Universal Hashing

https://cs.pomona.edu/classes/cs140/

Hash Tables

Operations:

- Insert
- Delete
- Look-up

O(1)

What are they not good for?

Guaranteed constant running time for those operations if:

1. If the hash table is properly implemented, and
2. The data is non-pathological.

Pathological Data Sets

- We want our hash functions to “spread-out” the data (i.e., minimize collisions)
- Unfortunately, no perfect hash function exists (it’s impossible)
- You can create a pathological data set for any hash function

Pathological Data Sets

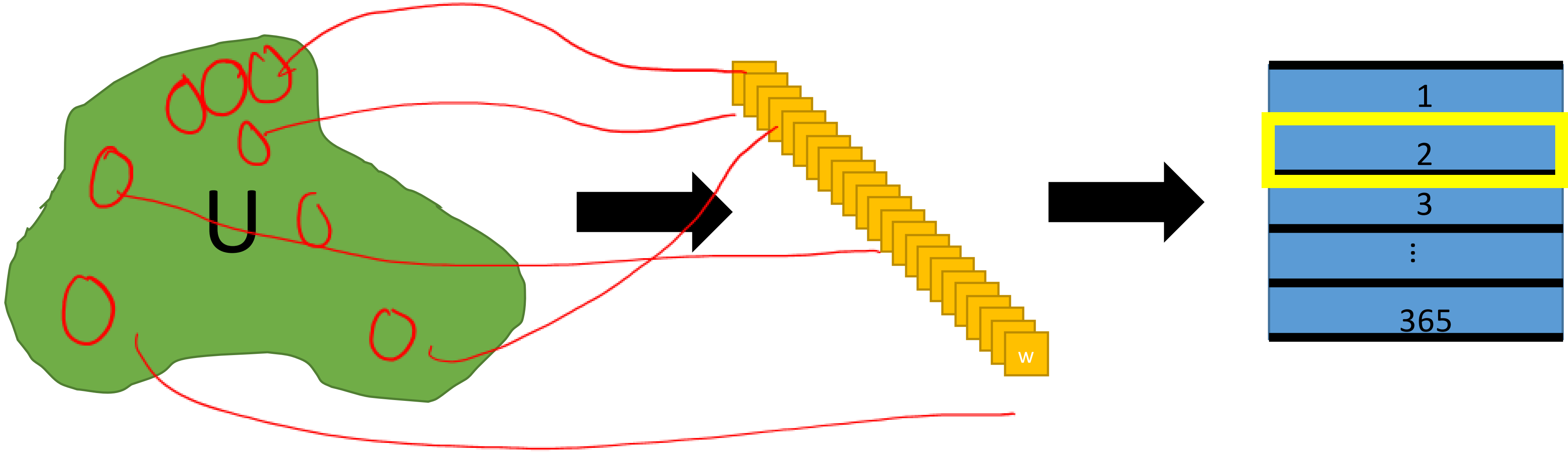
Purposefully select only the elements that map to the same bucket.

Universe of all possible objects

Fix (create) the hash function $h(x) \rightarrow \{0, 1, \ldots, n-1\}$,
where n is the number of buckets in the hash table and $n \ll |U|$

With the pigeonhole principle, there must exist a bucket i, such that at least $|U|/n$ elements of U hash to i under h

$h(x_k) = i$



Pathological Data Set Example

- We want to store student student ID numbers in a hash table.
- We will store about 30 students worth of data
- Let's use a hash table with 87 buckets

  100
- Let's use the final three numbers as the hash

```python
s = 30
n = 87

def hash_fcn(id_number):
    return id_number % n

id_numbers = [randint(1000000, 9999999) for _ in range(s)]
hash_values = map(hash_fcn, id_numbers)
print('Number of unique student IDs:', len(set(id_numbers)))
print('Number of unique hash values:', len(set(hash_values)))


id_numbers_pathological = [round(num,  -2) for num in id_numbers]
hash_values_pathological = map(hash_fcn, id_numbers_pathological)
print('Number of unique student IDs:', len(set(id_numbers_pathological)))
print('Number of unique hash values:', len(set(hash_values_pathological)))
```

Output:

```
Number of unique student IDs: 30
Number of unique hash values: 28

Number of unique student IDs: 30
Number of unique hash values: 1
```

Real World Pathological Data

Distributed DoS

- Denial of service attack (DOS)
- A study in 2003 found that they could interrupt the service of any server with the following attributes:

  1. The server used an open-source hash table
  2. The hash table uses an easy-to-reverse-engineer hash function

- How does reverse engineering the hash function help an attacker?

Create a pathological data set of IPs

Solutions to Pathological Data

$h(x) = \_\_\_\_$

Use a cryptographic hash function

- Infeasible to create pathological data for such a function (but not theoretically impossible)

Use randomization (Can still be an open-source implementation!)

1. Create a family of hash functions
2. Randomly pick one at runtime

Universal Hashing

Let H be a **set** of hash functions mapping U to {0, 1, …, n-1}

The family H is universal if and only if for all x, y in U

$\Pr(h(x) = h(y)) \leq 1/n$ Probability of a collision given any hash function

where h is chosen uniformly at random from H

Hash functions do not consistently map a set of inputs to the same bucket

Example: Hashing IP Addresses

$|U| = 2^{32} = 256^4$
$=$
$4{,}294{,}967{,}296$

- What is U? And how big is U?
- U includes all IP addresses, which we'll denote as 4-tuples
  example: $X = (x_1, x_2, x_3, x_4)$ where x_i is in $[0, 255]$
- Let n = some prime number that is near a multiple of the number of objects we expect to store
  example: $|S| = 500$, we set $n = 997$
- Let H be our set of hash functions
  example: $h(x) = A \text{ dot } X \bmod n = (a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4) \bmod n$
  where $A = (a_1, a_2, a_3, a_4)$ and a_i is in $[0, n-1]$
  H includes all combinations the coefficients in A

IP Address

How large is the family of hash function?

$n \cdot n \cdot n \cdot n$

$|H| = n^4$
$=$
988 billion

$h(x) = (a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4)$ % n

Here are some members of H

- $h_\alpha(x) = (1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4)$ % n
- $h_\beta(x) = (0 \cdot x_1 + 127 \cdot x_2 + 91 \cdot x_3 + 88 \cdot x_4)$ % n
- $h_\gamma(x) = (14 \cdot x_1 + 13 \cdot x_2 + 12 \cdot x_3 + 11 \cdot x_4)$ % n

3 of the family H

```python
n = 997

def ip_hash_fcn(X, A):
    return sum([x * a for x, a in zip(X, A)]) % n

ip_address = [randrange(256) for _ in range(4)] # i.e., 192.168.3.7
hash_coeff = [randrange(n) for _ in range(4)]

print("IP address          :", ".".join(map(str, ip_address)))
print("Hash coefficients :", hash_coeff)
print("Hash value          :", ip_hash_fcn(ip_address, hash_coeff))
```

$$\begin{array}{lll} & & \quad x_1 \quad\; x_2 \quad\; x_3 \quad\; x_4 \\ \texttt{IP address} & : & \texttt{227.75.113.191} \end{array}$$

$$\begin{array}{lll} & & \quad\; a_1 \qquad a_2 \qquad a_3 \qquad a_4 \\ \texttt{Hash coefficients} & : & \texttt{[394, 429, 328, 78]} \end{array}$$

```
Hash value          : 97
```

Example: Hashing IP Addresses

Theorem: the family H is universal

$$\boxed{\frac{\#\ of\ functions\ that\ map\ x\ and\ y\ to\ the\ same\ location}{total\ \#\ of\ functions} \leq \frac{1}{n}}$$

- Let H be a **set** of hash functions mapping U to {0, 1, …, n-1}
- The family H is universal if and only if for all x, y in U
- `Pr(h(x) = h(y)) ≤ 1/n`
- where h is chosen uniformly at random from H

Hashing IP Addresses Proof

- Consider two *distinct* IP addresses X and Y
- Assume that $x_4 \neq y_4$ (they might differ in other places as well)
  - The same argument will hold regardless of which part of the tuple we consider
- Based on our choice of h_i, what is the probability of a collision?
  - What fraction of hash functions (h_i) cause a collision? `Pr[h(X) = h(Y)]`
- Where h_i is any of the hash function from H

- We want to show that ≤ 1/n of the billions of hash functions have a collision for X and Y

Theorem: for any possible hash function, the probability of a collision between objects X and Y is $\leq \frac{1}{n}$

Hash functions are selected from the hash family by randomly generating four values for A

Collision between objects X and Y

$$h(X) = h(Y)$$

$$(A \cdot X)\ mod\ n = (A \cdot Y)\ mod\ n$$

$$(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4)\ mod\ n = (a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4)\ mod\ n$$

$$0 = [a_1(y_1 - x_1) + a_2(y_2 - x_2) + a_3(y_3 - x_3) + a_4(y_4 - x_4)]mod\ n$$

Theorem: for any possible hash function, the probability of a collision between objects X and Y is $\leq \frac{1}{n}$

Hash functions are selected from the hash family by randomly generating four values for A

$$0 = [a_1(y_1 - x_1) + a_2(y_2 - x_2) + a_3(y_3 - x_3) + a_4(y_4 - x_4)] mod\ n$$

Something must be different between X and Y. Let's assume that $x_4 \neq y_4$

R

$$a_4(x_4 - y_4)\ mod\ n = [a_1(y_1 - x_1) + a_2(y_2 - x_2) + a_3(y_3 - x_3)] mod\ n$$

Non-zero value that depends on a_4

Assume n is prime.

From here we are going to **fix** our choices of a_1, a_2, and a_3 and let a_4 continue to be a random variable

Principle of Deferred Decisions

We want to show that for any value of a_4 we have a $\frac{1}{n}$ chance of a collision.

Theorem: for any possible hash function, the probability of a collision between objects X and Y is $\leq \frac{1}{n}$

Something must be different between X and Y. Let's assume that $x_4 \neq y_4$

Non-zero value that depends on a_4

Assume n is prime.

$$a_4(x_4 - y_4) \bmod n = [a_1(y_1 - x_1) + a_2(y_2 - x_2) + a_3(y_3 - x_3)] \bmod n$$

[0, n-1] [0, n-1]

From here we are going to **fix** our choices of a_1, a_2, and a_3 and let a_4 be a random variable

Principle of Deferred Decisions

We want to show that for any value of a_4 we have a $\frac{1}{n}$ chance of a collision.

How many choices of a_4 satisfy the above equation?

TTYNs

- Our RHS is some constant! It is just some number in $[0, n\text{-}1]$ because X, Y, and a_1, a_2, a_3 are fixed
- If n is a prime number, then the LHS is equally likely to be any number from $[0, n\text{-}1]$
  - This claim requires some number theory to properly prove

Unique multiplicative

Thus, based on our choice for a_4, we have that $Pr(h(X) = h(Y)) = 1/n$

Prime number for n

$X = (x_1, x_2, x_3, x_4)$ where x_i is in [0, 255]

$Y = (y_1, y_2, y_3, y_4)$ where y_i is in [0, 255]

$A = (a_1, a_2, a_3, a_4)$ and a_i is in [0, n-1]

|S| = 500

n = 997

$h(x) = (A \cdot X) \bmod n$

And H includes all combinations for the coefficients in A

$n = 7, x_4 = 3, y_4 = 1$

a_4	$a_4(x_4 - y_4) \bmod n$
0	0 · (2) % 7 = 0
1	1 · (2) % 7 2
2	2 · (2) % 7 4
3	3 · (2) % 7 6
4	4 · (2) % 7 1
5	5 · (2) % 7 3
6	6 · (2) % 7 5

Different hash functions from the family H

What do we want in the second column?

Different values indicate different hash values, which is good.

$$a_4(x_4 - y_4) \bmod n = [a_1(y_1 - x_1) + a_2(y_2 - x_2) + a_3(y_3 - x_3)] \bmod n$$

Prime number for n

Different hash functions from the family H

n = 7, x_4 = 3, y_4 = 1

a_4	$a_4(x_4 - y_4) \ mod \ n$
0	0
1	2
2	4
3	6
4	1
5	3
6	5

n = 7, x_4 = 4, y_4 = 1

a_4	$a_4(x_4 - y_4) \ mod \ n$
0	0
1	3
2	6
3	2
4	5
5	1
6	4

Non-Prime number for n

x4-y4 shares factors with n

$n = 8, x_4 = 3, y_4 = 1$

a_4	$a_4(x_4 - y_4) \bmod n$
0	0
1	2
2	4
3	6
4	0
5	2
6	4
7	6

Different hash functions from the family H

$n = 8, x_4 = 4, y_4 = 1$

a_4	$a_4(x_4 - y_4) \bmod n$
0	0
1	3
2	6
3	1
4	4
5	7
6	2
7	5

Summary

- We cannot create a hash function that prevents creation of a pathological dataset

- As long as the hash function is known, a pathological dataset can be created

- We can create families of hash functions that make it infeasible to guess which hash function is in use