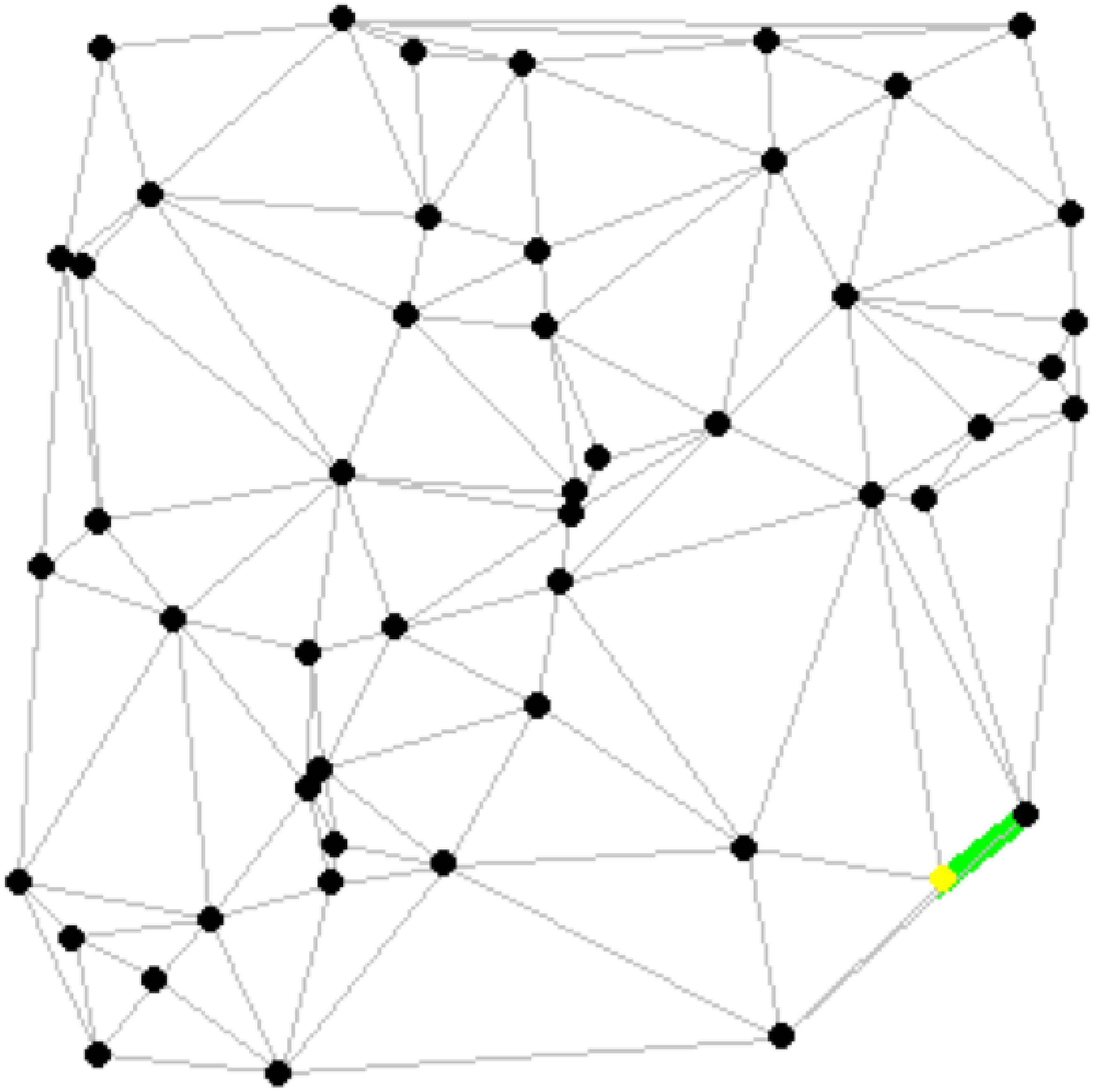
Dijkstra's Algorithm

Find the shortest path between a start vertex s and every other vertex in the graph G

Can halt the algorithm if you only want to find shortest path to a specific vertex (for example, a destination city)

Uses:

- Network routing
- Path planning
- Etc.



www.combinatorica.com

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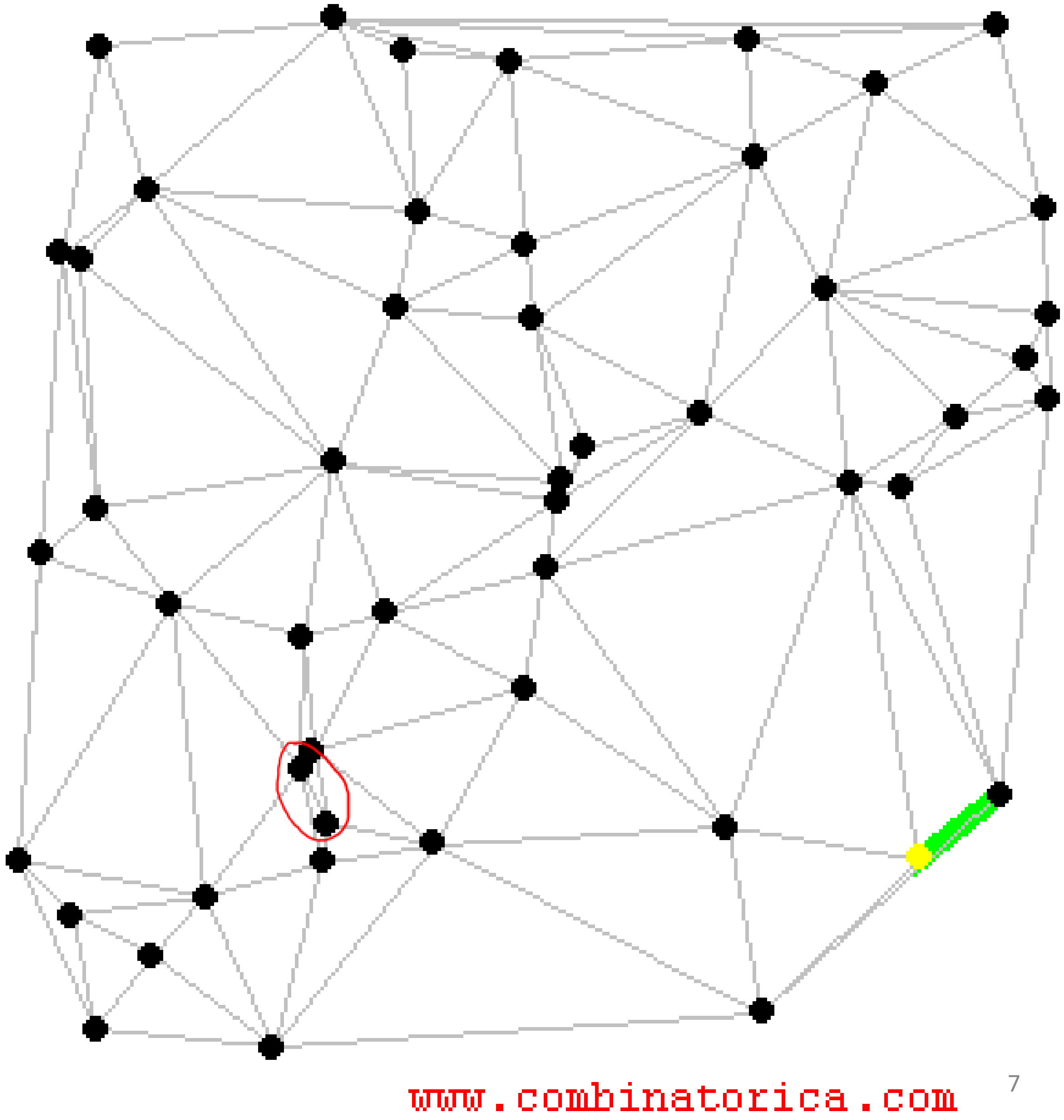


Dijkstra's Algorithm

Input

- A weighted graph G = (V,E) and
- A source vertex s

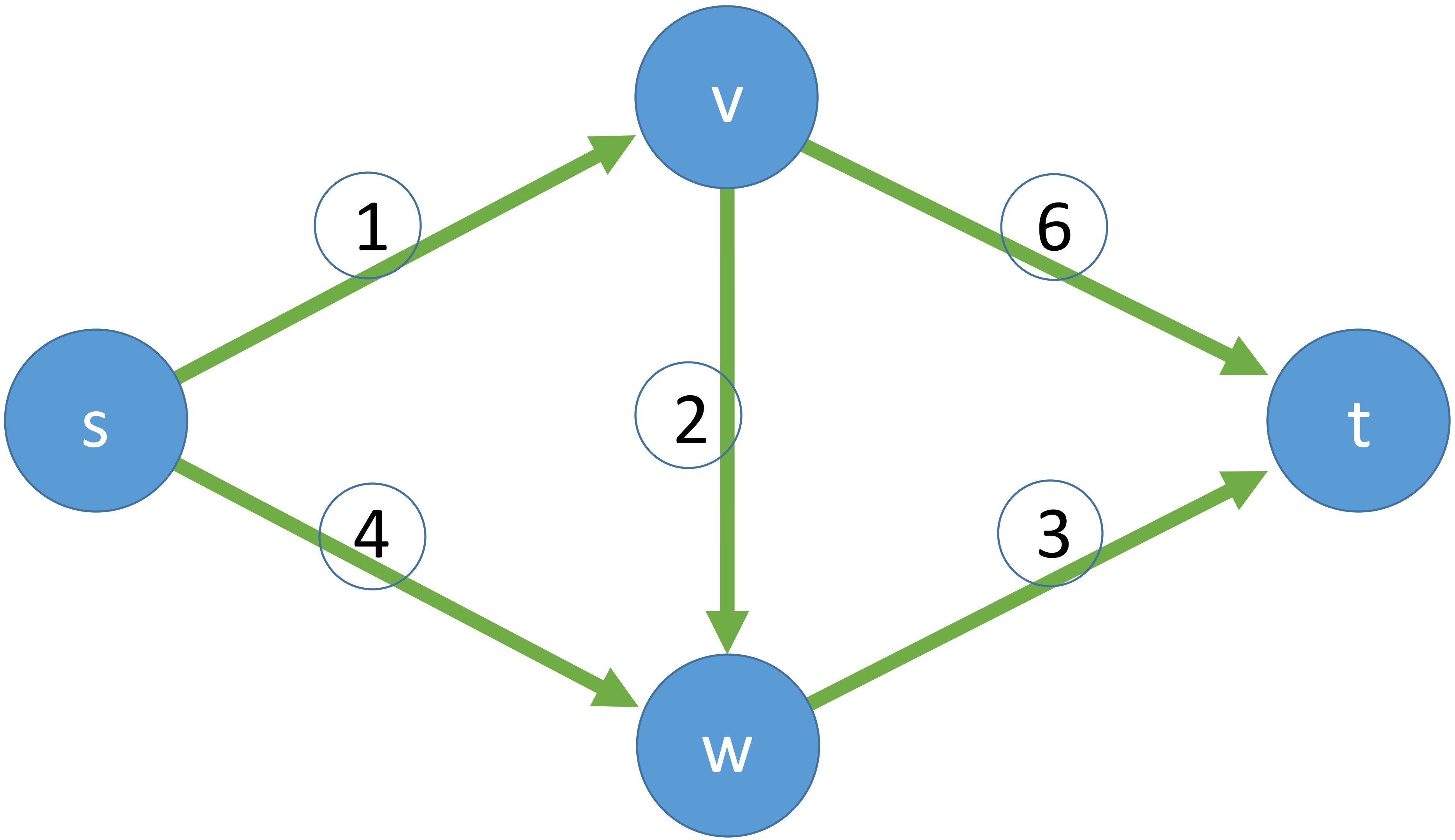
Output

- for all v in V we output the length of the **shortest path** from s → v
- you can also output the actual path, but we'll just worry about length for now

Assumptions

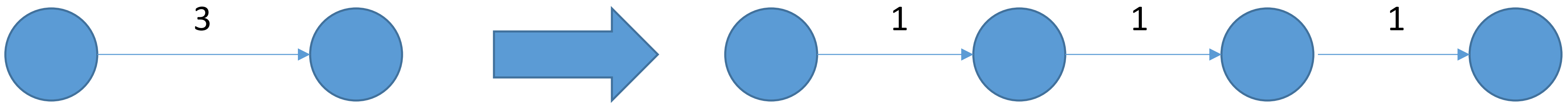
- A path exists from s to every other node (how can we check this property?)
- All edge weights are non-negative

What is the shortest path from S to all other vertices?



How did we do shortest path before?

- BFS
- How can we modify that process to work for graphs with weighted edges?



- Why would we not want to do that?

Initialize min_length

```
FUNCTION Dijkstra(G, start_vertex)
    found = {}
    lengths = {v: INFINITY FOR v IN G.vertices}

    found.add(start_vertex)
    lengths[start_vertex] = 0

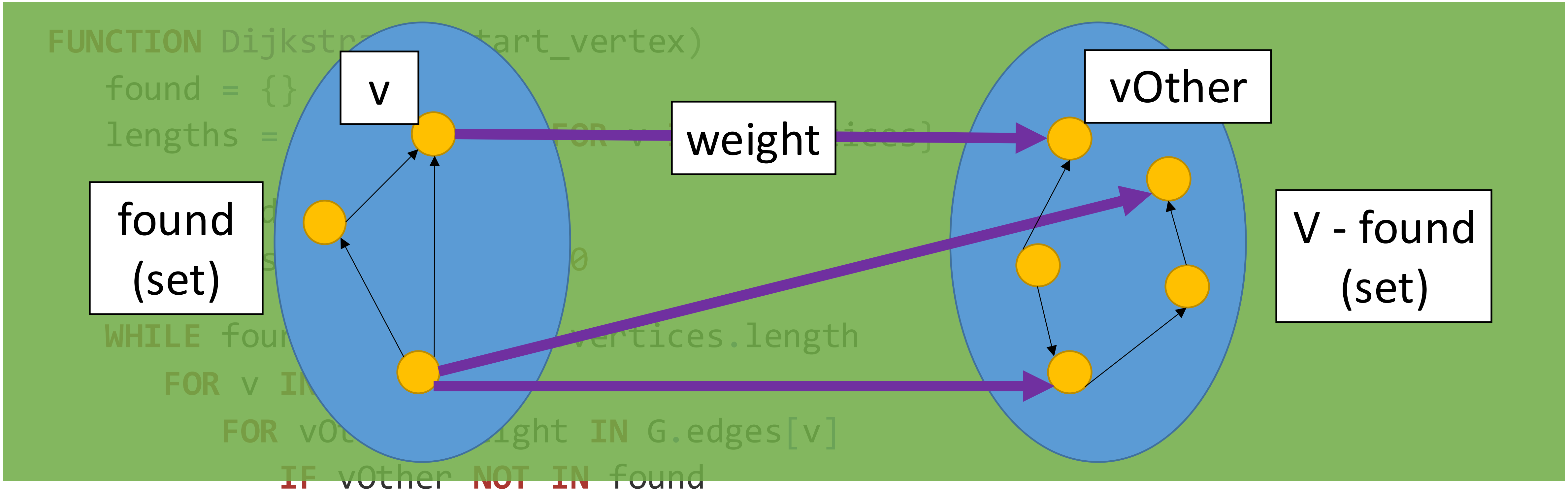
    WHILE found.length != G.vertices.length
        FOR v IN found
            FOR vOther, weight IN G.edges[v]
                IF vOther NOT IN found
                    vOther_length = lengths[v] + weight
                    IF vOther_length < min_length
                        min_length = vOther_length
                        vMin = vOther
        found.add(vMin)
        lengths[vMin] = min_length

    RETURN lengths
```

This is now a set instead of a dictionary

Dijkstra's greedy criterion

Computed in previous iterations




```
            IF vOther NOT IN found
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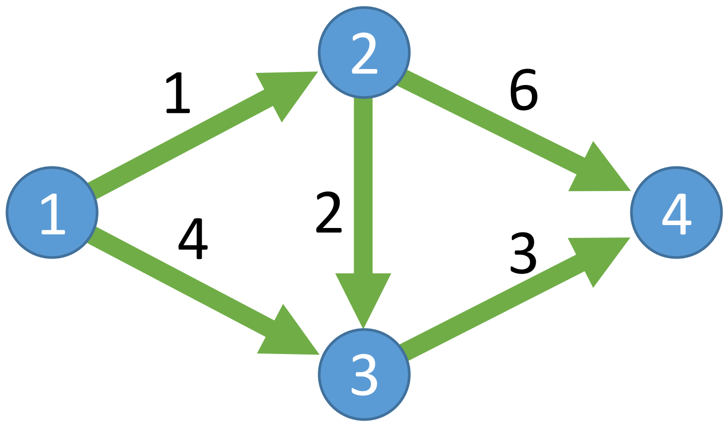

Iteration 1:

```
FUNCTION Dijkstra(G, start_vertex)
    found = {}
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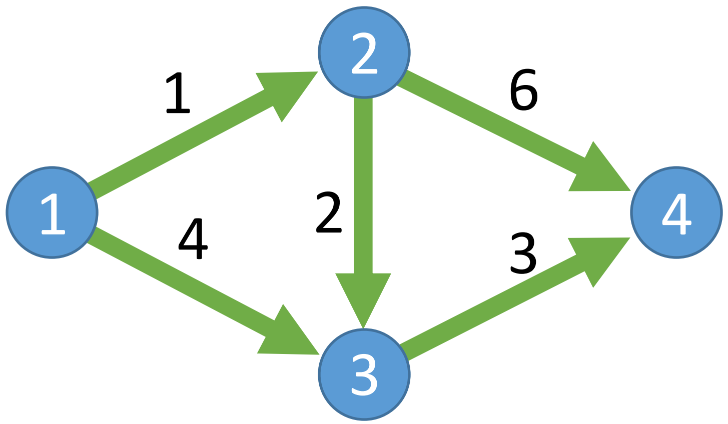
    RETURN lengths
```



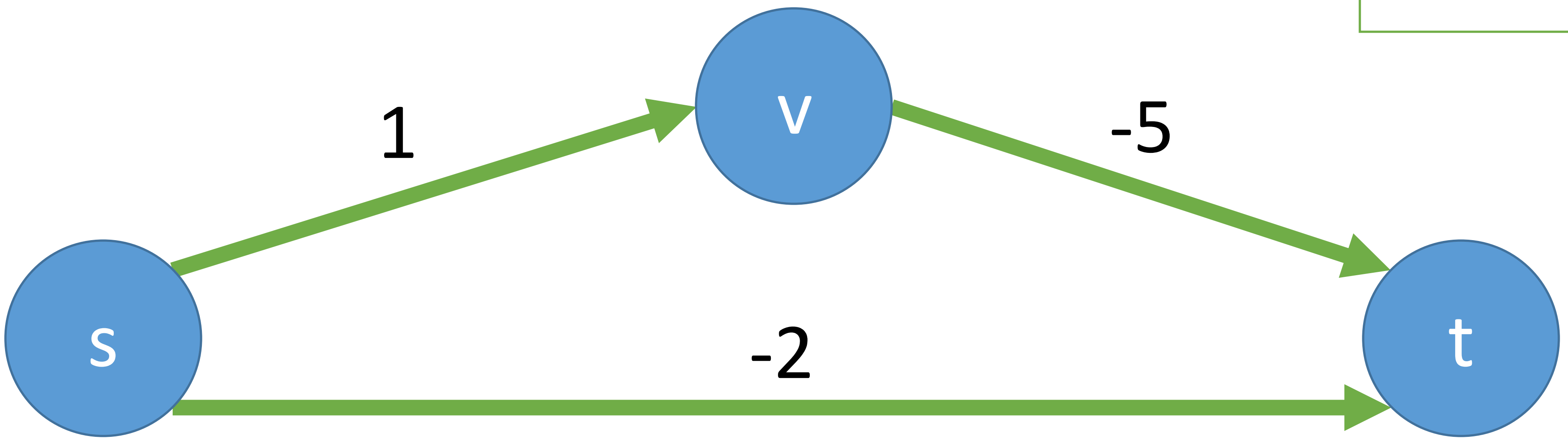
Iteration 2:

Exercise

Dijkstra's Algorithm with negative edges

- How might you deal with negative edges?
- How about adding some value to every edge?

What is the shortest path from s to t?



Dijkstra's Algorithm with negative edges

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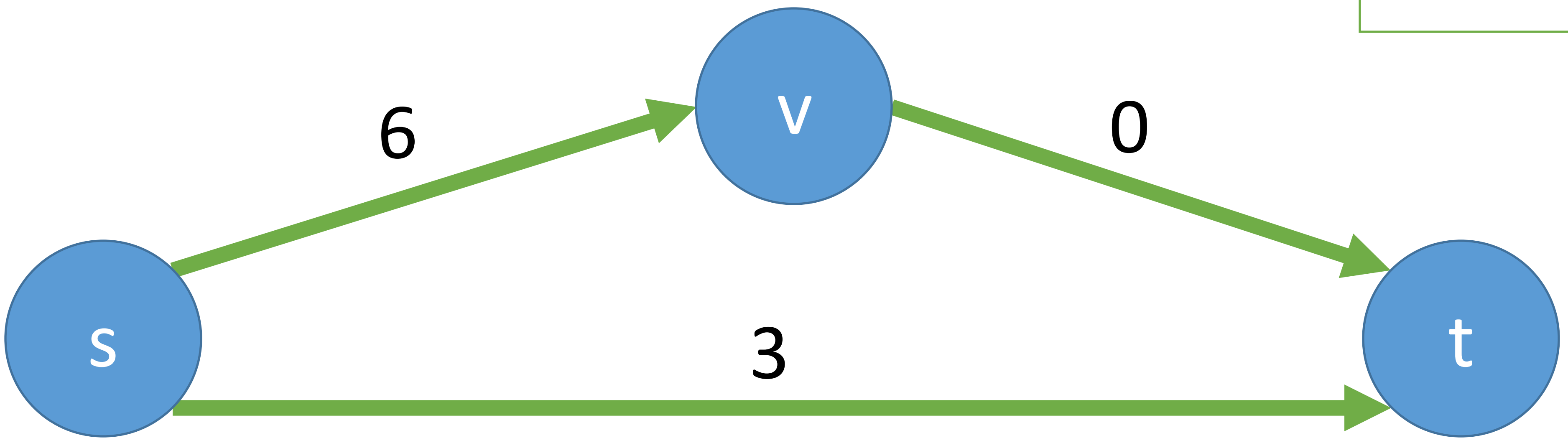

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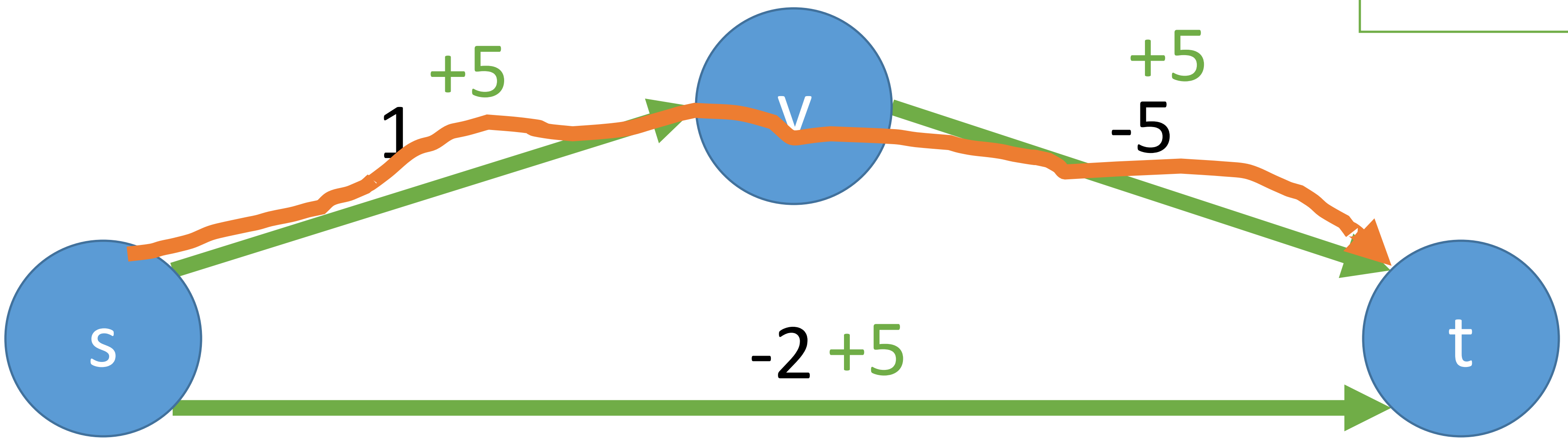


We would add a different amount to each **path**!

Dijkstra’s Algorithm

- What have we done so far?
- We’ve only shown that it works for the given example.
- This is not enough to prove correctness.

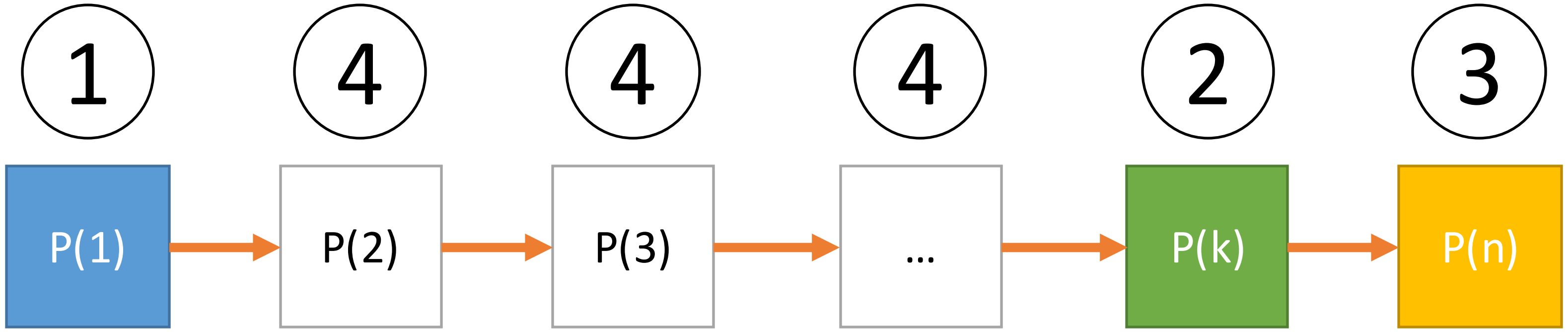
- In general, examples are good for:
    - Demonstration
    - Contradictions

- They are not good for proving correctness.

Proof by Induction Cheat-sheet

Proof by induction that P(n) holds for all n

1. P(1) holds because <something about the code/problem>
2. Let's assume that P(k) (where k < n) holds.
3. P(n) holds because of P(k) and <something about the code>
4. Thus, by induction, P(n) holds for all n



Correctness

Proof by induction that P(n) holds for all n

- P(1) holds because …
- Let's assume that P(k) (where k < n) holds.
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Theorem for Dijkstra's algorithm:

For every graph with non-negative edge lengths, Dijkstra's algorithm computes all shortest path distances from `start_vertex` to every other vertex

Base Case:

- `lengths[start_vertex] = 0`

Correctness

Proof by induction that P(n) holds for all n

- P(1) holds because …
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Theorem for Dijkstra's algorithm:

For every graph with non-negative edge lengths, Dijkstra's algorithm computes all shortest path distances from `start_vertex` to every other vertex

Inductive Hypothesis:

- Assume all previous iterations produce correct shortest paths
- For all v in `found`, `lengths[v]` = shortest path length from `start_vertex` to v

```
FUNCTION Dijkstra(G, start_vertex)
    found = {}
    lengths = {v: INFINITY FOR v IN G.vertices}

    found.add(start_vertex)
    lengths[start_vertex] = 0

    WHILE found.length != G.vertices.length
        FOR v IN found
            FOR vOther, weight IN G.edges[v]
                IF vOther NOT IN found
                    vOther_length = lengths[v] + weight
                    IF vOther_length < min_length
                        min_length = vOther_length
                        vMin = vOther
        found.add(vMin)
        lengths[vMin] = min_length

    RETURN lengths
```

Proof by induction that P(n) holds for all n

- P(1) holds because …
- Let's assume that P(k) (where k < n) holds.
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Inductive Step (look at code)

Inductive Step

Proof by induction that P(n) holds for all n

- P(1) holds because …
- Let's assume that P(k) (where k < n) holds.
- P(n) holds because of P(k) and …
- Thus, by induction, P(n) holds for all n

In the current iteration:

- We pick an edge (`v*`, `vMin`) based on Dijkstra's greedy criterion
- add `vMin` to `found`
- Set the path length of `vMin` → `lengths[vMin]` = `lengths[v*]` + $\texttt{weight}_{\texttt{v*},\texttt{vMin}}$

What do we know about `lengths[v*]`?

Our inductive hypothesis states that it is the minimal path length

- Optimal path to `v*`, and we won't find a better path to `vMin`

How do we prove this?

Loop Invariant

Proof by induction that P(n) holds for all n

- P(1) holds because …
- Let's assume that P(k) (where k < n) holds.
- P(n) holds because of P(k) and …
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Inductive Step

In the current iteration:

- We pick an edge (v*, vMin) based on Dijkstra's greedy criterion
- add vMin to found
- Set the path length of vMin → lengths[vMin] = lengths[v*] + $\text{weight}_{v*,\text{vMin}}$

What do we know about lengths[v*]?

Our inductive hypothesis states that it is the minimal path length

- Optimal path to v*, and we won't find a better path to vMin

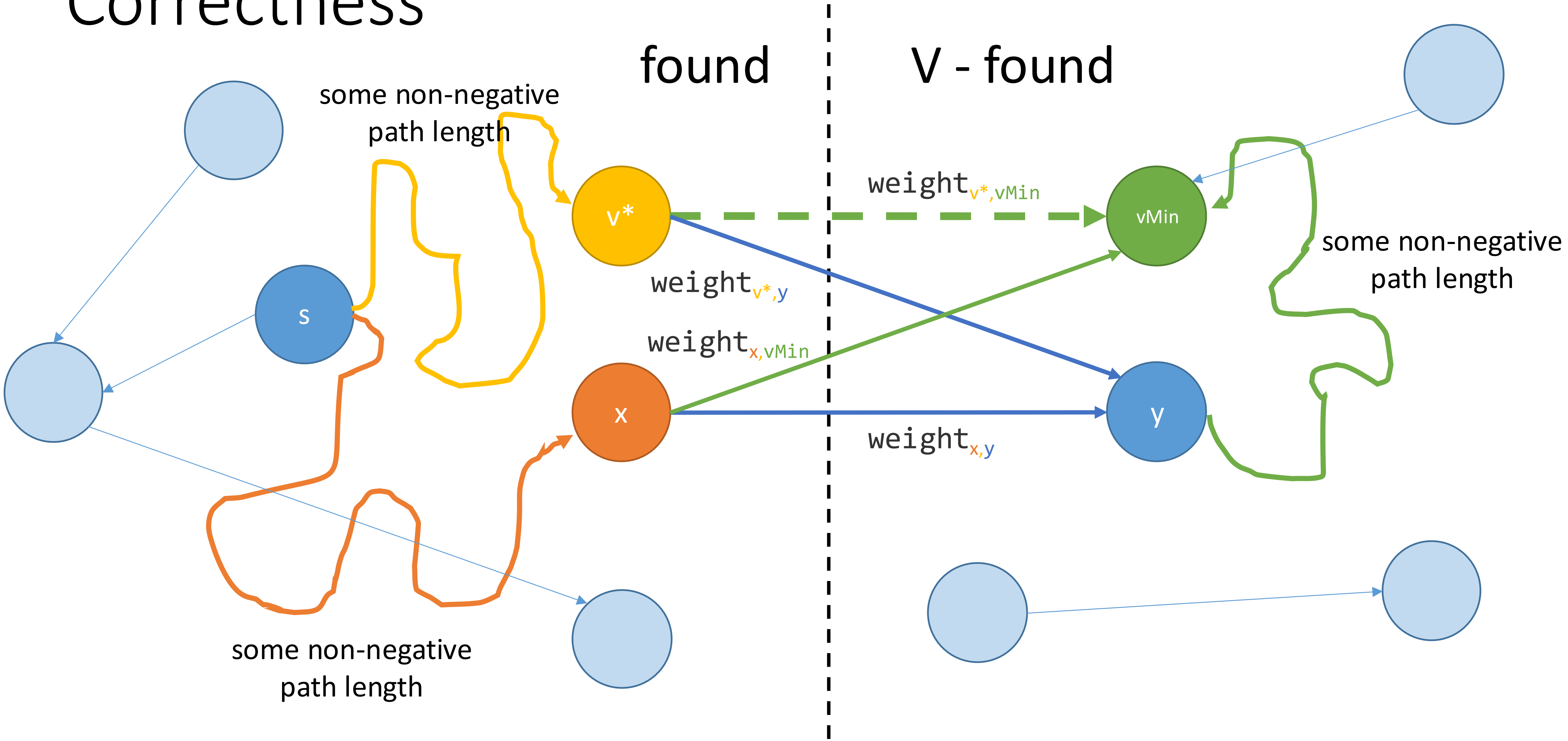
How do we prove this?

Loop Invariant

By our inductive hypothesis, our theorem for Dijkstra's is correct

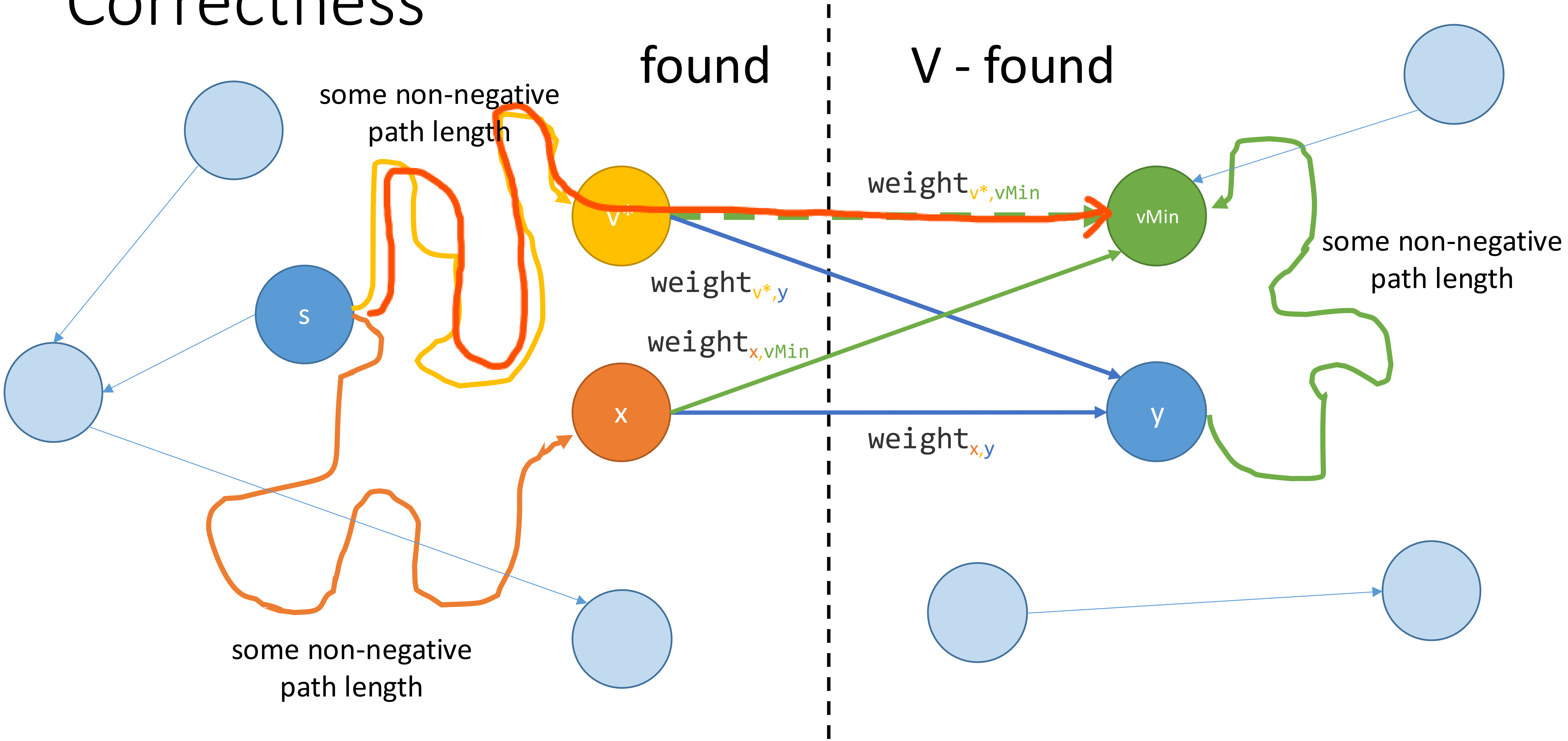
How many different types of paths do we consider each iteration?

Correctness

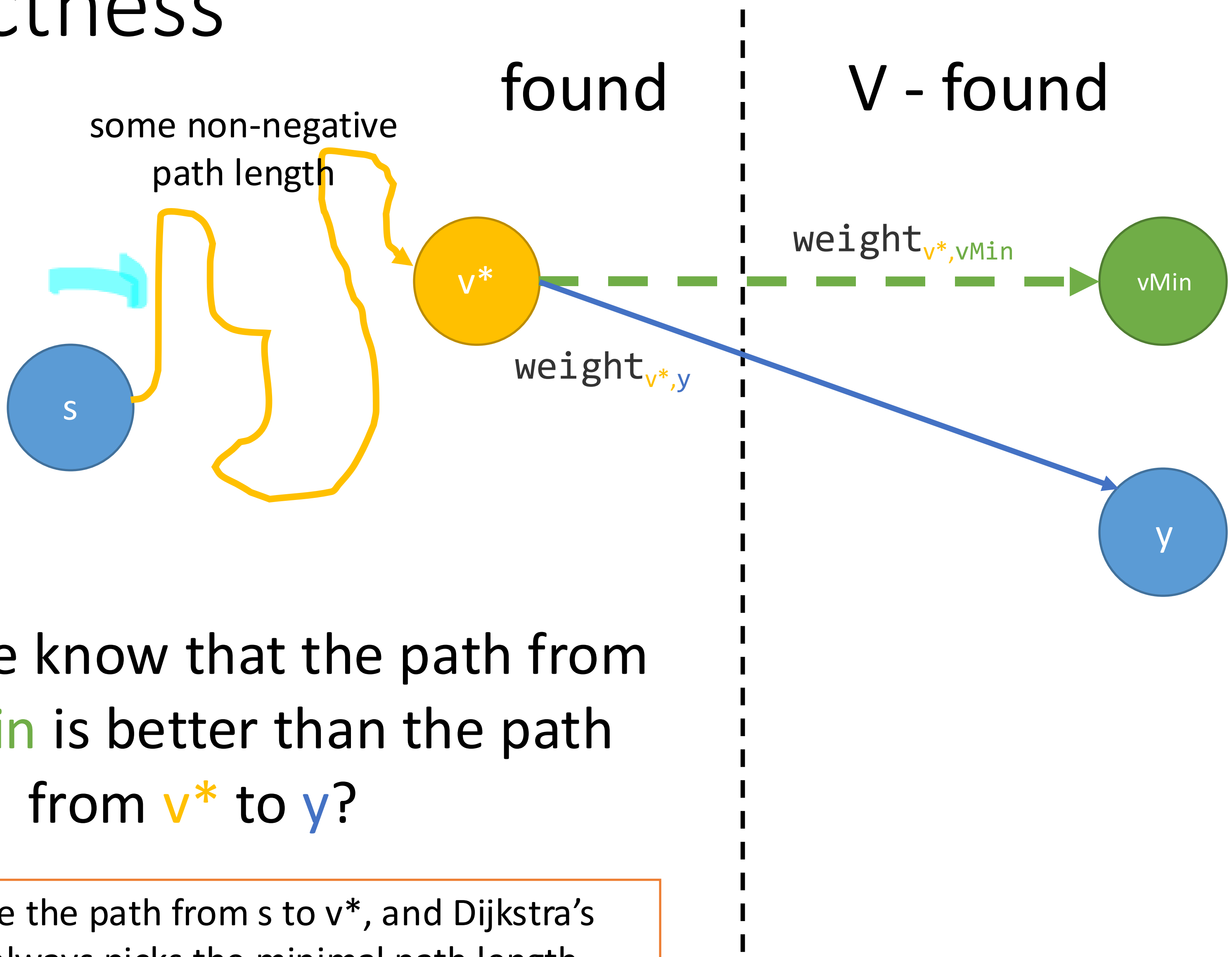


Correctness

Dijkstra's says that this is the best available path.



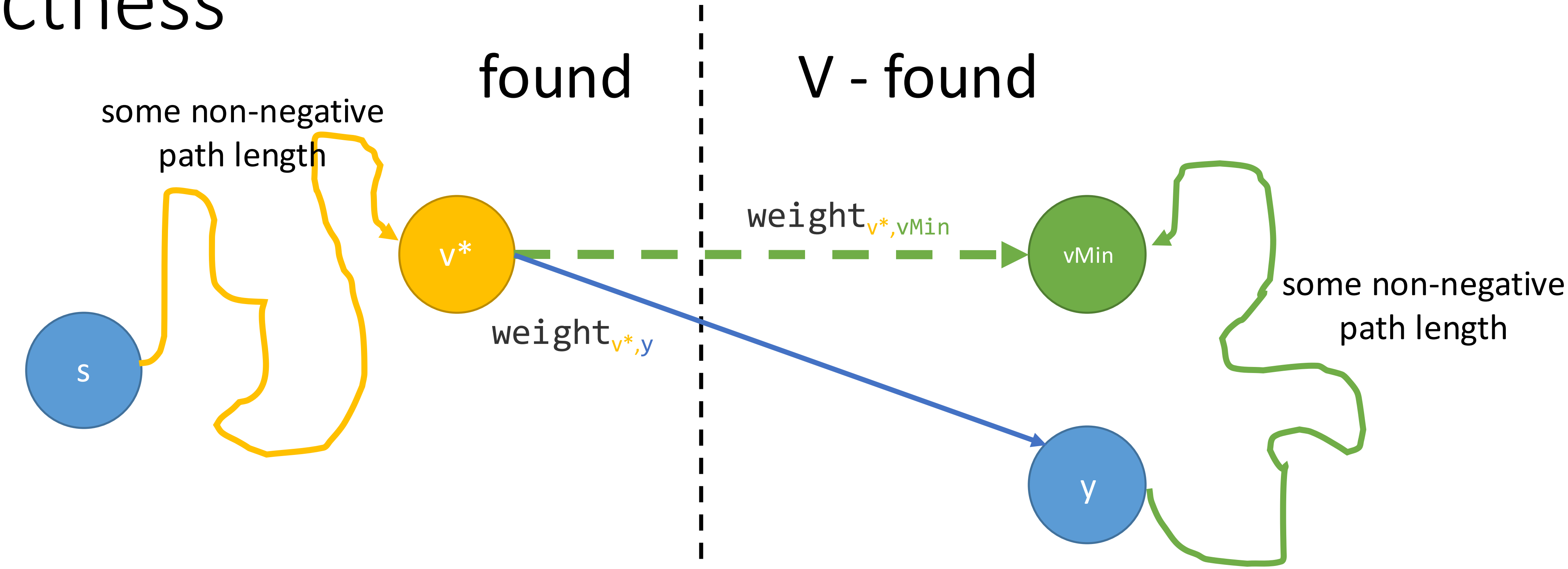
Correctness



How do we know that the path from v* to vMin is better than the path from v* to y?

Both include the path from s to v*, and Dijkstra's Algorithm always picks the minimal path length.

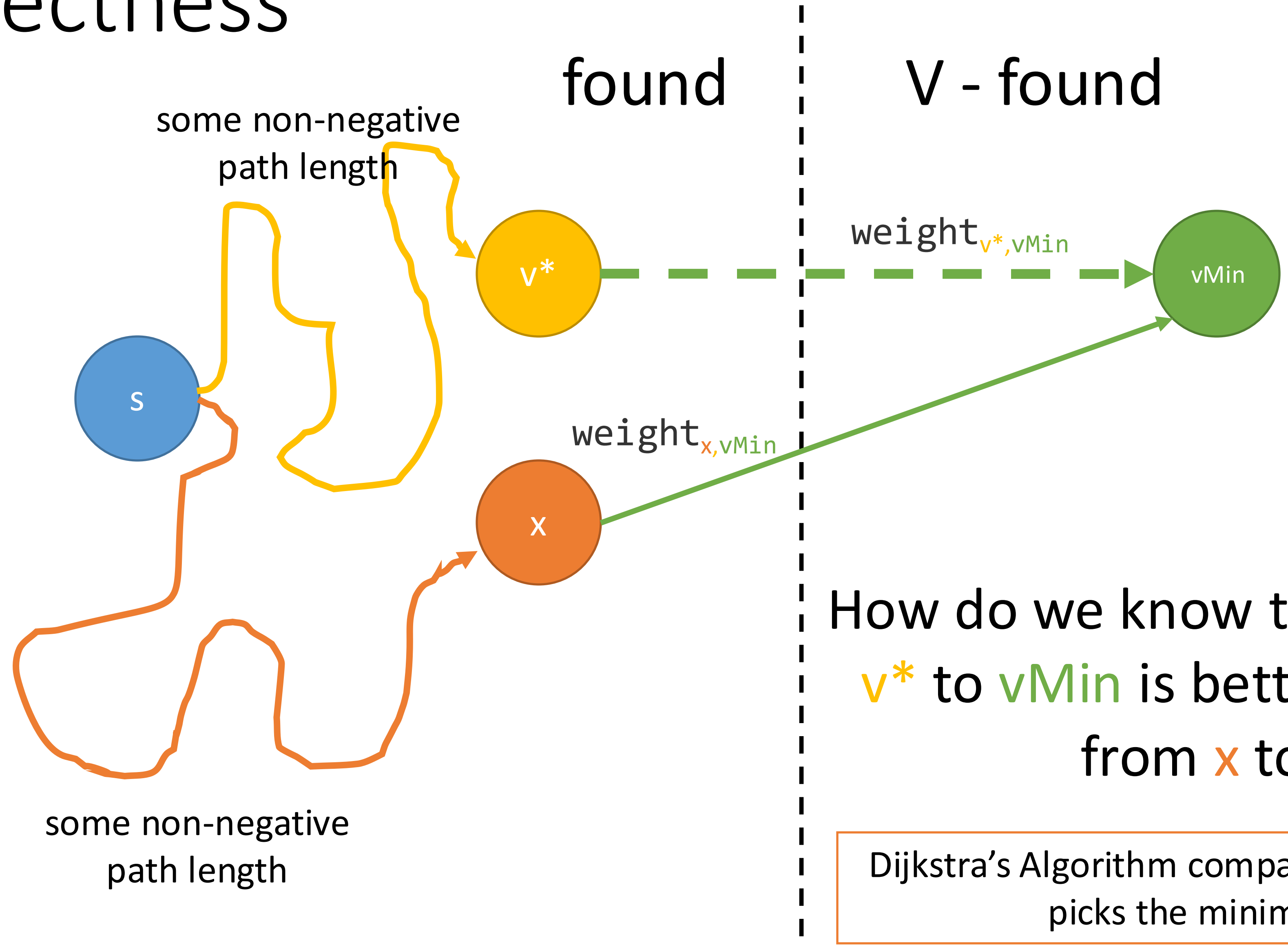
Correctness



How do we know that the path from v* to y to vMin is not even better than the path from v* to vMin?

Dijkstra's Algorithm only operates on graphs with non-negative edge weights.
Thus, this new path must be greater than or equal to the (v*, vMin) edge.

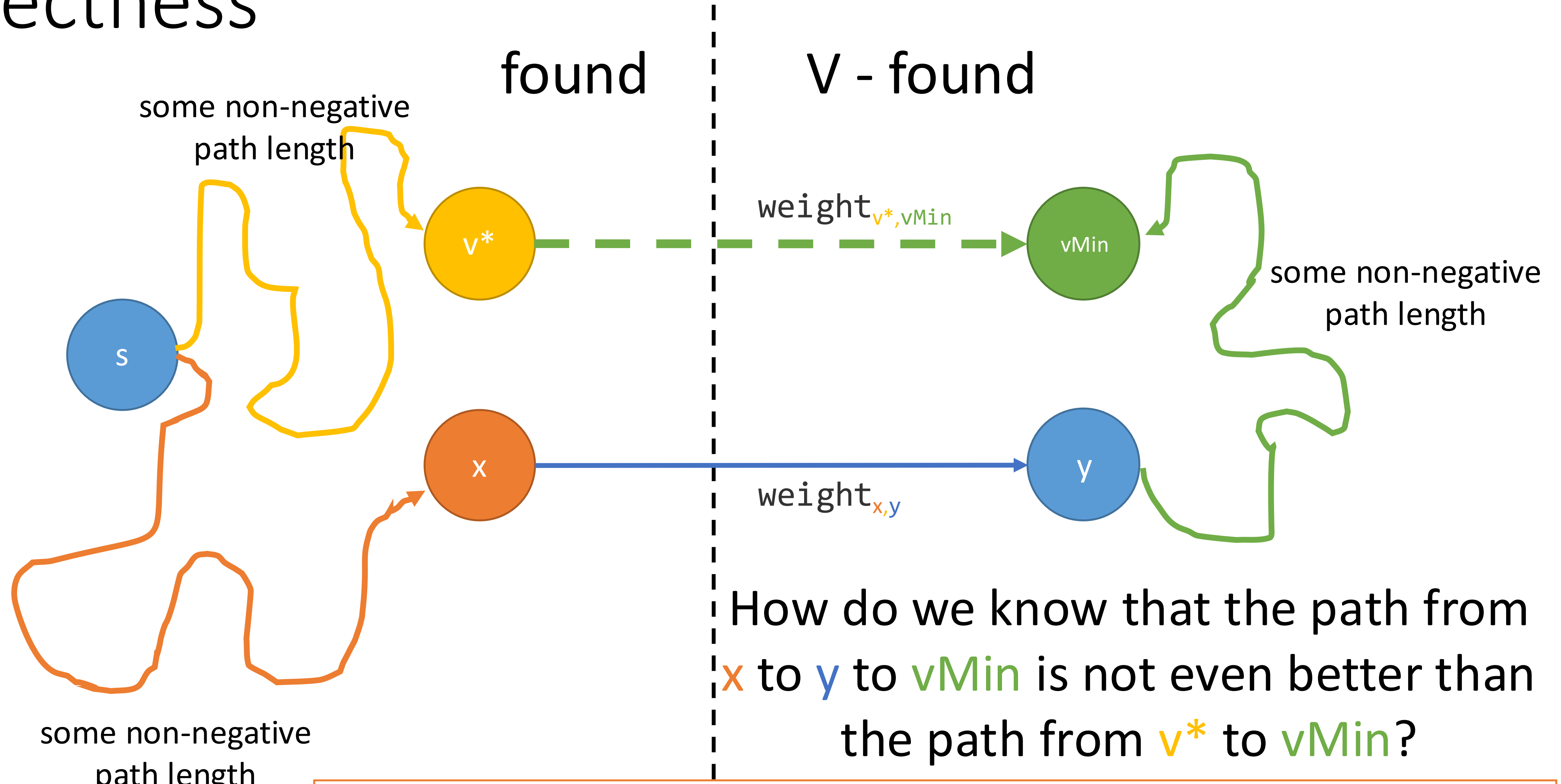
Correctness



How do we know that the path from v* to vMin is better than the path from x to vMin?

Dijkstra's Algorithm compares these two options and picks the minimal path length.

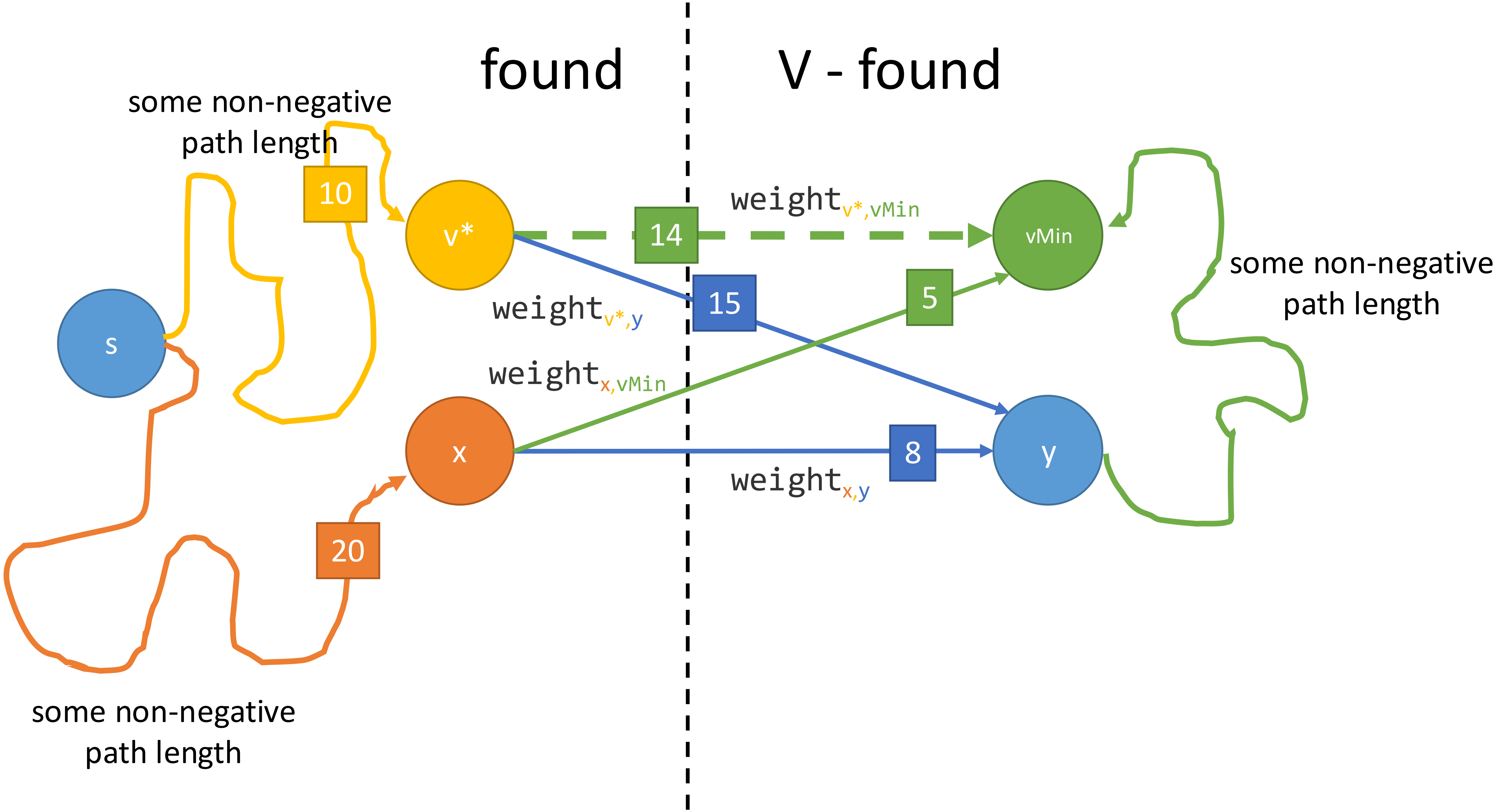
Correctness



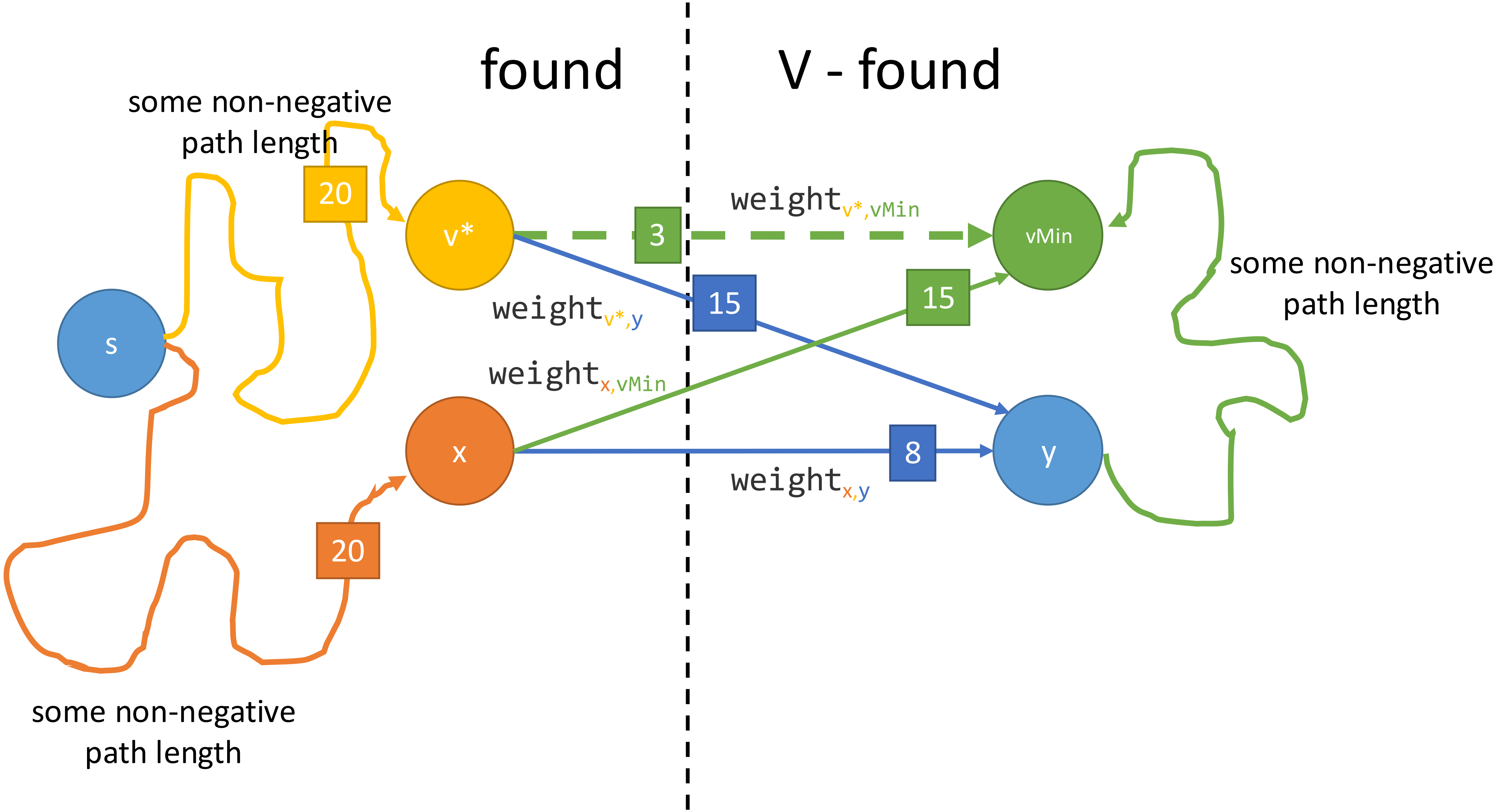
How do we know that the path from x to y to vMin is not even better than the path from v* to vMin?

Dijkstra's Algorithm only operates on graphs with non-negative edge weights. Thus, this new path must be greater than or equal to the (v*, vMin) edge.

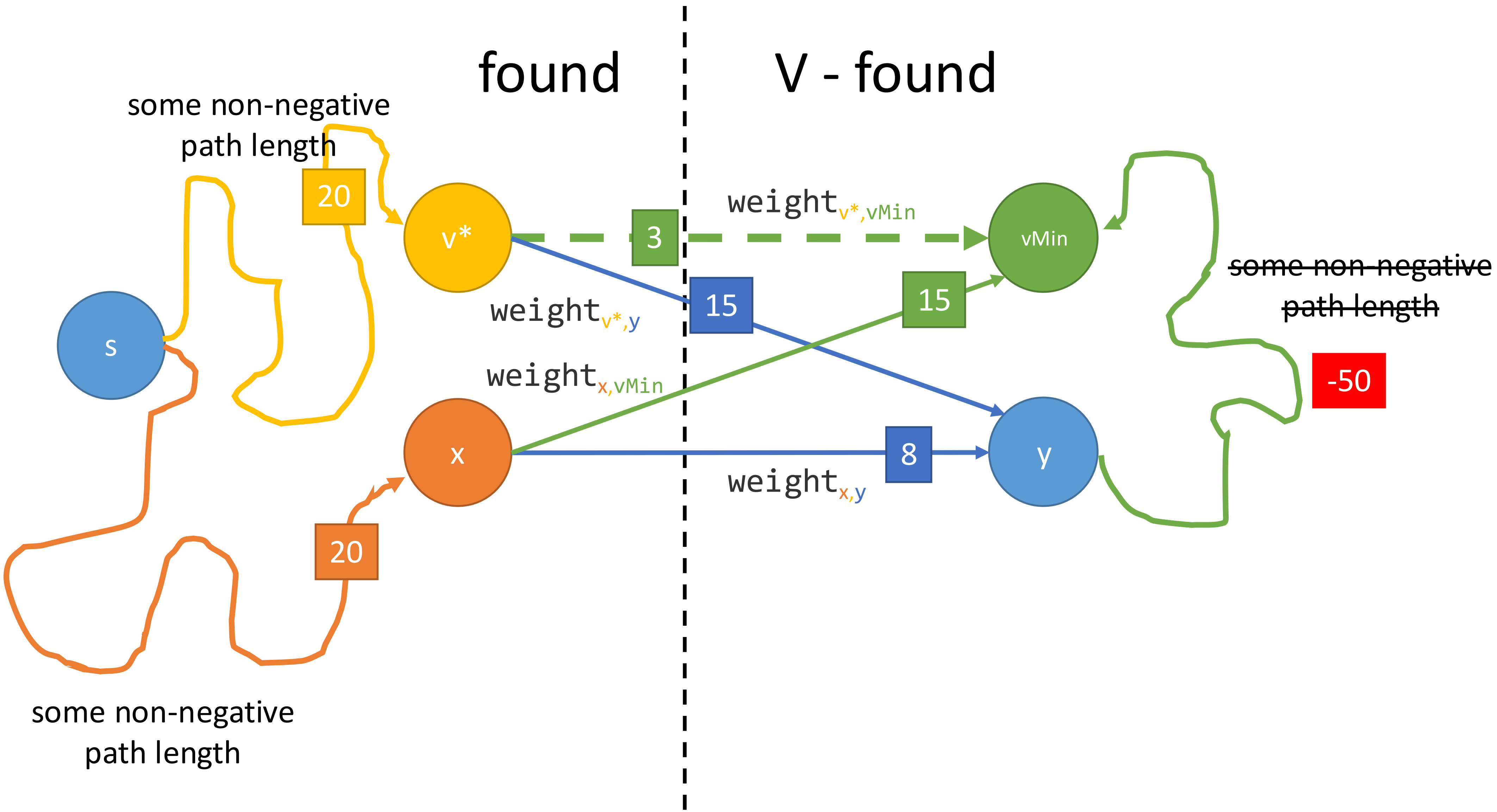
Not taking the shortest edge. We are taking the shortest path!



Sometimes the the shortest edge is on the shortest path.



Why doesn't Dijkstra's work on graphs with negative edges?



Correctness (summary)

- Given our assumption that we do not have negative edges
- And our inductive hypothesis that our path to v* is the shortest
- And our analysis of Dijkstra’s greedy criterion

- We have shown that

$\texttt{lengths[vMin]} = \texttt{lengths[v*]} + \texttt{weight}_{\texttt{v*,vMin}}$ is the best available path length

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    lengths = {v: INFINITY FOR v IN G.vertices}

    found.add(start_vertex)
    lengths[start_vertex] = 0

    WHILE found.length != G.vertices.length
        FOR v IN found
            FOR vOther, weight IN G.edges[v]
                IF vOther NOT IN found
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                    IF vOther_length < min_length
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    RETURN lengths
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What is the running time?

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What is the running time?

How many times does the outer loop run?

O(n)

How many times do the inner two loops run?

O(m)

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$\boxed{O(nm)}$