Kosaraju’s Algorithm for Strongly Connected Components

https://cs.pomona.edu/classes/cs140/

Outline

Topics and Learning Objectives

- Review topological orderings
- Discuss strongly connected components
- Cover Kosaraju's Algorithm

Exercise

- Work through Kosaraju's Algorithm

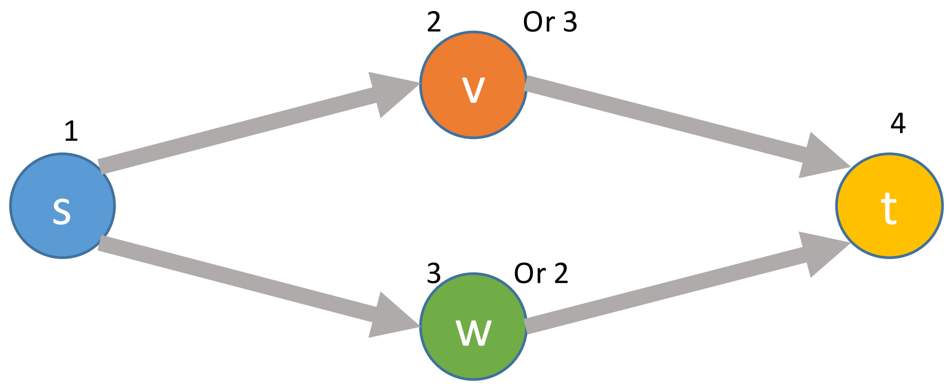
Extra Resources

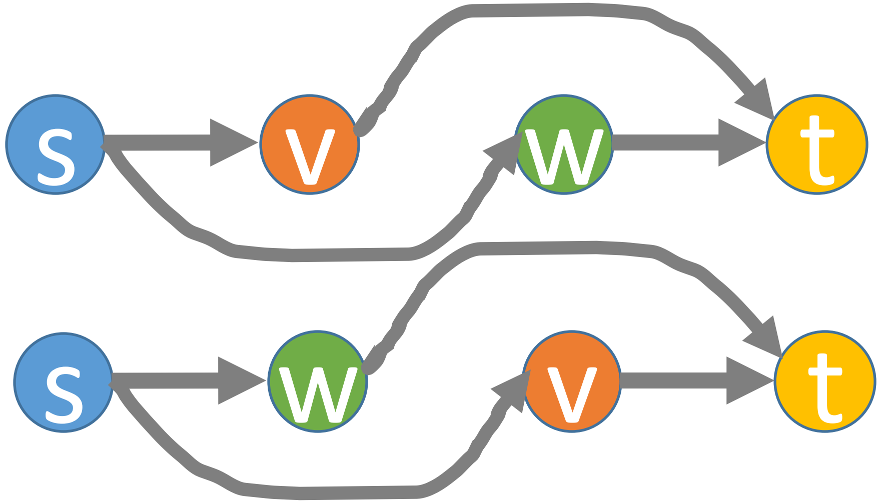
- Introduction to Algorithms, 3rd, chapter 22
- Algorithms Illuminated Part 2: Chapter 8

Topological Orderings

Definition: a topological ordering of a directed acyclic graph is a labelling f of the graph's vertices such that:

1. The f-values are of the set {1, 2, …, n}
2. For an edge (u, v) of G, f(u) < f(v)





Solve with DFS

```
FUNCTION TopologicalOrdering(G)
    found = {v: FALSE FOR v IN G.vertices}
    fValues = {v: INFINITY FOR v IN G.vertices}
    f = G.vertices.length
    FOR v IN G.vertices
        IF found[v] == FALSE
            DFSTopological(G, v, found, f, fValues)
    RETURN fValues
```

```
FUNCTION DFSTopological(G, v, found, f, fValues)
    found[v] = TRUE
    FOR vOther IN G.edges[v]
        IF found[vOther] == FALSE
            DFSTopological(G, vOther, found, f, fValues)
    fValues[v] = f
    f = f - 1
```

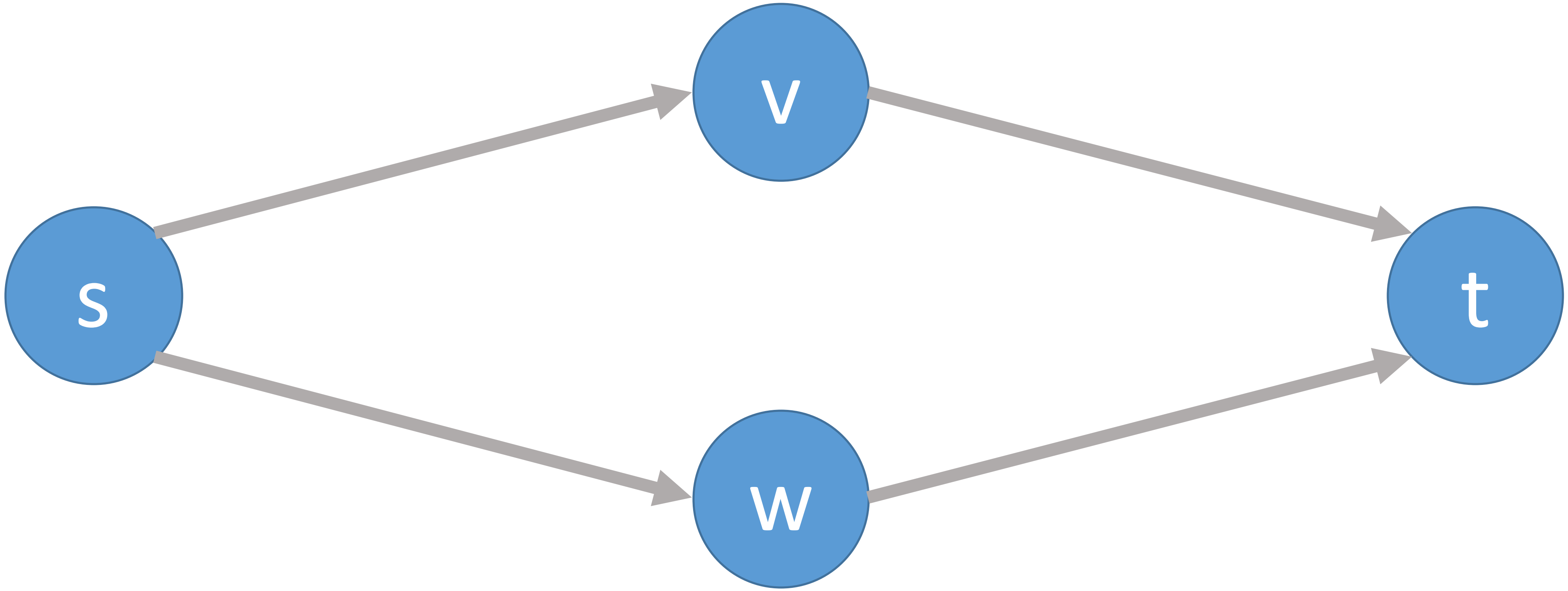
Strongly Connected Components

- Topological orderings are useful in their own right, but they also let us efficiently calculate the strongly connected components (SCCs) of a graph

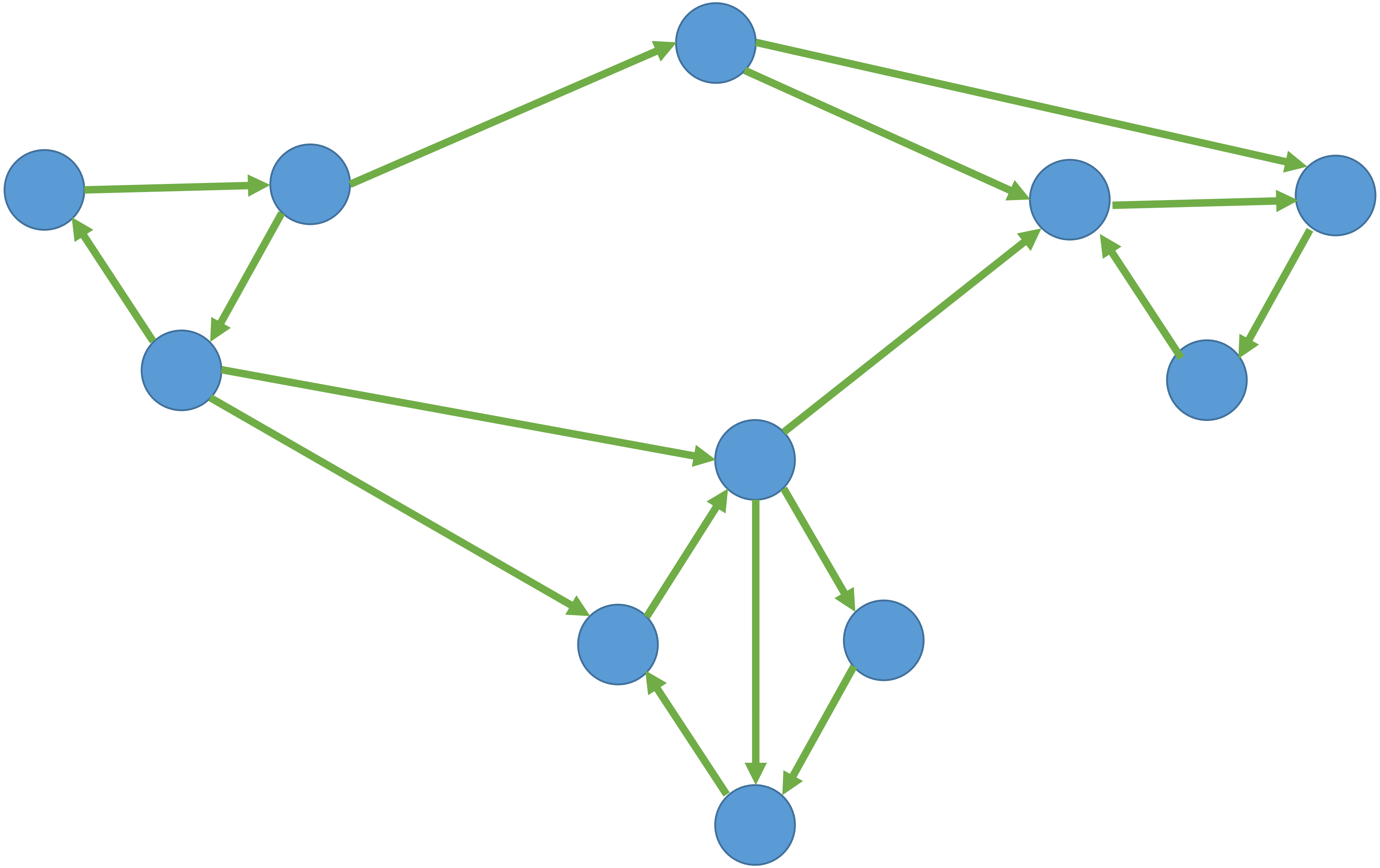
- A component (set of vertices) of a graph is strongly connected if we can find a path from any vertex to any other vertex
- This is a concept for **directed** graphs only
- (just *connected components* for undirected graphs)

Why are SCCs useful?

What are the strongly connected components of this graph?



What are the strongly connected components of this graph?



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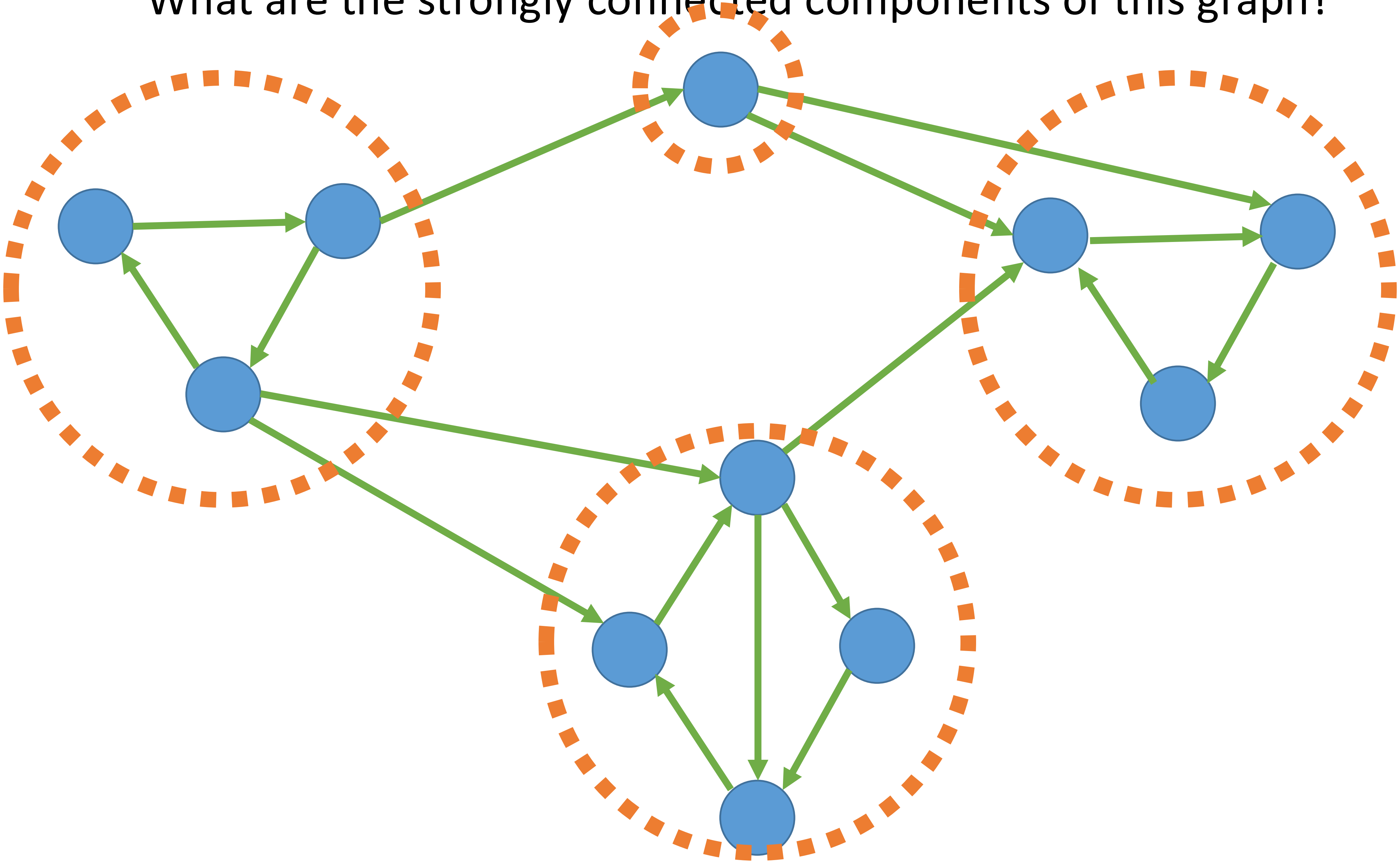
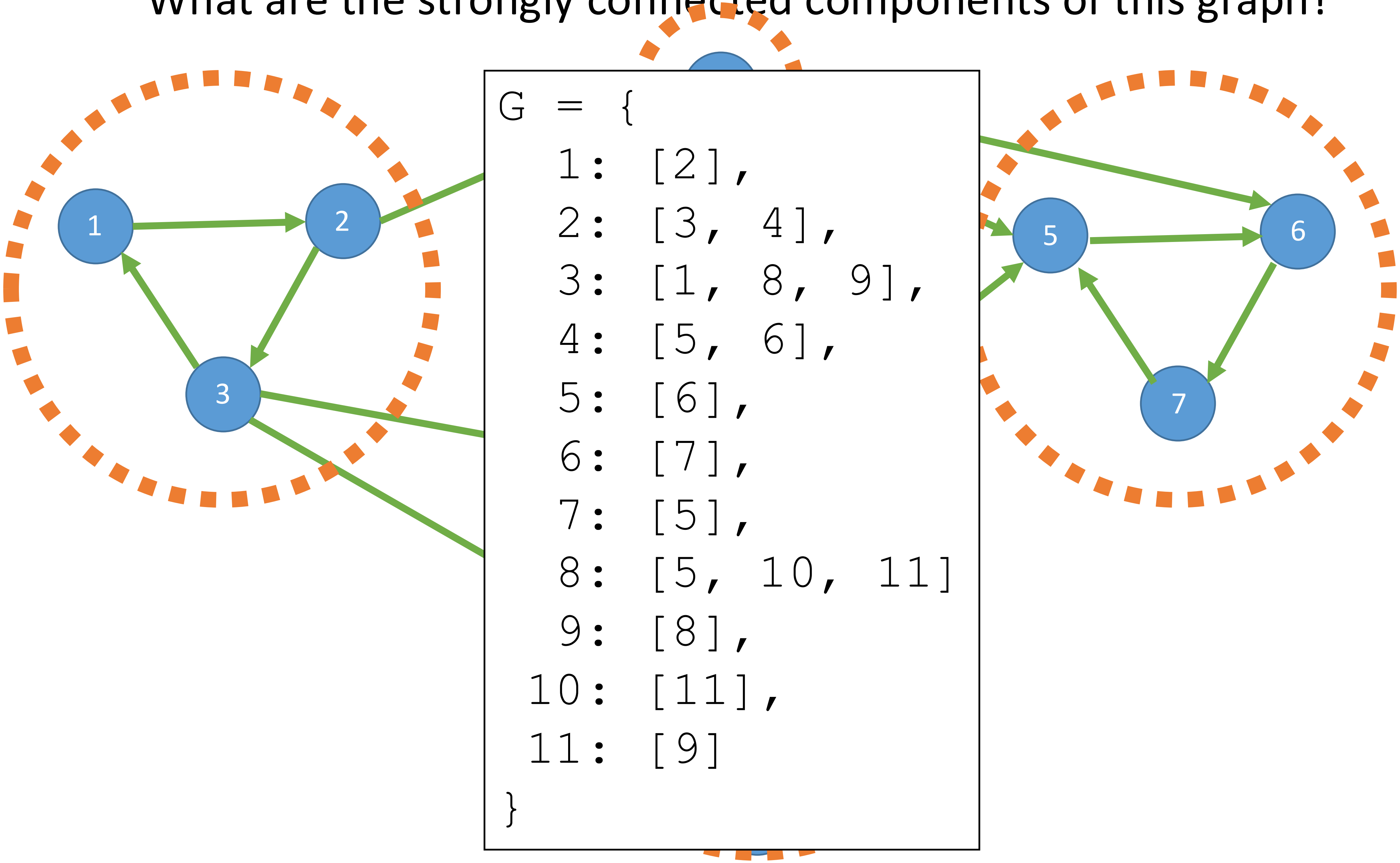


What are the strongly connected components of this graph?

```
G = {
   1: [2],
   2: [3, 4],
   3: [1, 8, 9],
   4: [5, 6],
   5: [6],
   6: [7],
   7: [5],
   8: [5, 10, 11]
   9: [8],
  10: [11],
  11: [9]
}
```

What are the strongly connected components of this graph?



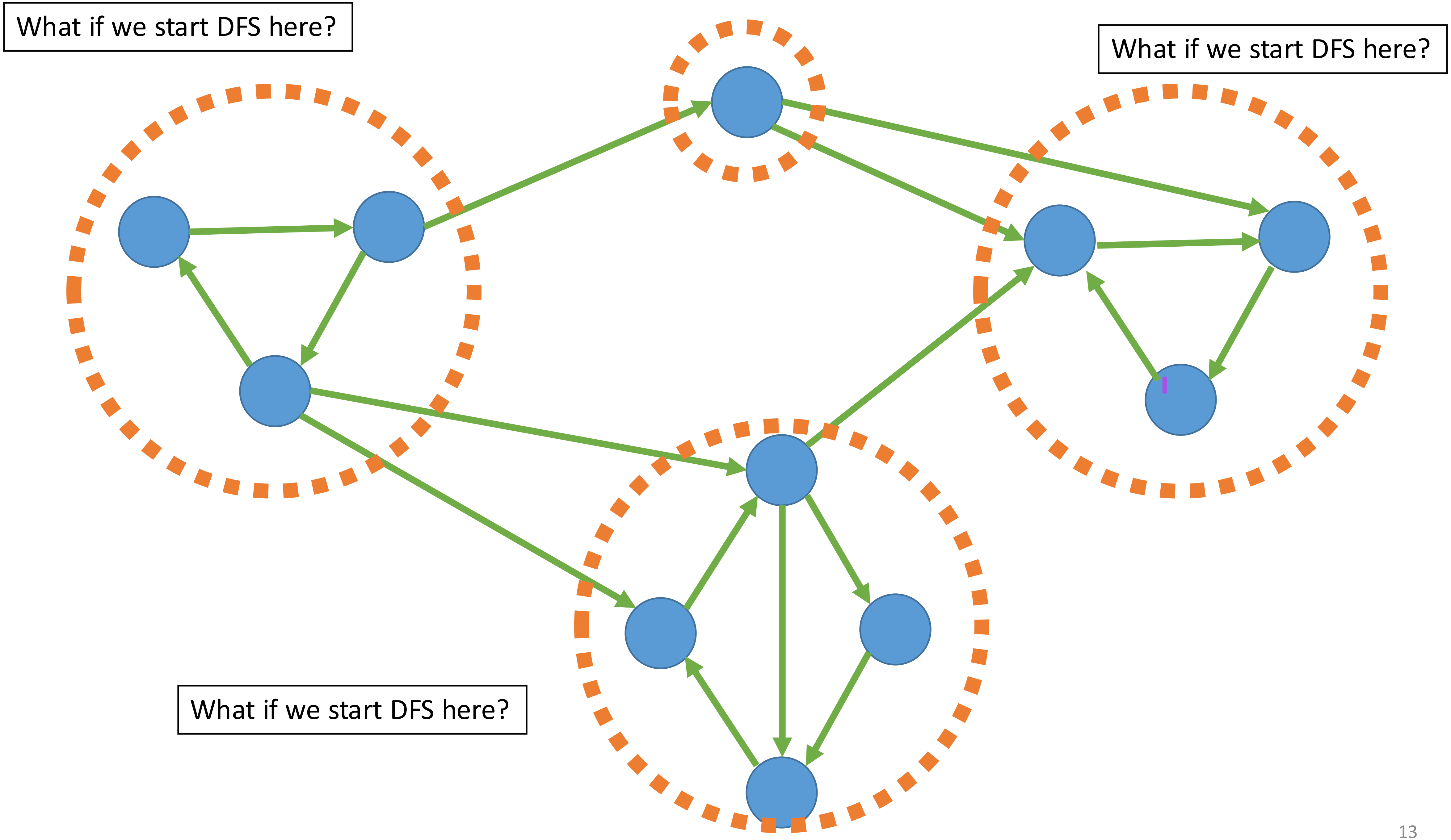

```
G = {
   1:  [2],
   2:  [3,  4],
   3:  [1,  8,  9],
   4:  [5,  6],
   5:  [6],
   6:  [7],
   7:  [5],
   8:  [5,  10,  11]
   9:  [8],
  10:  [11],
  11:  [9]
}
```

Can we use DFS?

What does a DFS do?

- Finds everything that is findable
- Does not visit any vertex more than once

So, what can we find from each of the different nodes?


What if we start DFS here?
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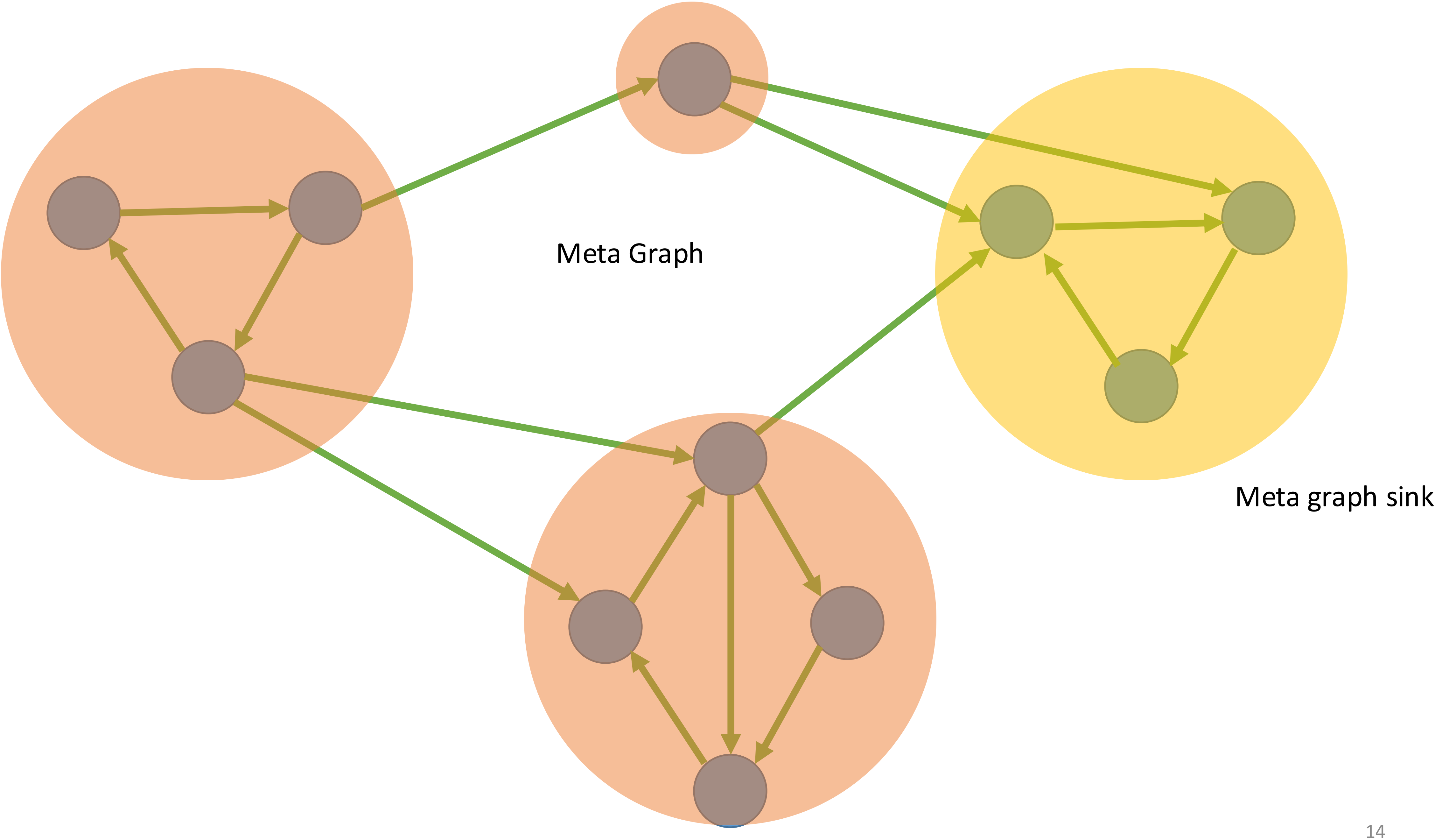
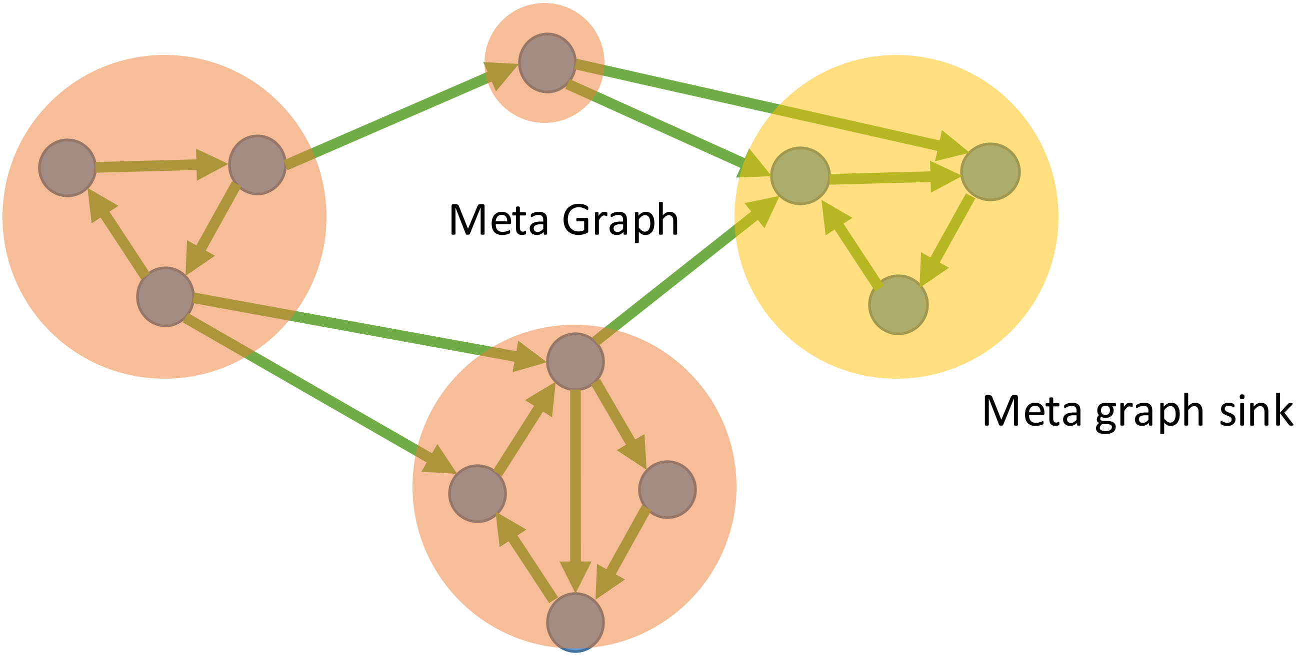


Meta Graph
Meta graph sink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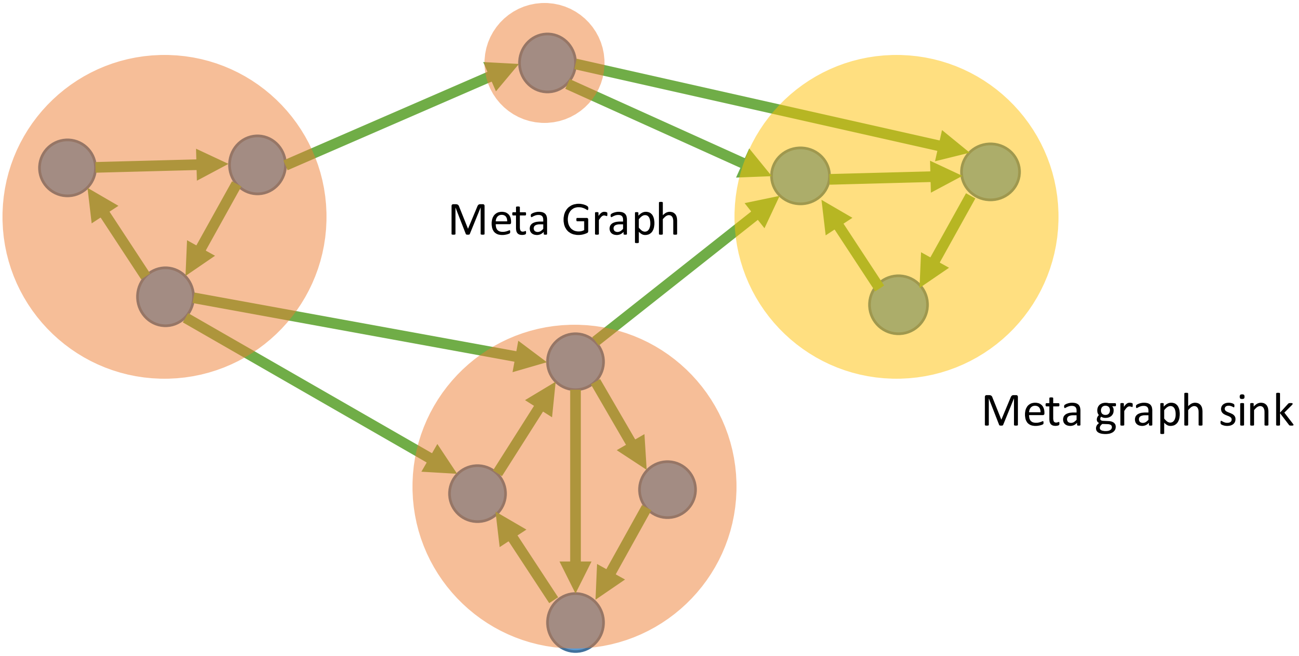
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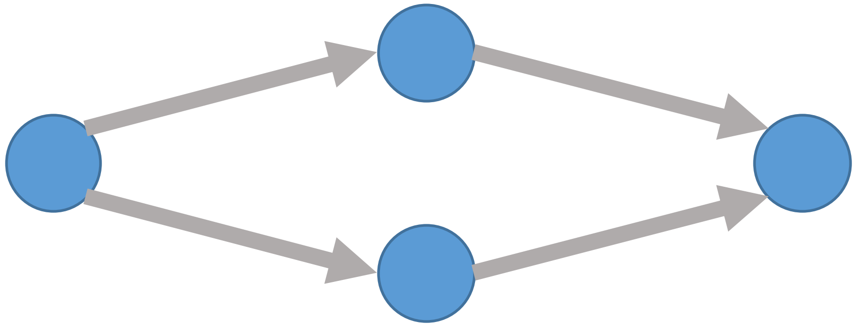
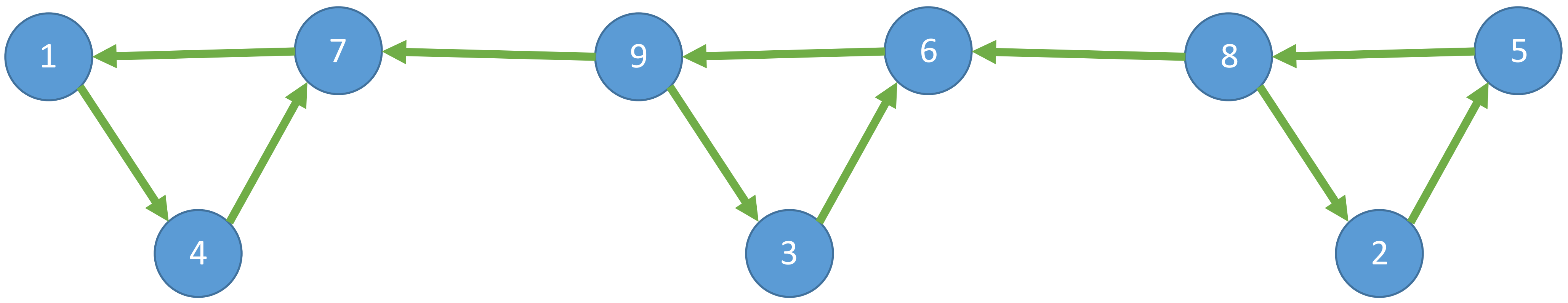


Kosaraju

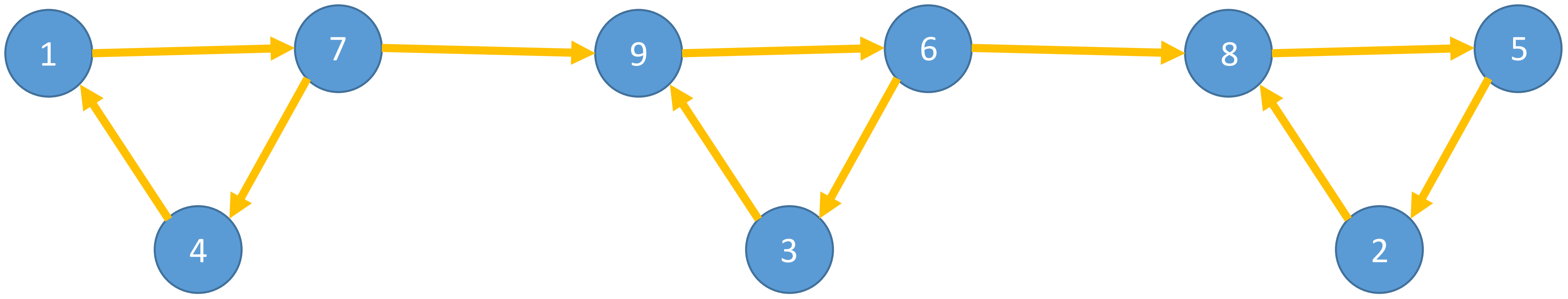
Computes the SCCs in O(m + n) time **(linear!)**

1. Create a reverse version of the G called G_reversed

G



G_reversed



Kosaraju

Computes the SCCs in O(m + n) time **(linear!)**

1. Create a reverse version of the G called G_reversed

2. Run `KosarajuLabels` on G_reversed

   Compute a topological order of the meta graph

3. Create a relabeled version of the G called G_relabeled

4. Run `KosarajuLeaders` on G_relabeled

   Explore vertices in the new order

```
FUNCTION Kosaraju(G)
    G_reversed = reverse_graph(G)
    new_labels = KosarajuLabels(G_reversed)

    G_relabeled = relabel_graph(G, new_labels)
    leaders = KosarajuLeaders(G_relabeled)

    RETURN leaders
```

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FUNCTION Kosaraju(G)
    G_reversed = reverse_graph(G)
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   leaders = KosarajuLeaders(G_relabeled)

   RETURN leaders
```

```
FUNCTION KosarajuLabels(G)
   found = {v: FALSE FOR v IN G.vertices}
   label = 0
   labels = {v: NONE FOR v IN G.vertices}

   FOR v IN G.vertices
      IF found[v] == FALSE
         DFSLabels(G, v, found, label, labels)

   RETURN labels
```

```
FUNCTION DFSLabels(G, v, found, label, labels)
   found[v] = TRUE
   FOR vOther IN G.edges[v]
      IF found[vOther] == FALSE
         DFSLabels(G, vOther, found, label, labels)
   label = label + 1
   labels[v] = label
```

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FUNCTION Kosaraju(G)
   G_reversed = reverse_graph(G)
   new_labels = KosarajuLabels(G_reversed)

   G_relabeled = relabel_graph(G, new_labels)
   leaders = KosarajuLeaders(G_relabeled)

   RETURN leaders
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```
FUNCTION KosarajuLeaders(G)
   found = {v: FALSE FOR v IN G.vertices}
   leaders = {v: NONE FOR v IN G.vertices}

   FOR v IN G.vertices.reverse_order
      IF found[v] == FALSE
         leader = v
         DFSLeaders(G, v, found, leader, leaders)

   RETURN leaders
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   found[v] = TRUE
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These are typically implemented in a single function

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```

```
FUNCTION DFSLeaders(G, v, found, leader, leaders)
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   FOR vOther IN G.edges[v]
      IF found[vOther] == FALSE
         DFSLeaders(G, vOther, found, leader, leaders)
```

These are typically implemented in a single function

Does both labels and leaders.

```
FUNCTION KosarajuLoop(G)
   found = {v: FALSE FOR v IN G.vertices}
   label = 0
   labels = {v: NONE FOR v IN G.vertices}
   leaders = {v: NONE FOR v IN G.vertices}

   FOR v IN G.vertices.reverse_order
      IF found[v] == FALSE
         leader = v
         KosarajuDFS(G, v, found, label, labels, leader, leaders)

   RETURN labels, leaders


FUNCTION KosarajuDFS(G, v, found, label, labels, leader, leaders)
   found[v] = TRUE
   leaders[v] = leader
   FOR vOther IN G.edges[v]
      IF found[vOther] == FALSE
         KosarajuDFS(G, v, found, label, labels, leader, leaders)
   label = label + 1
   labels[v] = label
```

```
FUNCTION Kosaraju(G)
    G_reversed = reverse_graph(G)
    new_labels = KosarajuLabels(G_reversed)

    G_relabeled = relabel_graph(G, new_labels)
    leaders = KosarajuLeaders(G_relabeled)

    RETURN leaders
```

```
FUNCTION Kosaraju(G)
    G_reversed = reverse_graph(G)
    new_labels, _ = KosarajuLoop(G_reversed)

    G_relabeled = relabel_graph(G, new_labels)
    _, leaders = KosarajuLoop(G_relabeled)

    RETURN leaders
```

Kosaraju

Computes the SCCs in O(m + n) time **(linear!)**

1. Create a reverse version of the G called G_reversed

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   Compute a topological order of the meta graph

3. Create a relabeled version of the G called G_relabeled

4. Run KosarajuLoop on G_relabeled

   Explore vertices in the new order

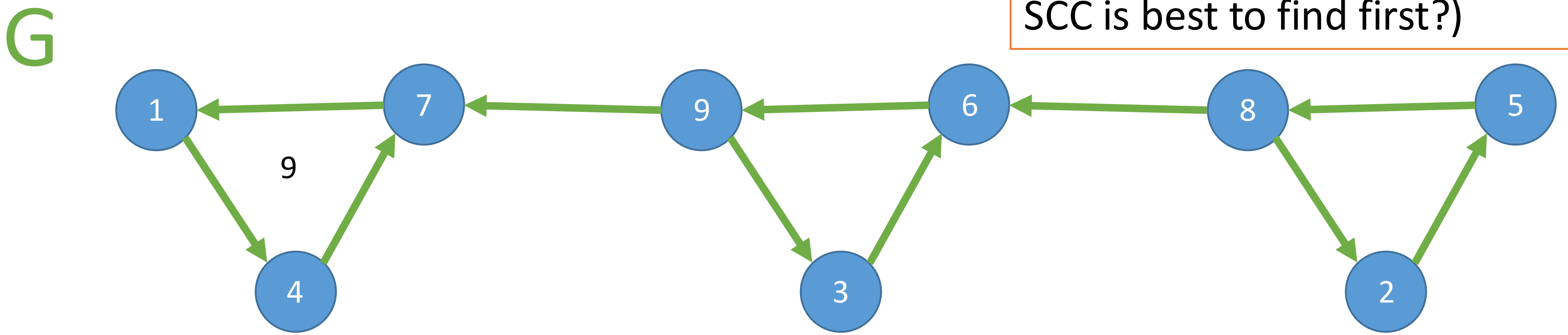
What are the SCCs?

```
FUNCTION Kosaraju(G)
    G_reversed = reverse_graph(G)
    new_labels, _ = KosarajuLoop(G_reversed)

    G_relabeled = relabel_graph(G, new_labels)
    _, leaders = KosarajuLoop(G_relabeled)

    RETURN leaders
```

Where do we want to start DFS if we are looking for SCCs? (Which SCC is best to find first?)



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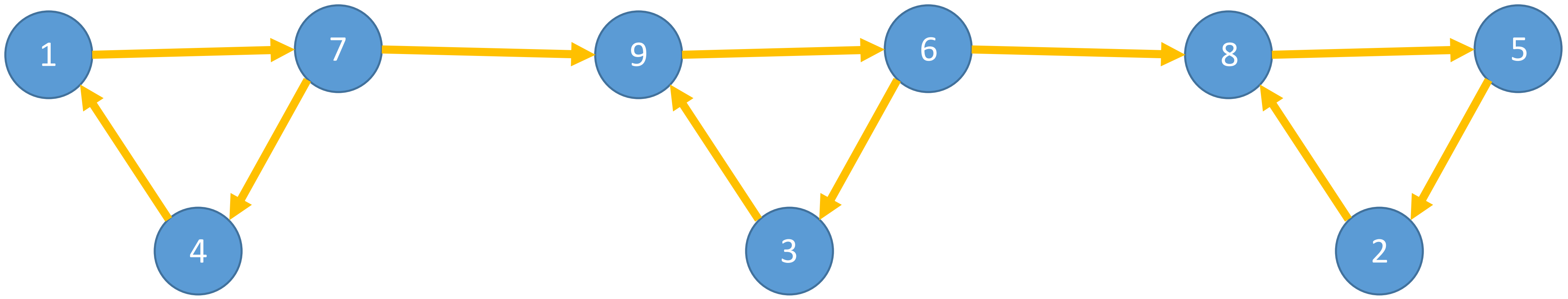

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    RETURN leaders
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Where do we want to start DFS if we are looking for SCCs? (Which SCC is best to find first?)

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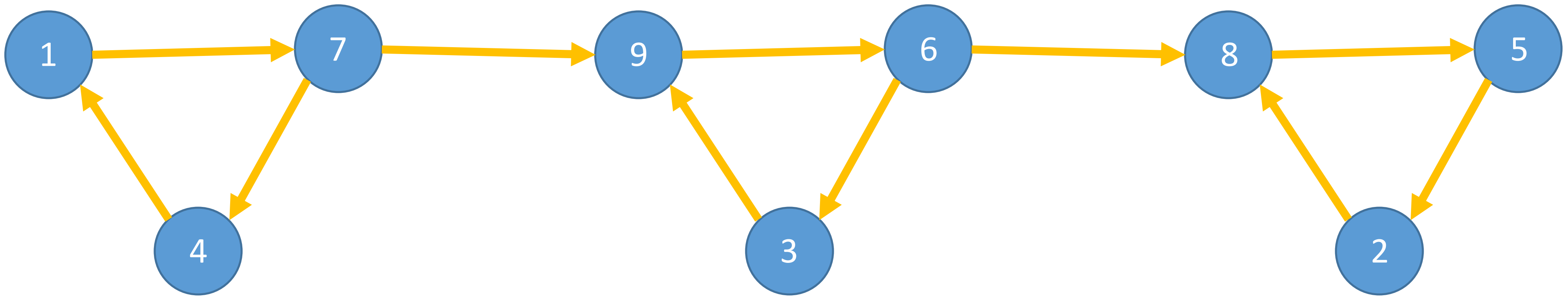

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    found = {v: FALSE FOR v IN G.vertices}
    label = 0
    labels = {v: NONE FOR v IN G.vertices}
    leaders = {v: NONE FOR v IN G.vertices}

    FOR v IN G.vertices.reverse_order
        IF found[v] == FALSE
            leader = v
            KosarajuDFS(…)

    RETURN labels, leaders
```
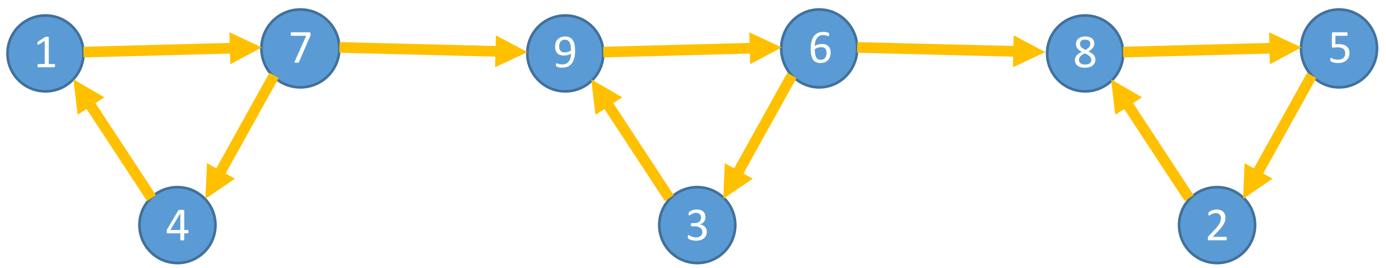
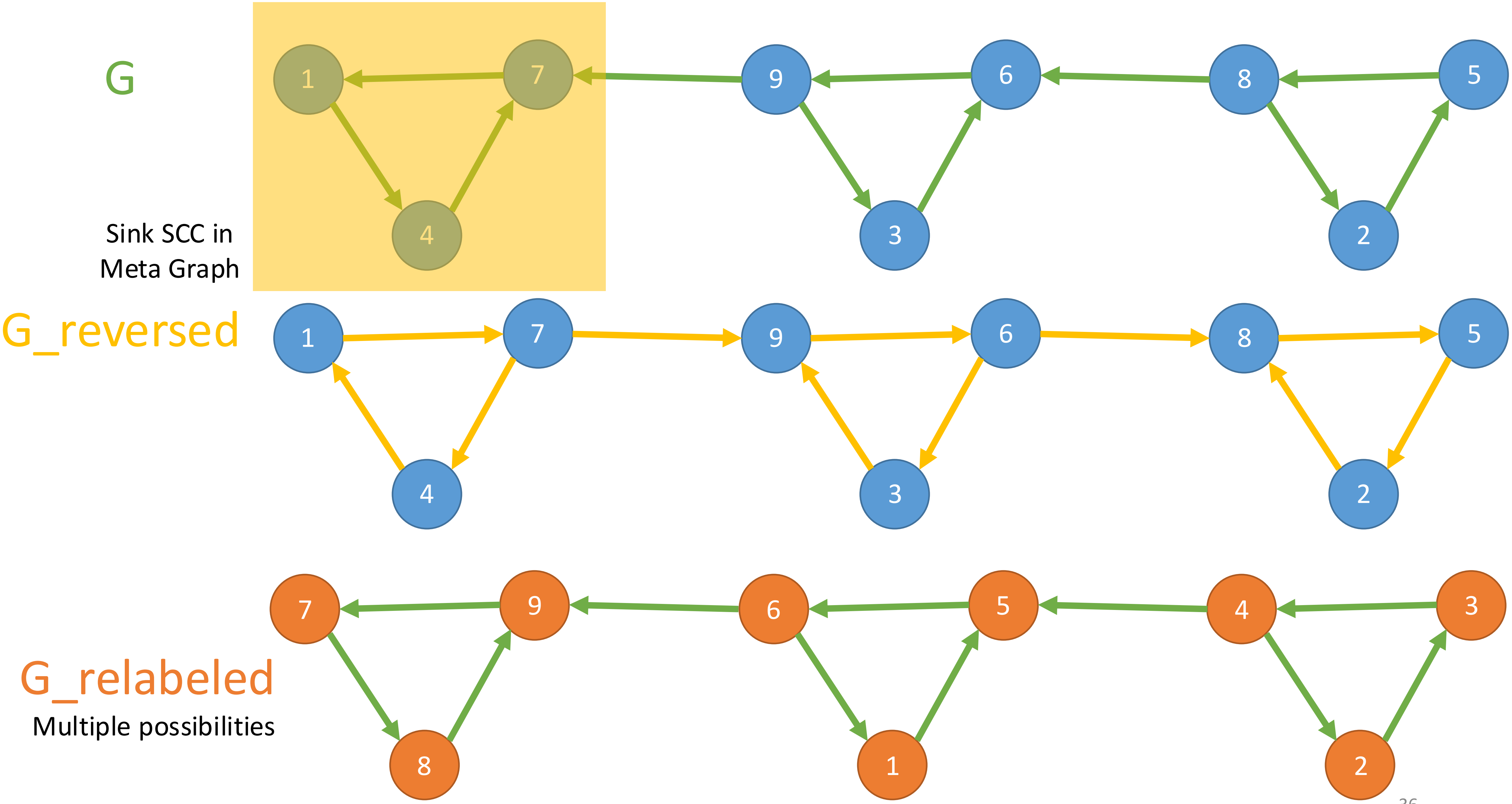

Can we start anywhere?

```
FUNCTION KosarajuDFS(…)
    found[v] = TRUE
    leaders[v] = leader
    FOR vOther IN G.edges[v]
        IF found[vOther] == FALSE
            KosarajuDFS(…)
    label = label + 1
    labels[v] = label
```



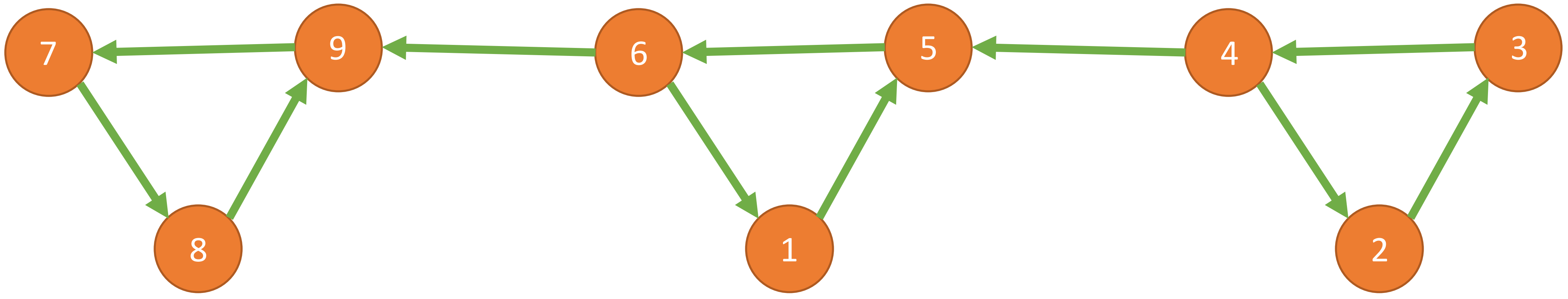
Ignore leaders the first pass

Ignore labels the second pass


G
Sink SCC in Meta Graph
G_reversed
G_relabeled
Multiple possibilities

```
FUNCTION Kosaraju(G)
    G_reversed = reverse_graph(G)
    new_labels, _ = KosarajuLoop(G_reversed)

    G_relabeled = relabel_graph(G, new_labels)
    _, leaders = KosarajuLoop(G_relabeled)

    RETURN leaders
```

G_relabeled



```
FUNCTION KosarajuLoop(G)
   found = {v: FALSE FOR v IN G.vertices}
   label = 0
   labels = {v: NONE FOR v IN G.vertices}
   leaders = {v: NONE FOR v IN G.vertices}

   FOR v IN G.vertices.reverse_order
      IF found[v] == FALSE
         leader = v
         KosarajuDFS(…)

   RETURN labels, leaders
```

```
FUNCTION KosarajuDFS(…)
   found[v] = TRUE
   leaders[v] = leader
   FOR vOther IN G.edges[v]
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         KosarajuDFS(…)
   label = label + 1
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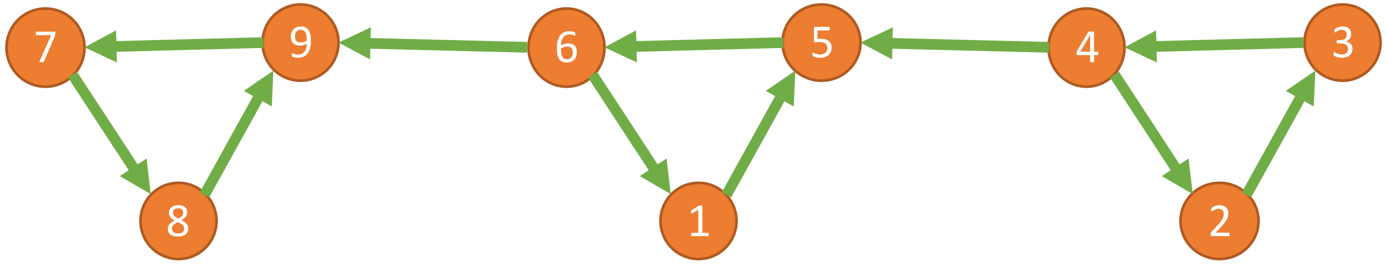

Ignore leaders the first pass

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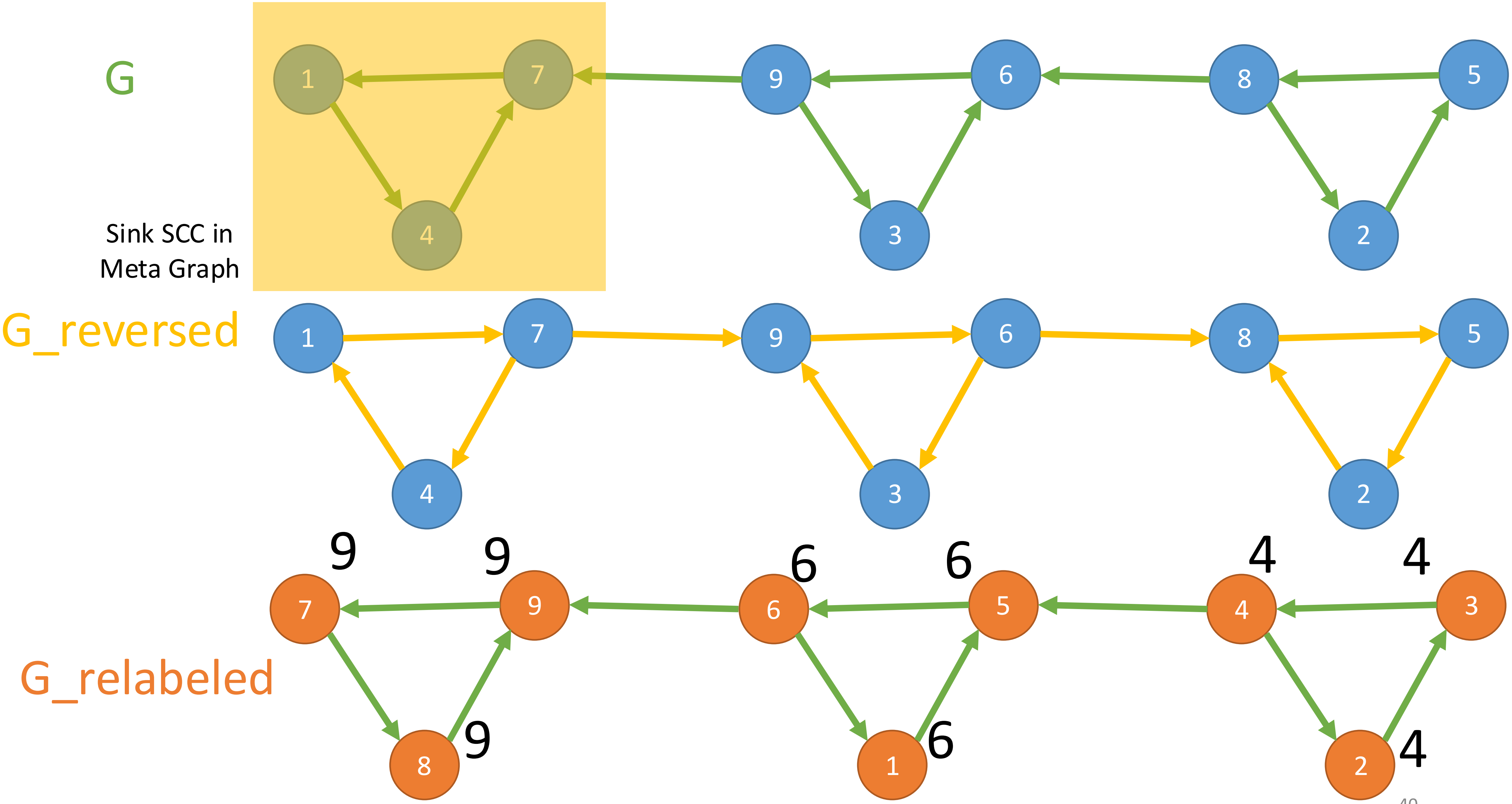

G
Sink SCC in
Meta Graph
G_reversed
G_relabeled

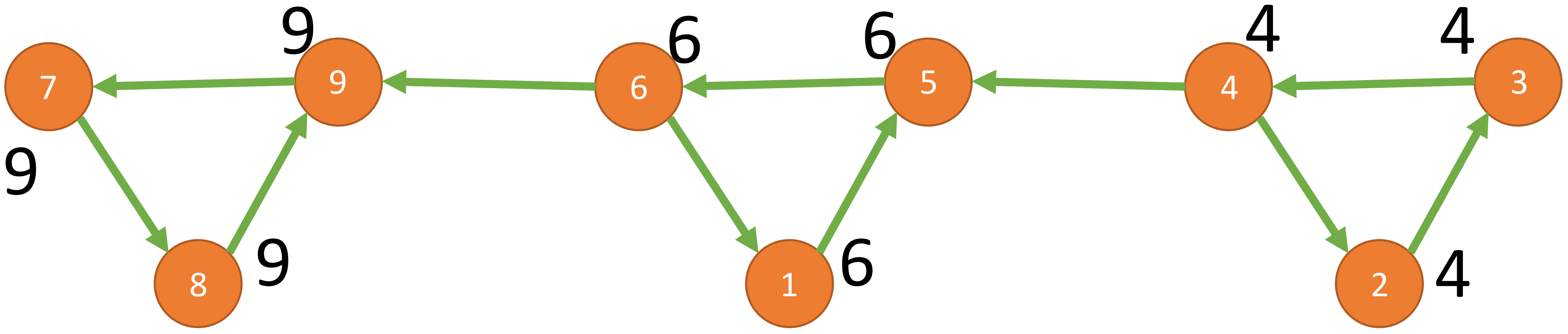
```
FUNCTION Kosaraju(G)
    G_reversed = reverse_graph(G)
    new_labels, _ = KosarajuLoop(G_reversed)

    G_relabeled = relabel_graph(G, new_labels)
    _, leaders = KosarajuLoop(G_relabeled)

    RETURN leaders
```

What could you do to make this API a bit nicer?

G_relabeled



Exercise

```
FUNCTION KosarajuLoop(G)
   found = {v: FALSE FOR v IN G.vertices}
   label = 0
   labels = {v: NONE FOR v IN G.vertices}
   leaders = {v: NONE FOR v IN G.vertices}

   FOR v IN G.vertices.reverse_order
      IF found[v] == FALSE
         leader = v
         KosarajuDFS(G, v, found, label, labels, leader, leaders)

   RETURN labels, leaders
```

```
FUNCTION Kosaraju(G)
   G_reversed = reverse_graph(G)
   new_labels, _ = KosarajuLoop(G_reversed)

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```

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   label = label + 1
   labels[v] = label
```
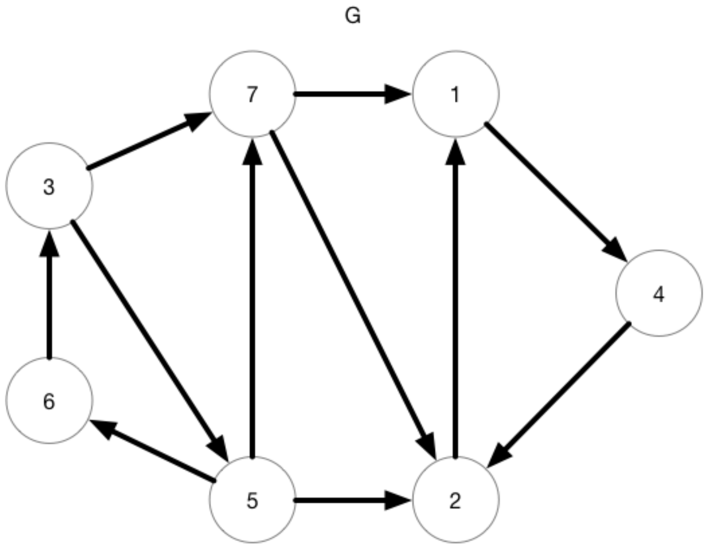
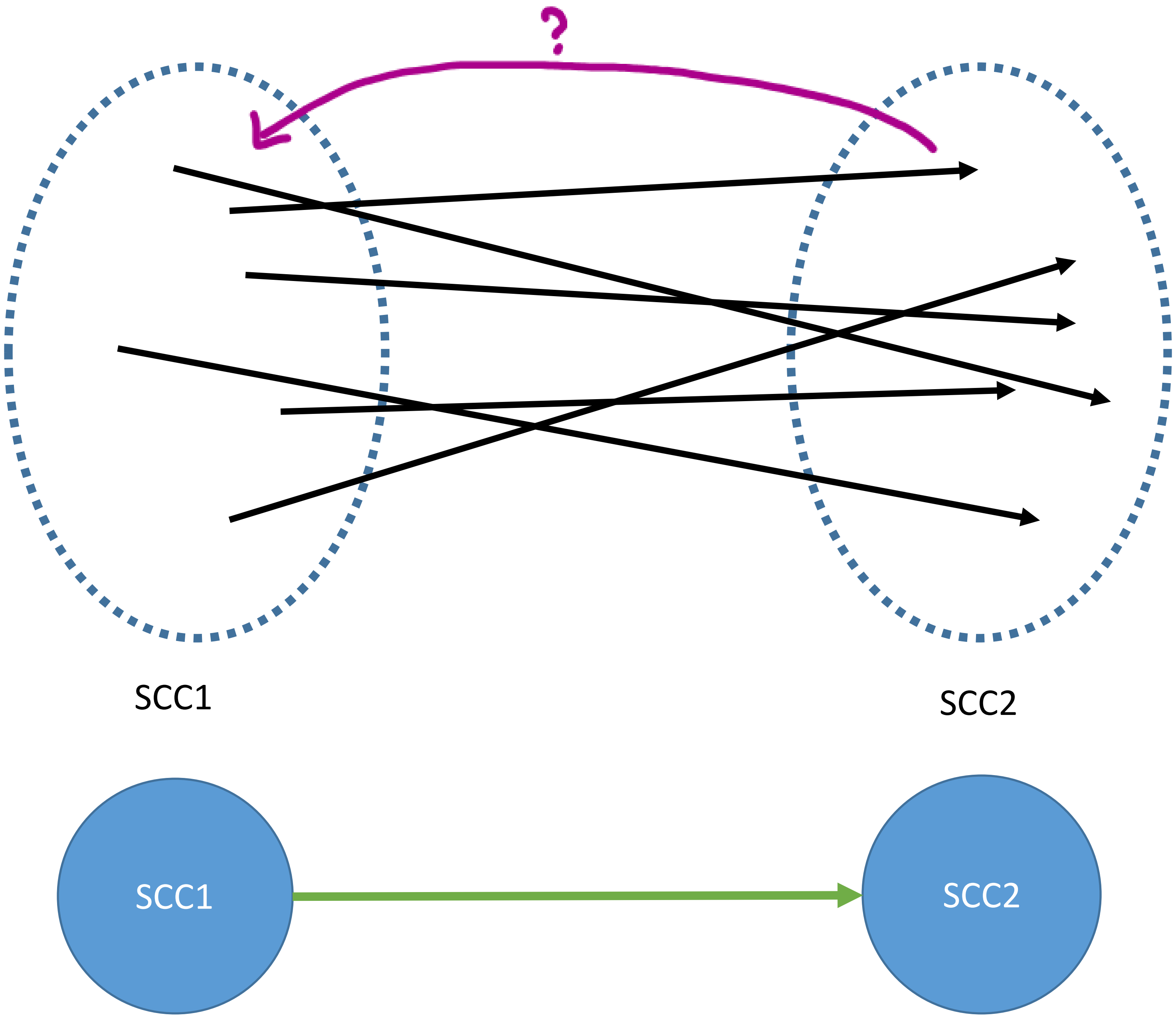
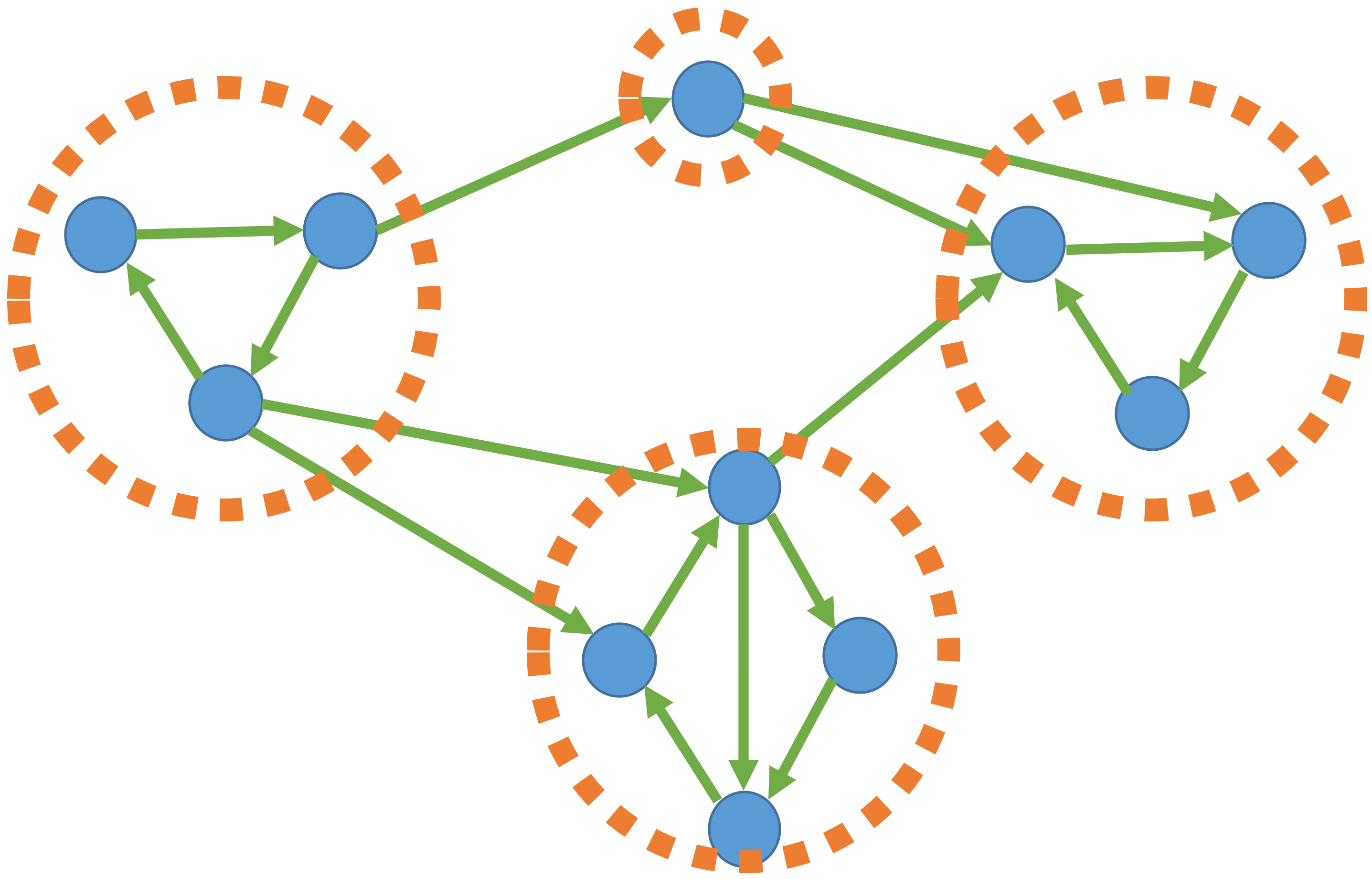
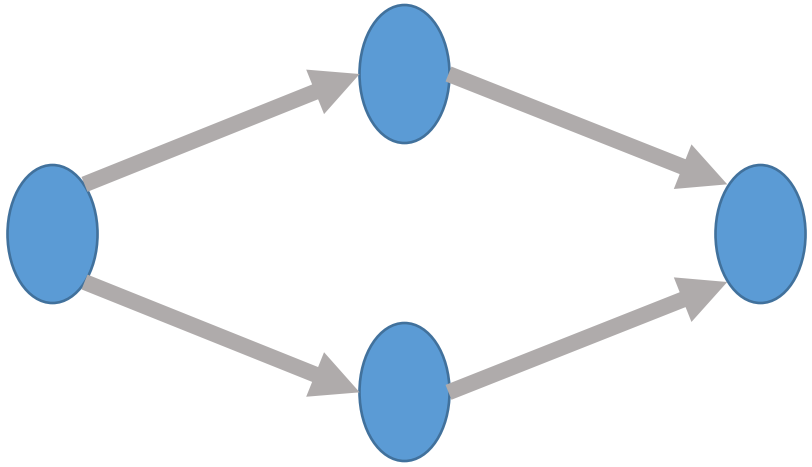



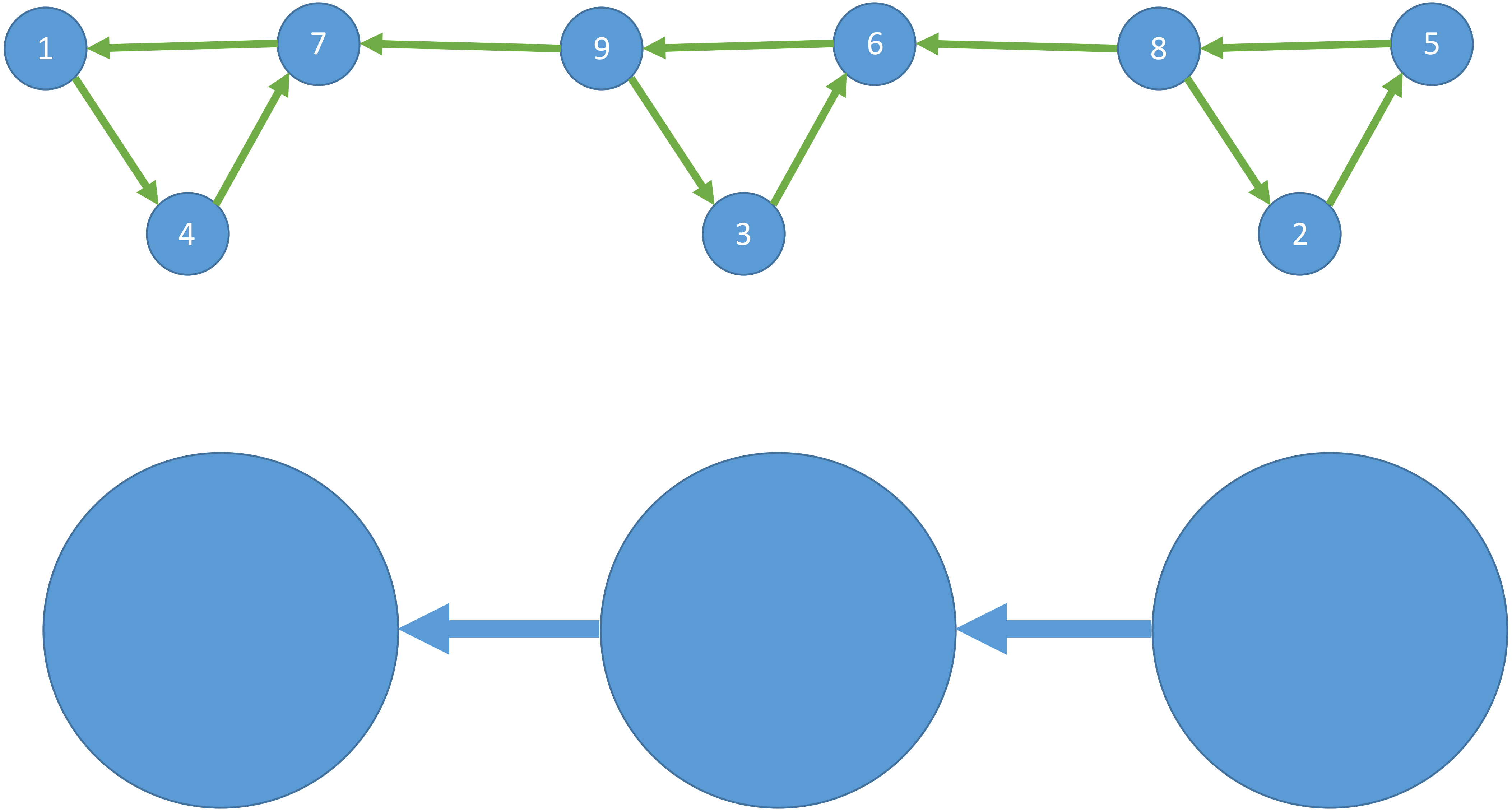
Why does this work?

- Does this work for all graphs, or just this example?
- The SCCs of G create an acyclic “meta-graph”
- For the “meta-graph”
  - Vertices correspond to the SCCs
  - Edges correspond to paths among the SCCs

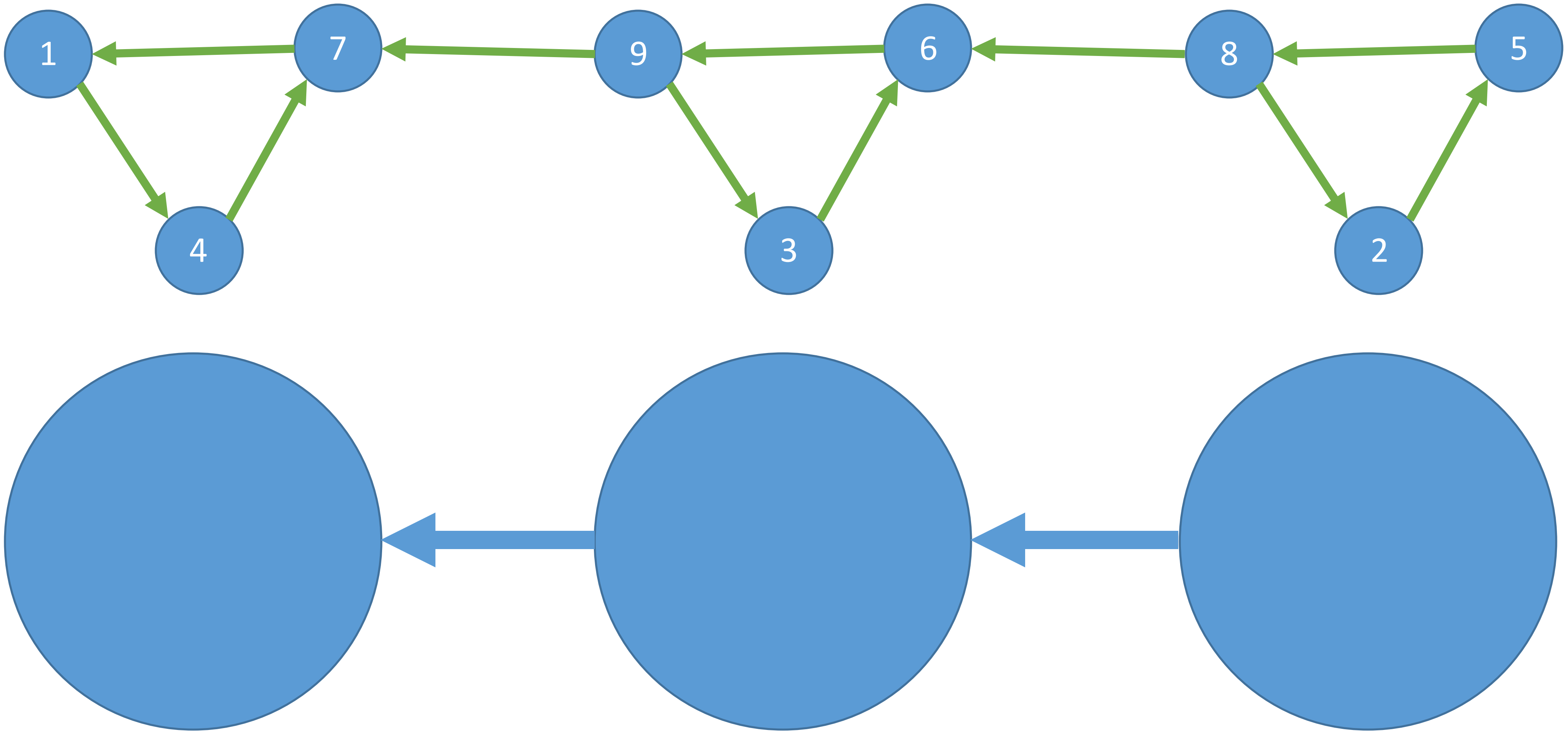

?
SCC1
SCC2
SCC1
SCC2



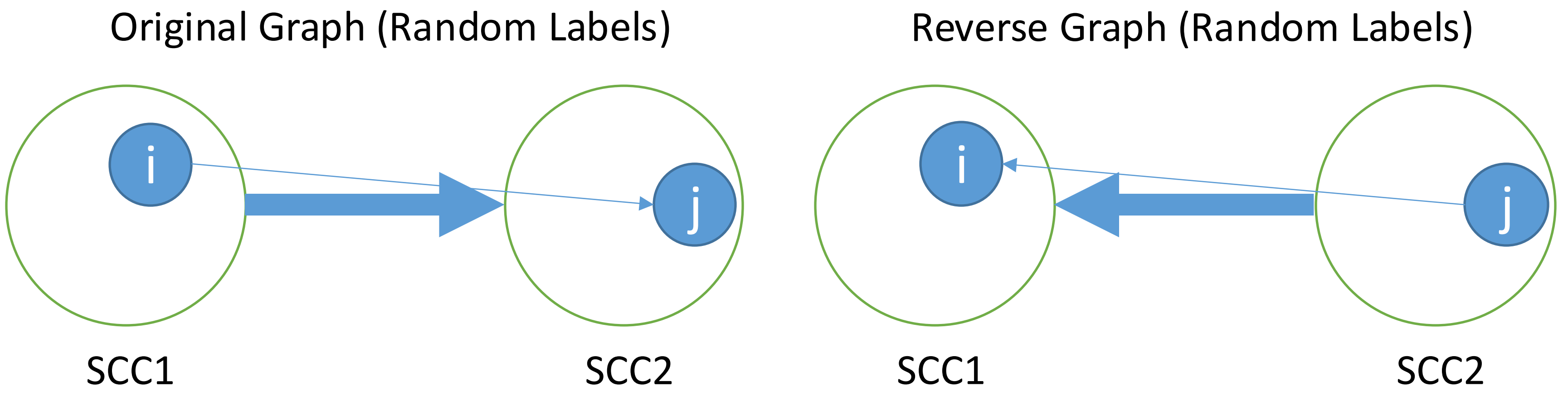



1
7
9
6
8
5
4
3
2

How do we know that the SCC based meta-graph is acyclic?



Key Lemma

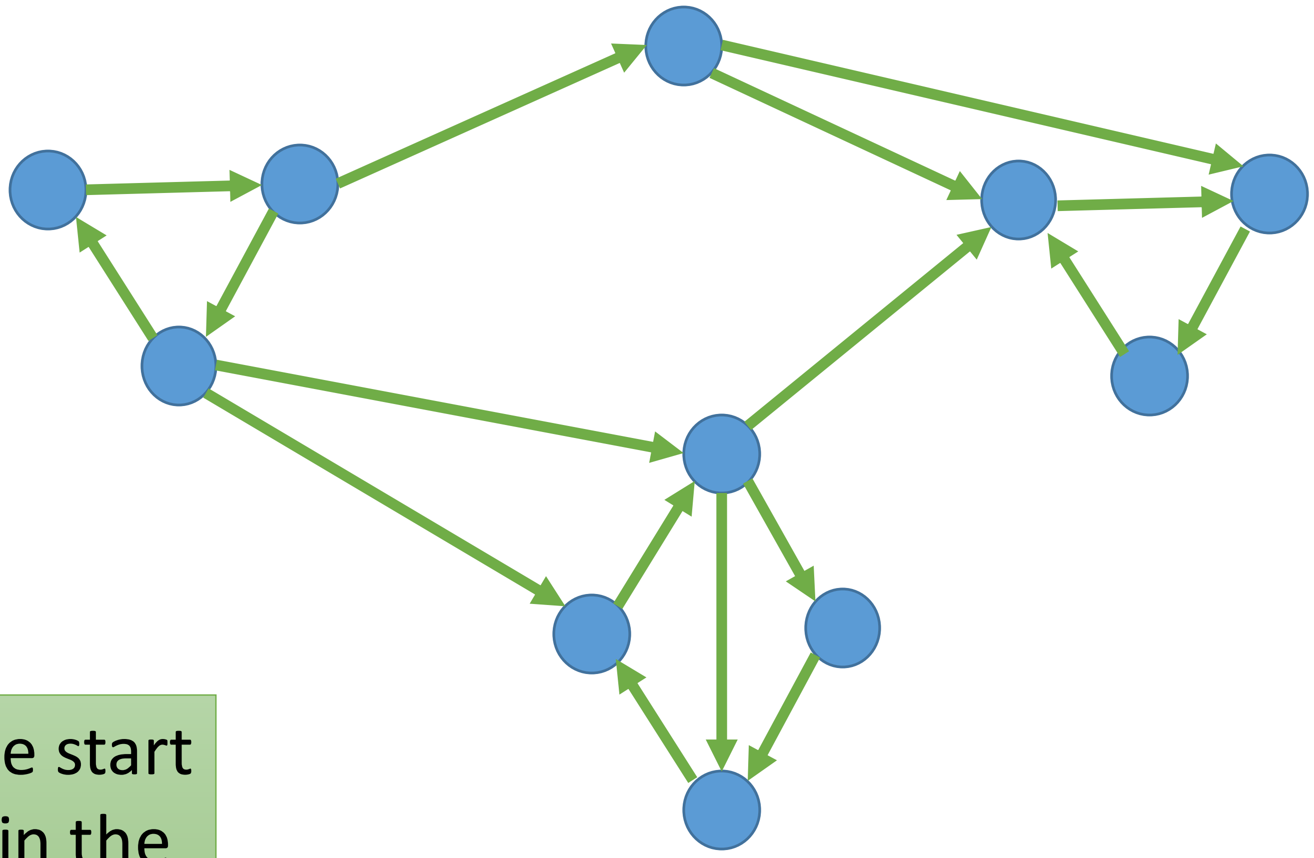


- Consider the two adjacent SCCs in the meta-graph above
- Now consider the re-labeling found from the reverse graph

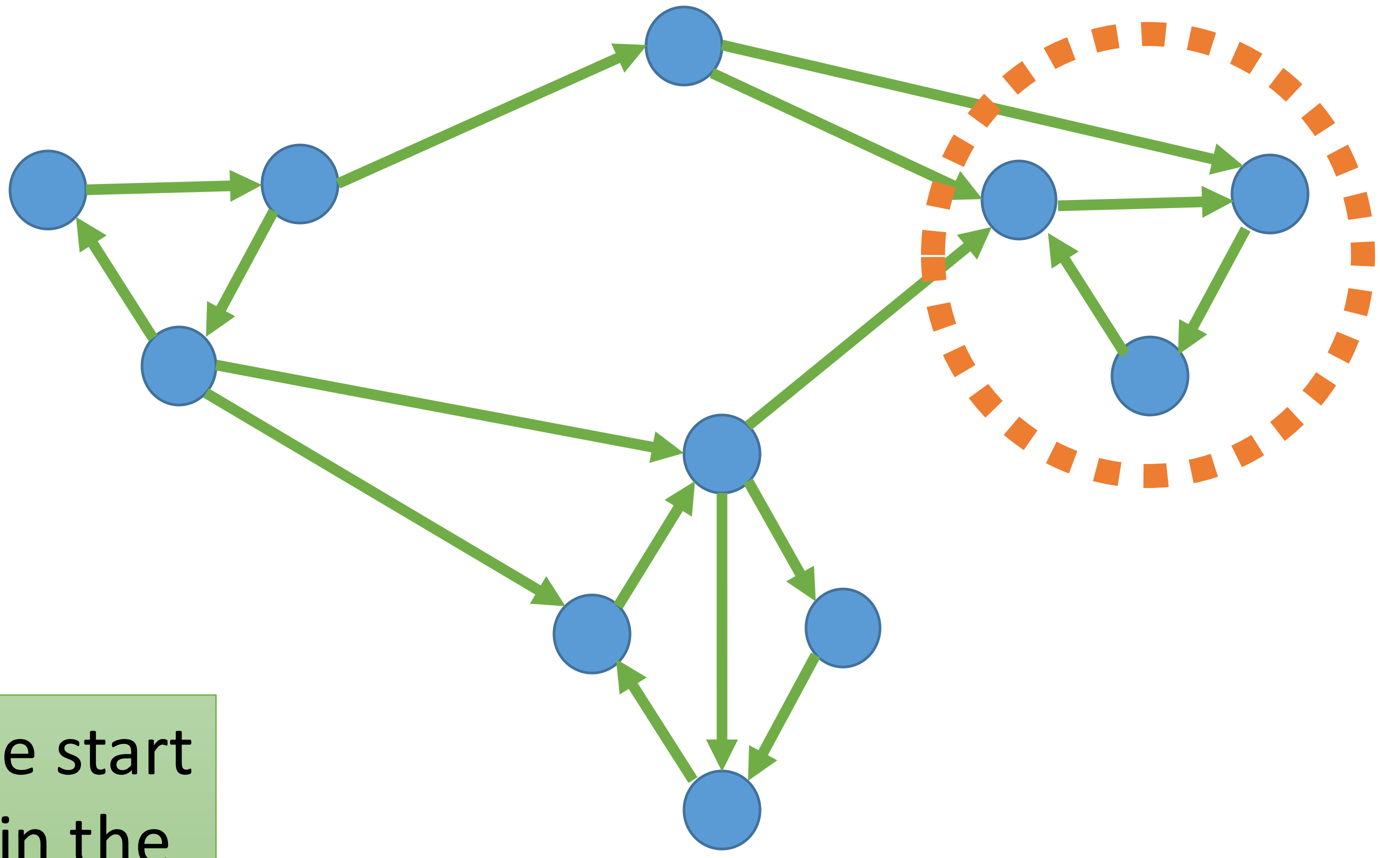
Where do we want to start DFS in the leaders pass?

- Let f(v) = the re-labeling resulting from `KosarajuLoop(G_reversed)`
- Then `max[f(.) in SCC1] < max[f(.) in SCC2]`
- Corollary: the maximum label must lie in a "sink SCC" of the original graph

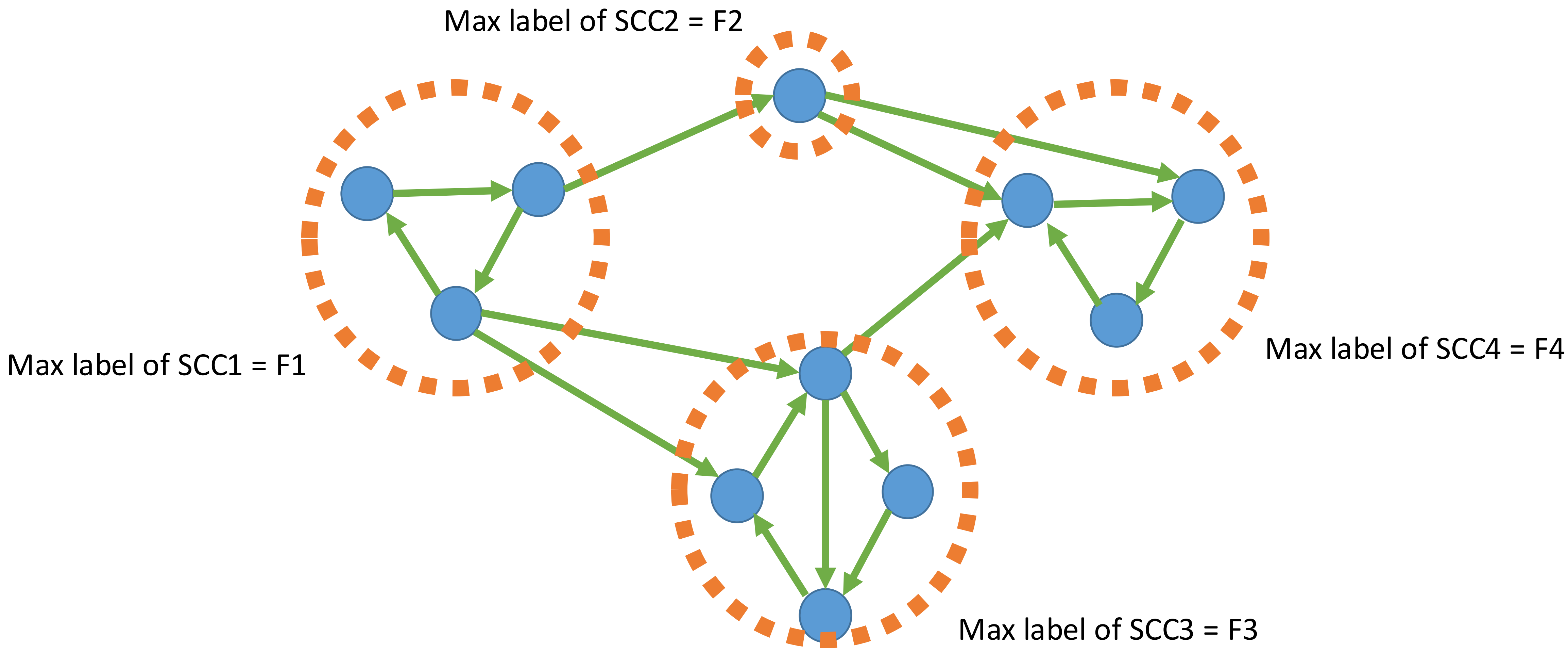
```
FUNCTION KosarajuDFS(…)
   found[v] = TRUE
   leaders[v] = leader
   FOR vOther IN G.edges[v]
      IF found[vOther] == FALSE
         KosarajuDFS(…)
   label = label + 1
   labels[v] = label
```



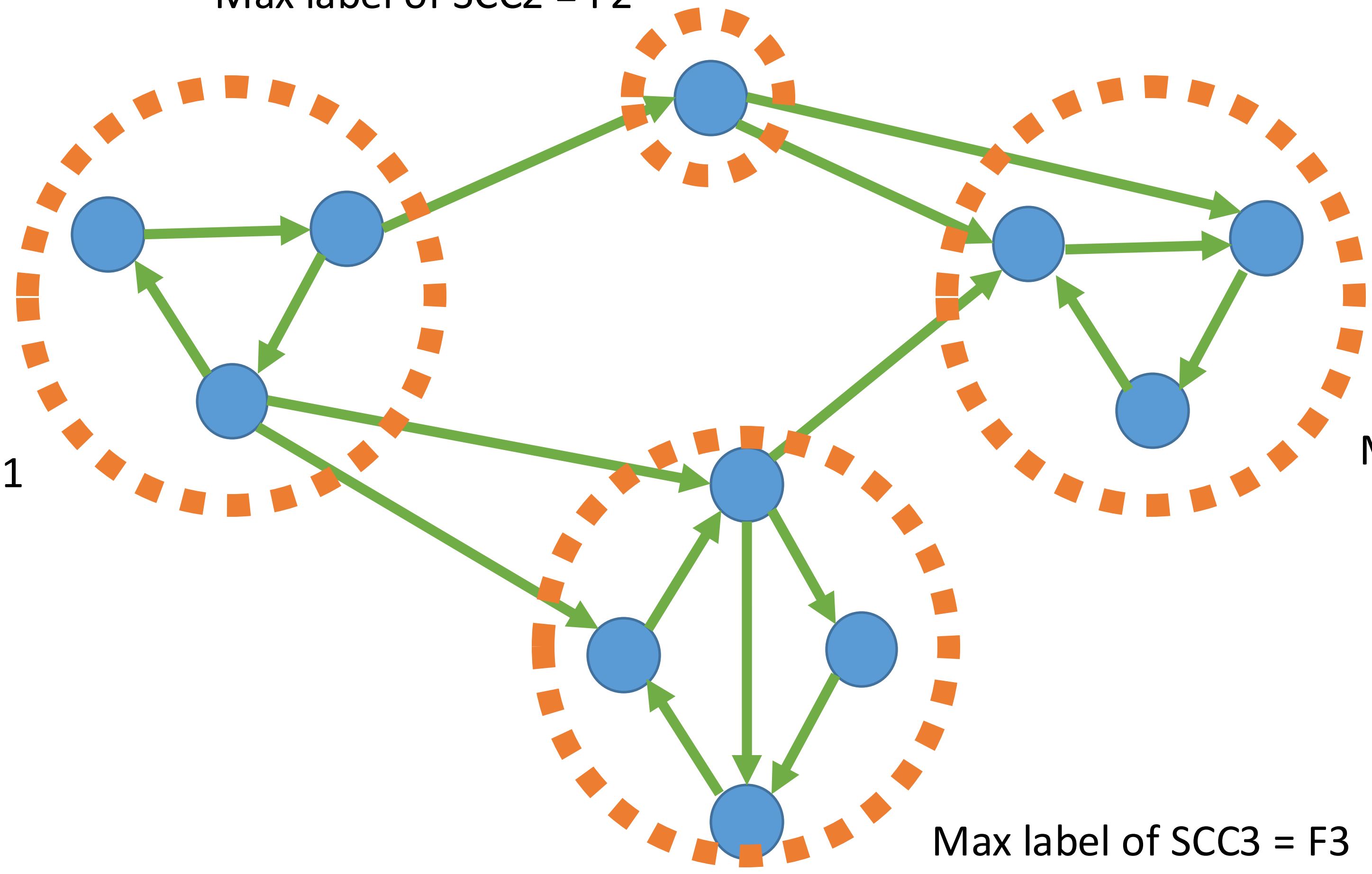
Where should we start labeling leaders in the second pass?



Where should we start labeling leaders in the second pass?



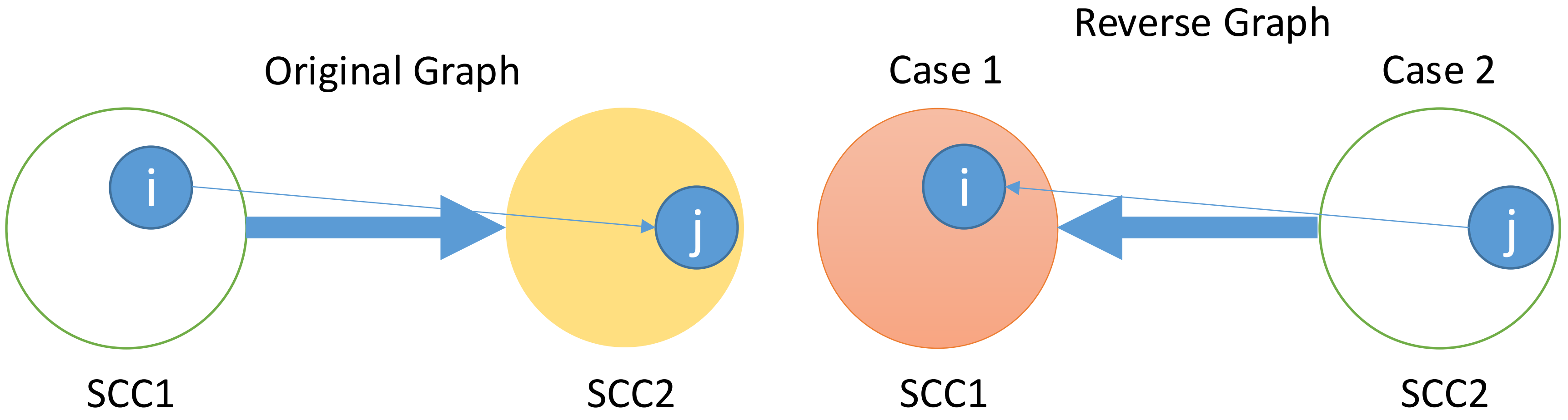
Then F1 < {F2, F3} < F4



Then F1 < {F2, F3} < F4

What would happen if SCC4 had a link back to SCC3?

Proof of Lemma

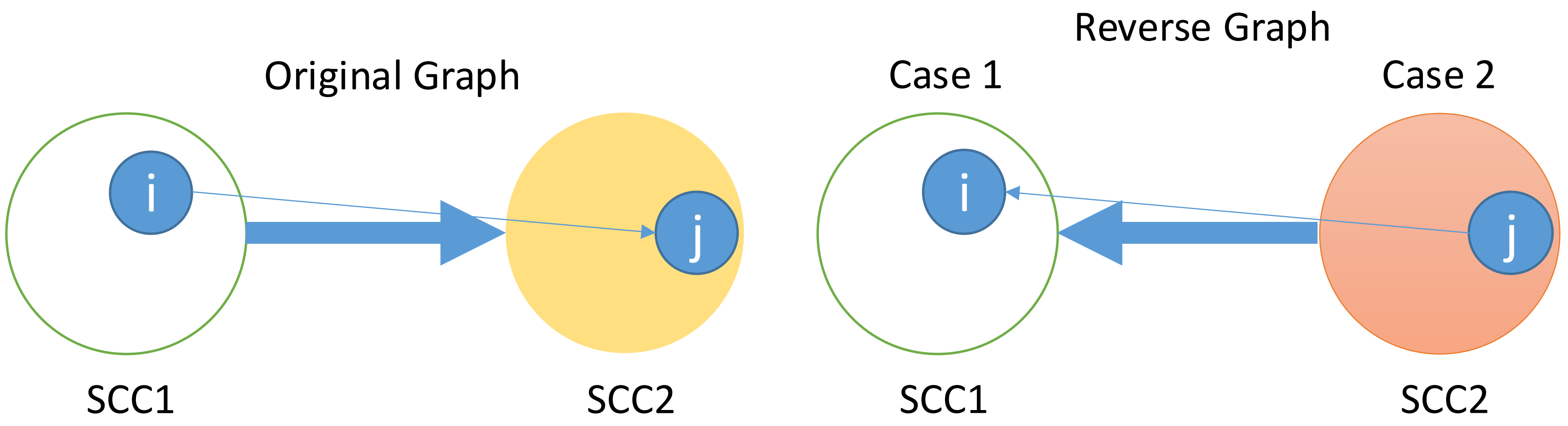


Case 1: consider the case when the first vertex that we explore is in SCC1

- Then all SCC1 is explored before SCC2
- Therefore, all labels in SCC1 are less than all labels in SCC2
- So, in the original graph we will start in SCC2 (the sink)

```
FUNCTION KosarajuDFS(…)
   found[v] = TRUE
   leaders[v] = leader
   FOR vOther IN G.edges[v]
      IF found[vOther] == FALSE
         KosarajuDFS(…)
   label = label + 1
   labels[v] = label
```

Proof of Lemma



Case 2: consider the case when the first vertex that we explore is in SCC2

- All other vertices in SSC2 are explored before vertex j
- All vertices in SSC1 are explored before vertex j
- Therefore, all labels in SSC1 and SSC2 are less than the label of vertex j
- So, in the original graph we will start at vertex j in SSC2 (the sink)

```
FUNCTION KosarajuDFS(…)
   found[v] = TRUE
   leaders[v] = leader
   FOR vOther IN G.edges[v]
      IF found[vOther] == FALSE
         KosarajuDFS(…)
   label = label + 1
   labels[v] = label
```

What does this mean?

- We’ll start the <u>second</u> `KosarajuLoop` at an “SCC sink”
- That sink will then be *removed* (by marking all vertices in the SCC as explored) and we’ll next move to the newly created sink
- And so on

Kosaraju's Algorithm Summary

Computes the SCCs in O(m + n) time **(linear!)**

1. Create a reverse version of the G called G_reversed

2. Run KosarajuLoop on G_reversed
    - Create a topological ordering on the meta graph

3. Create a relabeled version of the G called G_relabeled

4. Run KosarajuLoop on G_relabeled
    - Find all nodes with the same "leader"